IJMMS 28:1 (2001) 25–40
PII. S0161171201011668
http://ijmms.hindawi.com

DESCRIPTION OF THE STRUCTURE OF SINGULAR SPECTRUM FOR FRIEDRICHS MODEL OPERATOR NEAR SINGULAR POINT

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(Received 28 January 2001)

Abstract. The study of the point spectrum and the singular continuous one is reduced to investigating the structure of the real roots set of an analytic function with positive imaginary part $M(\lambda)$. We prove a uniqueness theorem for such a class of analytic functions. Combining this theorem with a lemma on smoothness of $M(\lambda)$ near its real roots permits us to describe the density of the singular spectrum.

2000 Mathematics Subject Classification. 47B06, 47B25.

1. Statement of the problem. We consider a selfadjoint operator A_2 given by

$$A_2 = t^2 \cdot + (\cdot,\varphi)\varphi \tag{1.1}$$

on the domain of functions $u(t) \in L_2(\mathbb{R})$ such that $t^2u(t) \in L_2(\mathbb{R})$. Here $\varphi \in L_2(\mathbb{R})$ and t is the independent variable. The action of the operator can be written as follows:

$$(A_2u)(t) = t^2 \cdot u(t) + \varphi(t)\int_{\mathbb{R}} u(x)\overline{\varphi(x)}dx. \tag{1.2}$$

The function φ is assumed to satisfy the smoothness condition

$$\left|\varphi(t+h)-\varphi(t)\right| \le \omega(|h|), \quad |h| \le 1, \tag{1.3}$$

where the function $\omega(t)$ (the modulus of continuity of the function φ) is monotone and satisfies a Dini condition

$$\omega(t) \downarrow 0 \quad \text{as} \quad t \downarrow 0, \quad \int_0^1 \frac{\omega(t)}{t}dt < \infty. \tag{1.4}$$

We are going to study the singular spectrum of the operator A_2. Note that we define the singular spectrum as the union of the point spectrum and the singular continuous one. The structure of the spectrum $\sigma_{\text{sing}}(S_1)$ (the singular spectrum of the operator $S_1 = t \cdot + (\cdot,\varphi)\varphi$) has been studied in detail (see [2, 3, 6, 7, 8, 9, 10, 12, 13, 14]). By using the simple change of variables $t^2 = x$, one can show that outside of any neighborhood of the origin the structure of the spectrum $\sigma_{\text{sing}}(A_2)$ is identical with the one of the operator S_1. This is due to the fact that this change of variables is smooth outside of any neighborhood of the origin. Suppose that conditions (1.3), (1.4), and also some additional conditions on the function φ are fulfilled only in a certain interval $(c,d) \subset \mathbb{R}$, then the main results of [2, 3, 6, 7, 8, 9, 10, 12, 13, 14] concerning the structure of $\sigma_{\text{sing}}(S_1)$ will remain true in any closed subinterval $\Delta \subset (c,d)$. At the same time, as it has been shown in [15], for the operator A_2 the behavior of the singular spectrum has quite different character in a neighborhood of the origin. Here we can also use the pointed change of variables but, since $(t^2)'|_0 = 0$, it is not smooth (i.e., not a diffeomorphism) near zero. Therefore, the point zero needs our special attention and we are going to study the singular spectrum just in a neighborhood of this singular point. Note that the origin is also a boundary point of the continuous spectrum of A_2 coinciding with the interval $[0,+\infty)$.

2. Analytic function $M(z)$ and the singular spectrum. One of the approaches to the investigation of the point and singular continuous spectra in the Friedrichs model is based on studying some properties of analytic functions with positive imaginary part. It is possible to define an analytic function in such a way that the singular spectrum of the perturbed operator embeds into its real roots.

Determine for $z \in \mathbb{C}\setminus[0,+\infty)$ an analytic function $M(z)$ as follows:

$$M(z) = 1 + \int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{t^2 - z}\,dt. \tag{2.1}$$

The proof of the following propositions is contained in [15].

