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OSCILLATION OF NONLINEAR DELAY DIFFERENCE EQUATIONS

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Abstract. We obtain some oscillation criteria for solutions of the nonlinear delay difference equation of the form $x_{n+1}-x_n+p_n\prod_{j=1}^m x_{n-k_j}^{\alpha_j}=0$.

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1. Introduction. Consider the nonlinear delay difference equation

$$x_{n+1}-x_n+p_n\prod_{j=1}^{m}x_{n-k_j}^{\alpha_j}=0,\quad n=0,1,2,\dots,\tag{1.1}$$

where $p_n\ge 0$, $n=0,1,2,\dots$, $0\le k_1\le k_2\le\cdots\le k_m$ are integers, $\alpha_j>0$ are rational numbers with denominator of positive odd integers for each $j=1,2,\dots,m$ and $\sum_{j=1}^m\alpha_j=1$.

Equation (1.1) is a discrete analogue of the following first-order nonlinear delay differential equation

$$x'(t)+p(t)\prod_{j=1}^{m}\left[x(t-\tau_j)\right]^{\alpha_j}=0,\tag{1.2}$$

where $p(t)\in C([t_0,\infty),[0,\infty))$, $0\le\tau_1\le\tau_2\le\cdots\le\tau_m$, and α_j are the same as in (1.1). For (1.2), the oscillation of its solutions has been extensively studied in the literature, see, for example [2, 4, 11, 12].

When $k_1=k_2=\cdots=k_m=k$, (1.1) reduces to the linear delay difference equation

$$x_{n+1}-x_n+p_nx_{n-k}=0,\quad n=0,1,2,\dots.\tag{1.3}$$

Recently, there has been a lot of activity concerning the oscillatory behavior of (1.3). See, for example, [1, 3, 5, 6, 7, 8, 9, 10]. In particular, [7] proved that every solution of (1.3) is oscillatory provided

$$\sum_{n=0}^{\infty}\left[\sum_{i=n}^{n+k}p_i\ln\left(\sum_{i=n}^{n+k}p_i+1-\operatorname{sign}\sum_{i=n}^{n+k}p_i\right)-\sum_{i=n+1}^{n+k}p_i\ln\left(\sum_{i=n+1}^{n+k}p_i+1-\operatorname{sign}\sum_{i=n+1}^{n+k}p_i\right)\right]=\infty.\tag{1.4}$$

Condition (1.4) improves many previous well-known results. Furthermore, (1.4) fits the case when $\sum_{i=n-k}^{n-1}p_i-(k/(k+1))^{k+1}$ oscillate or $\sum_{i=n-k}^{n-1}p_i\le(k/(k+1))^{k+1}$.

Our main aim in this note is to generalize condition (1.4) to (1.1).

2. Main results

Theorem 2.1. *Assume that*

$$\begin{aligned}\sum_{n=0}^{\infty}\Bigg\{&\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i\ln\Bigg[\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i+1-\operatorname{sign}\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i\Bigg)\Bigg]\\&-\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i\ln\Bigg[\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i+1-\operatorname{sign}\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i\Bigg)\Bigg]\Bigg\}=\infty.\end{aligned}\tag{2.1}$$

Then every solution of (1.1) oscillates.

Proof. Assume, by way of contradiction, that (1.1) has an eventually positive solutions $\{x_n\}$. Then there exists an integer $n_1 > 0$ such that $x_{n-k_m} > 0$, $x_{n+1} \le x_n$, $n \ge n_1$. We define the functions $p(t)$ and $x(t)$ as follows:

$$p(t)=p_n,\quad x(t)=x_n+(t-n)(x_{n+1}-x_n),\quad n\le t<n+1,\ n=0,1,2,\dots.\tag{2.2}$$

Let $x'(t)$ denote derivation on the right. Then

$$\begin{gathered}x(t)>0,\quad x'(t)\le 0,\quad t\ge n_1,\\ x'(t)=x_{n+1}-x_n\quad\text{for } n\le t<n+1,\ n=0,1,2,\dots.\end{gathered}\tag{2.3}$$

Hence, (1.1) can be rewritten as

$$x'(t)+p(t)\prod_{j=1}^{m}(x[t-k_j])^{\alpha_j}=0,\quad t\ge 0,\tag{2.4}$$

where here and in the sequel, $[\cdot]$ denotes the greatest integer function.