Proposition 2.1. *If conditions (1.3) and (1.4) are fulfilled, then the analytic function $M(z)$ defined in the complex plane with the slit $(0,+\infty)$ has continuous boundary values on the edges of the slit.*

We determine for $\lambda > 0$ the value $M(\lambda) := M(\lambda + i0)$ and let $N := \{\lambda > 0 : M(\lambda) = 0\}$ be the set of roots of the analytic function $M(z)$. The set N is bounded [15].

Proposition 2.2. *If the function φ satisfies conditions (1.3) and (1.4), then the singular spectrum of the operator A_2, defined by (1.1), embeds into the set N plus the origin, that is, $\sigma_{\text{sing}}(A_2) \subset N \cup \{0\}$.*

So the investigation of $\sigma_{\text{sing}}(A_2)$ is reduced to the description of the set of roots N. (It is not difficult to show that zero is not an eigenvalue of the operator $A_2 = t^2 \cdot +(\cdot,\varphi)\varphi$ [15].) It follows that we need to study the behavior of the function $M(z)$ in a neighborhood of its real roots. (The behavior of boundary functions and, in particular, their sets of uniqueness were studied by many authors. See, for example, [1].) For this purpose we prove a certain uniqueness theorem for this function, which imposes some restrictions on the admissible structure of the set of its roots. This uniqueness theorem may be applied in fact to the whole class of analytic functions. The functions from this class admit a representation in a specific form. We start Section 3 with the description of this class of functions.

3. Uniqueness theorem. It is self-evident that, using the change of variables $t^2 = \tau$, the function $M(z)$ can be written in the form

$$M(z) = 1 + \int_0^{+\infty} \frac{\psi(\tau)}{\tau - z}\,d\tau, \quad z \notin [0,+\infty), \tag{3.1}$$

where

$$\psi(\tau) = \frac{|\varphi^2(\sqrt{\tau})| + |\varphi^2(-\sqrt{\tau})|}{2\sqrt{\tau}}. \tag{3.2}$$

The following lemma describes a class of analytic functions. It is for this class that a uniqueness theorem will be formulated.

Lemma 3.1. *Let the function $f(z)$ be written in the form*

$$f(z) = 1 + \int_0^{+\infty} \frac{d\nu(t)}{t-z}, \quad z \in \mathbb{C} \setminus [0,+\infty), \tag{3.3}$$

with a positive finite measure $d\nu(t)$,

$$d\nu(t) \geq 0, \quad \int_0^{+\infty} d\nu(t) < \infty. \tag{3.4}$$

Then the function $(f(z))^{-1}$ possesses the representation

$$(f(z))^{-1} = 1 - \int_0^{+\infty} \frac{d\mu(t)}{t-z}, \tag{3.5}$$

where the positive finite measure $d\mu(t)$ has the following properties:

$$\int_0^1 \frac{d\mu(t)}{t} \leq 1, \tag{3.6}$$

$$\int_0^{+\infty} \frac{y\, d\mu(t)}{t^2+y^2} \leq 1 \quad \text{for } y > 0. \tag{3.7}$$

Proof. The function $\varphi(z) := f(z) - 1$ has the integral representation

$$\varphi(z) = \int_{-\infty}^{+\infty} \frac{d\nu(t)}{t-z} \tag{3.8}$$

with the positive finite measure $d\nu(t)$ (in addition in our case $d\nu(t) = 0$ for $t < 0$), that is, according to the definition (see [8, 9]), $\varphi(z)$ is an analytic R_0-function. Recall that for the function to belong to the class R_0 it is necessary and sufficient, for example, that

$$\operatorname{Im}\varphi(z) \geq 0 \quad \text{for } \operatorname{Im} z > 0, \qquad \varphi(iy) \longrightarrow 0 \quad \text{as } y \longrightarrow +\infty, \tag{3.9}$$

$$\lim_{y\to+\infty} y \operatorname{Im}\varphi(iy) < \infty. \tag{3.10}$$

If this is the case, the following relation is easily established

$$\lim_{y\to+\infty} y \operatorname{Im}\varphi(iy) = \int_{-\infty}^{+\infty} d\nu(t). \tag{3.11}$$

Note that $f(z)$ has no zeros in $\mathbb{C} \setminus [0,+\infty)$. In fact, if $\operatorname{Im} z_0 > 0$ and $f(z_0) = 0$, then by the maximum principle for harmonic functions $\operatorname{Im} f(z) = \int_0^{+\infty} y/((t-x)^2 + y^2)\, d\nu(t) \geq 0$ is identically equal to zero in $\mathbb{C}_+$. This is possible provided that the spectral function $\nu(t)$ is constant. Then from the integral representation $f(z) = 1$

for all $z \in \mathbb{C}_+$. The case $\mathbb{C}_-$ is treated analogously. Now if $z = x_0 < 0$, then $f(x_0) = 1 + \int_0^{+\infty} d\nu(t)/(t - x_0) \geq 1$. At the same time under certain smoothness conditions on $\nu(t)$ the function $f(z)$ can be continuously extended to the positive half of the real axis $(0, +\infty)$, where it can already have zeros. Studying the density of this zero set as a closed set of Lebesgue measure zero is the main purpose of this paper.

Verify that the function

$$g(z) := 1 - \frac{1}{f(z)} = \frac{\varphi(z)}{1+\varphi(z)}, \quad z \in \mathbb{C} \setminus [0, +\infty), \tag{3.12}$$

is also an analytic R_0-function. Conditions (3.9) are obviously fulfilled for $g(z)$. For checking condition (3.10), note that $\operatorname{Im} g(z) = \operatorname{Im} \varphi(z)/|1 + \varphi(z)|^2$. Then clearly

$$\lim_{y \to +\infty} y \operatorname{Im} g(iy) = \lim_{y \to +\infty} y \frac{\operatorname{Im} \varphi(iy)}{|1+\varphi(iy)|^2} = \lim_{y \to +\infty} y \operatorname{Im} \varphi(iy) < \infty. \tag{3.13}$$

Hence,

$$g(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \tag{3.14}$$

with a finite positive measure $d\mu(t)$. If $x > 0$, the function $g(-x) = 1 - 1/f(-x)$ take real values, therefore by the Stiltjes inversion formula the spectral function $\mu(t)$ has no points of growth in the interval $(-\infty, 0)$. By letting $\mu(t)$ be left continuous at zero, we can write

$$g(z) = \int_0^{+\infty} \frac{d\mu(t)}{t - z}. \tag{3.15}$$

Then

$$\frac{1}{f(z)} = 1 - \int_0^{+\infty} \frac{d\mu(t)}{t - z}. \tag{3.16}$$

Besides, using (3.11) and (3.13), we get

$$\int_0^{+\infty} d\mu(t) = \lim_{y \to +\infty} y \operatorname{Im} g(iy) = \lim_{y \to +\infty} y \operatorname{Im} \varphi(iy) = \int_0^{+\infty} d\nu(t). \tag{3.17}$$

When $x > 0$, the following equality holds

$$\int_0^{+\infty} \frac{d\mu(t)}{t + x} = g(-x) = 1 - \frac{1}{1 + \int_0^{+\infty} d\nu(t)/(t+x)}. \tag{3.18}$$

Letting $x \to 0^+$ in it, we find

$$\int_0^{+\infty} \frac{d\mu(t)}{t} = 1 - \frac{1}{1 + \int_0^{+\infty} d\nu(t)/t} \leq 1. \tag{3.19}$$

Since $\operatorname{Re} \varphi(iy) = \int_0^{+\infty} (t/(t^2 + y^2)) d\nu(t) \geq 0$ for $y > 0$, we obviously have

$$\operatorname{Im} g(iy) = \frac{\operatorname{Im} \varphi(iy)}{|1+\varphi(iy)|^2} \leq 1. \tag{3.20}$$