Set $\lambda(t)=-x'(t)/x(t)$ for $t\ge n_1$. Then $\lambda(t)\ge 0$ for $t\ge n_1$, and from (2.4) we have

$$\lambda(t)=p(t)\exp\Bigg(\sum_{j=1}^{m}\alpha_j\int_{[t-k_j]}^{t}\lambda(s)ds\Bigg),\quad t\ge n_1+k_m,\tag{2.5}$$

or

$$\begin{aligned}&\lambda(t)\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]}p(s)ds\\&\quad\ge p(t)\Bigg(\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]}\lambda(s)ds\Bigg)\exp\Bigg(\sum_{j=1}^{m}\alpha_j\int_{[t-k_j]}^{t}\lambda(s)ds\Bigg),\quad t\ge n_1+k_m.\end{aligned}\tag{2.6}$$

One can easily show that

$$\phi(r)e^x\ge\phi(r)x+\phi(r)\ln(er+1-\operatorname{sign}r),\quad r\ge 0,\ x\ge R,\tag{2.7}$$

where $\phi(0)=0$ and $\varphi(r)\ge 0$ for $r>0$. By the definition of $p(t)$, we see that $p(t)$ is nonnegative and right-continuous. Therefore, it follows from

$$\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]}p(s)ds=0 \tag{2.8}$$

that $p(t)=0$. Applying inequalities (2.7) to the right side of (2.6), we obtain

$$\lambda(t)\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]}p(s)ds\geq p(t)\sum_{j=1}^{m}\alpha_j\int_{[t-k_j]}^{t}\lambda(s)ds+p(t)\ln A(t),\quad n\geq n_1+k_m, \tag{2.9}$$

where

$$A(t)=e\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]}p(s)ds+1-\operatorname{sign}\left(\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]}p(s)ds\right). \tag{2.10}$$

Set $n_2=n_1+k_m$. Integrating both sides of (2.9) from n_2 to $N>n_2+2k_m$, we have

$$\begin{aligned}&\sum_{j=1}^{m}\alpha_j\int_{n_2}^{N}\lambda(t)\int_t^{[t+k_j+1]}p(s)ds\,dt\\&\qquad\geq\sum_{j=1}^{m}\alpha_j\int_{n_2}^{N}p(t)\int_{[t-k_j]}^{t}\lambda(s)ds\,dt+\int_{n_2}^{N}p(t)\ln A(t)dt.\end{aligned} \tag{2.11}$$

Interchanging the order of integration, we get

$$\begin{aligned}\int_{n_2}^{N}p(t)\int_{[t-k_j]}^{t}\lambda(s)ds\,dt&\geq\int_{n_2}^{N-k_j}\lambda(s)\int_s^{[s+k_j+1]}p(t)dt\,ds\\&=\int_{n_2}^{N-k_j}\lambda(t)\int_t^{[t+k_j+1]}p(s)ds\,dt.\end{aligned} \tag{2.12}$$

Substituting this into (2.11), we have

$$\sum_{j=1}^{m}\alpha_j\int_{N-k_j}^{N}\lambda(t)\int_t^{[t+k_j+1]}p(s)ds\,dt\geq\int_{n_2}^{N}p(t)\ln A(t)dt. \tag{2.13}$$

From (1.1) we have

$$x_{n+1}-x_n+p_nx_n^{1-\alpha_m}x_{n-k_m}^{\alpha_m}\leq 0,\quad n\geq n_1. \tag{2.14}$$

Set $y_n=x_n^{\alpha_m}$ for $n\geq n_1$. Then

$$y_{n+1}-y_n+\alpha_mp_ny_{n-k_m}\leq 0,\quad n\geq n_1+k_m. \tag{2.15}$$

It follows that $\alpha_m\sum_{i=n-k_m}^{n}p_i\leq 1$, and so

$$\alpha_m\int_t^{[t+k_m+1]}p(s)ds\leq\alpha_m\sum_{i=[t]}^{[t]+k_m}p_i\leq 1,\quad t\geq n_2. \tag{2.16}$$

Recall that $k_1 \le k_2 \le \cdots \le k_m$, therefore

$$\alpha_j \int_t^{[t+k_j+1]} p(s)\,ds \le \frac{\alpha_j}{\alpha_m}, \quad t \ge n_2,\ j = 1,2,\dots,m. \tag{2.17}$$

Substituting this into (2.13), we obtain

$$\sum_{j=1}^{m} \frac{\alpha_j}{\alpha_m} \int_{N-k_j}^{N} \lambda(t)\,dt \ge \int_{n_2}^{N} p(t)\ln A(t)\,dt, \tag{2.18}$$

or

$$\ln\left(\prod_{j=1}^{m}\left(\frac{x(N-k_j)}{x(N)}\right)^{\alpha_j}\right) \ge \alpha_m \int_{n_2}^{N} p(t)\ln A(t)\,dt. \tag{2.19}$$

It follows that

$$\lim_{N\to\infty} \prod_{i=1}^{m}\left(\frac{x(N-k_j)}{x(N)}\right)^{\alpha_j} \ge \exp\left(\alpha_m \int_{n_2}^{\infty} p(t)\ln A(t)\,dt\right). \tag{2.20}$$