It follows that

$$\int_0^{+\infty} \frac{y}{t^2+y^2}\, d\mu(t) = \operatorname{Im} g(iy) \le 1. \tag{3.21}$$

This completes the proof. □

The proof of a uniqueness theorem, which is formulated below, is based on Lemma 3.1 and on the following remark. As it was shown in [4], if a positive locally integrable (with respect to Lebesgue measure) function $\sigma(t)$ defined on the real axis satisfies the following condition:

$$\sup_{I\subset\mathbb{R}} \left\{ \left(\frac{1}{|I|} \int_I \sigma(x)\,dx \right) \cdot \operatorname*{ess\,sup}_{x\in I} \frac{1}{\sigma(x)} \right\} < \infty, \tag{3.22}$$

where I is an arbitrary finite interval of the real axis, then for the Hilbert transform $\hat{H}$ of any $g \in L_{1,\sigma}(\mathbb{R})$ the following weighted norm inequality

$$\int_{\{|\hat{H}g|>a\}} \sigma(t)\,dt \le \frac{C}{a} \cdot \int_{-\infty}^{+\infty} |g(t)|\,\sigma(t)\,dt, \quad a > 0, \tag{3.23}$$

holds with a constant C independent of g and a. (Here, and later, we denote by C various absolute constants.)

Note that in the sequel we use the notation $\sigma - \operatorname{mes} I := \int_I \sigma(x)\,dx$ for $I \subset \mathbb{R}$.

Theorem 3.2 (uniqueness theorem). *Let $\sigma(t)dt$ be the measure on the real axis with the positive weight function $\sigma(t)$ being even, monotonically decreasing on the positive half of the real axis*

$$\sigma(t) = \sigma(-t); \quad \sigma(t)\downarrow \quad \text{as } t \in (0,+\infty), \tag{3.24}$$

and satisfying condition (3.22). Let the analytic function $f(z)$ be written in the form (3.3) and (3.4). Then the estimate

$$\sigma - \operatorname{mes}\{x > 0 : |f(x+iy)| < d\} \le C\,d \tag{3.25}$$

holds for all sufficiently small $d > 0$ with a constant C independent of $y > 0$.

Proof. For $a = 1/d$ we have

$$\sigma - \operatorname{mes}\{x > 0 : |f(x+iy)| < d\} = \sigma - \operatorname{mes}\{x > 0 : |f^{-1}(x+iy)| > a\}. \tag{3.26}$$

By Lemma 3.1,

$$\begin{aligned} f^{-1}(x+iy) &= 1 - \int_0^{+\infty} \frac{(t-x)\,d\mu(t)}{(t-x)^2+y^2} - i\int_0^{+\infty} \frac{y\,d\mu(t)}{(t-x)^2+y^2} \\ &=: 1 - u(x+iy) - iv(x+iy). \end{aligned} \tag{3.27}$$

Clearly,

$$\int_{\mathbb{R}} v(x+iy)\frac{dx}{1+|x|} \le \int_0^{+\infty} d\mu(t)\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}dx = \pi\int_0^{+\infty} d\mu(t) < +\infty. \tag{3.28}$$

Therefore (cf. [5, Chapter 6]), using the properties of the Poisson kernel, for $\tau \in \mathbb{R}$ and $\delta > 0$ we have

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\tau-x}{(\tau-x)^2+\delta^2} v(x+iy)dx \\
&\quad = \int_0^{+\infty} d\mu(t)\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tau-x}{(\tau-x)^2+\delta^2}\cdot\frac{y}{(t-x)^2+y^2}dx \\
&\quad = \int_0^{+\infty}\frac{\tau-t}{(\tau-t)^2+(y+\delta)^2}d\mu(t) \\
&\quad = u(\tau+i(y+\delta)).
\end{aligned} \tag{3.29}$$