Let $E = \{n \ge n_2 \mid p_n > 0\}$. Then

$$\begin{aligned}
&\int_{n_2}^{\infty} p(t)\ln A(t)\,dt \\
&\quad= \sum_{n=n_2}^{\infty}\int_n^{n+1} p(t)\ln\left(e\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]} p(s)\,ds + 1 - \operatorname{sign}\left(\sum_{j=1}^{m}\alpha_j\int_t^{[t+k_j+1]} p(s)\,ds\right)\right)dt \\
&\quad= \sum_{n=n_2}^{\infty} p_n \int_n^{n+1} \ln\left(e\sum_{j=1}^{m}\alpha_j\left(\int_n^{n+k_j+1} p(s)\,ds - \int_n^{t} p(s)\,ds\right) + 1\right. \\
&\qquad\qquad \left. - \operatorname{sign}\left(\sum_{j=1}^{m}\alpha_j\left(\int_n^{n+k_j+1} p(s)\,ds - \int_n^{t} p(s)\,ds\right)\right)\right)dt \\
&\quad= \sum_{n=n_2}^{\infty} p_n \int_n^{n+1} \ln\left(e\sum_{j=1}^{m}\alpha_j\left(\sum_{i=n}^{n+k_j} p_i - p_n(t-n)\right) + 1\right. \\
&\qquad\qquad \left. - \operatorname{sign}\left(\sum_{j=1}^{m}\alpha_j\left(\sum_{i=n}^{n+k_j} p_i - p_n(t-n)\right)\right)\right)dt \\
&\quad= \sum_{E} p_n \int_n^{n+1} \ln\left(e\sum_{j=1}^{m}\alpha_j\left(\sum_{i=n}^{n+k_j} p_i - p_n(t-n)\right)\right)dt
\end{aligned}$$

$$\begin{aligned}
&= \sum_{E}\Bigg(\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i\ln\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i\Bigg)\Bigg) \\
&\qquad -\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i\ln\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i+1-\operatorname{sign}\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i\Bigg)\Bigg)\Bigg) \\
&= \sum_{n=n_2}^{\infty}\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i\ln\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i+1-\operatorname{sign}\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n}^{n+k_j}p_i\Bigg)\Bigg) \\
&\qquad -\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i\ln\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i+1-\operatorname{sign}\Bigg(\sum_{j=1}^{m}\alpha_j\sum_{i=n+1}^{n+k_j}p_i\Bigg)\Bigg)\Bigg).
\end{aligned} \tag{2.21}$$

From (2.1) and (2.20), we have

$$\lim_{N\to\infty}\prod_{j=1}^{m}\left(\frac{x(N-k_j)}{x(N)}\right)^{\alpha_j}=\infty. \tag{2.22}$$

On the other hand, it follows from (2.1) that

$$\limsup_{n\to\infty} p_n > 0. \tag{2.23}$$

By (2.15), we have

$$\limsup_{n\to\infty} p_n \le \limsup_{n\to\infty}\frac{y_n}{\alpha_m y_{n-k_m}}=\frac{1}{\alpha_m}\frac{1}{\liminf_{n\to\infty}(y_{n-k_m}/y_n)}, \tag{2.24}$$

which, together with (2.23) yields

$$\liminf_{n\to\infty}\frac{y_{n-k_m}}{y_n}<\infty, \tag{2.25}$$

that is,

$$\liminf_{N\to\infty}\left(\frac{x(N-k_m)}{x(N)}\right)^{\alpha_m}<\infty, \tag{2.26}$$

and so

$$\liminf_{N\to\infty}\prod_{j=1}^{m}\left(\frac{x(N-k_j)}{x(N)}\right)^{\alpha_j}\le\liminf_{N\to\infty}\frac{x(N-k_m)}{x(N)}<\infty, \tag{2.27}$$

which contradicts (2.22) and so the proof is completed. □

From Theorem 2.1 we have immediately.

Corollary 2.2. *Assume that there exists an integer* $N \geq 0$ *such that*

$$\sum_{j=1}^{m} \alpha_j \sum_{i=n}^{n+k_j} p_i > 0 \quad \textit{for } n \geq N, \tag{2.28}$$

and that

$$\sum_{n=N}^{\infty} \left[\sum_{j=1}^{m} \alpha_j \sum_{j=n}^{n+k_j} p_i \ln\left(\sum_{j=1}^{m} \alpha_j \sum_{i=n}^{n+k_j} p_i \right) - \sum_{j=1}^{m} \alpha_j \sum_{i=n+1}^{n+k_j} p_i \ln\left(\sum_{j=1}^{m} \alpha_j \sum_{i=n+1}^{n+k_i} p_i \right) \right] = \infty. \tag{2.29}$$

Then every solution of (1.1) oscillates.

Clearly, when $k_1 = k_2 = \cdots = k_m$, condition (2.1) reduces to (1.4).

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