Consequently,

$$u(\tau+iy) = \lim_{\delta\to 0+}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tau-x}{(\tau-x)^2+\delta^2}v(x+iy)dx = \hat{H}_{x\to\tau}v(x+iy), \tag{3.30}$$

that is, for any fixed $y > 0$ the next relation is valid

$$(\hat{H}_x v)(x+iy) = u(x+iy). \tag{3.31}$$

Hence, by (3.23), for every $y > 0$

$$\sigma - \operatorname{mes}\{x : |u(x+iy)| > a\} \le \frac{C}{a}\int_{\mathbb{R}} v(x+iy)\sigma(x)dx. \tag{3.32}$$

We will estimate the integral

$$\int_{\mathbb{R}} v(x+iy)\sigma(x)dx = \int_0^{+\infty} d\mu(t)\int_{\mathbb{R}}\frac{y\sigma(x)}{(t-x)^2+y^2}dx. \tag{3.33}$$

For this we split the domain of inner integration into three parts

$$\int_{\mathbb{R}}\frac{y\sigma(x)}{(t-x)^2+y^2}dx = \left(\int_{-\infty}^{-1}+\int_{-1}^{t/2}+\int_{t/2}^{+\infty}\right)\frac{y\sigma(x)}{(t-x)^2+y^2}dx. \tag{3.34}$$

First observe that by (3.24) $\sigma(|x|) \le \sigma(1)$ if $|x| \ge 1$. Besides, substituting $I = (0,1)$ into (3.22), we find that for $x \in (0,1)$

$$C \ge \int_0^1 \sigma(t)dt\cdot\frac{1}{\sigma(1)} \ge \int_0^x \sigma(t)dt\cdot\frac{1}{\sigma(1)} \ge \frac{\sigma(x)\cdot x}{\sigma(1)}, \tag{3.35}$$

that is,

$$\sigma(x) \le C\sigma(1)\cdot\frac{1}{x}, \quad x \in (0,1). \tag{3.36}$$

Using the first inequality, we obtain

$$\begin{aligned}
&\int_{-\infty}^{-1} \frac{y\sigma(x)}{(t-x)^2+y^2}dx \le \sigma(1)\int_{-\infty}^{+\infty} \frac{y}{(t-x)^2+y^2}dx = \pi\sigma(1),\\
&\int_{-1}^{t/2} \frac{y\sigma(x)}{(t-x)^2+y^2}dx \le \frac{y}{(t/2)^2+y^2}\int_{-1}^{t/2}\sigma(x)dx\\
&\qquad\qquad \le \frac{y}{(t/2)^2+y^2}\left(2\int_0^1\sigma(x)dx+\sigma(1)\cdot\frac{t}{2}\right),\\
&\int_{t/2}^{+\infty} \frac{y\sigma(x)}{(t-x)^2+y^2}dx \le \sigma\left(\frac{t}{2}\right)\int_{-\infty}^{+\infty} \frac{y}{(t-x)^2+y^2}dx = \pi\sigma\left(\frac{t}{2}\right).
\end{aligned} \tag{3.37}$$

Thus we have

$$\begin{aligned}
\int_{\mathbb{R}} v(x+iy)\sigma(x)dx \le \int_0^{+\infty} d\mu(t)\Big[\pi\sigma(1)+\pi\sigma\left(\frac{t}{2}\right)\\
+\frac{y}{(t/2)^2+y^2}\left(2\int_0^1\sigma(x)dx+\sigma(1)\frac{t}{2}\right)\Big].
\end{aligned} \tag{3.38}$$

We estimate each summand separately using the properties of the measure $d\mu(t)$ proved in Lemma 3.1. Combining (3.4) for $d\mu(t)$ and (3.7), we get

$$\pi\sigma(1)\int_0^{+\infty}d\mu(t)+2\int_0^1\sigma(x)dx\cdot 2\int_0^{+\infty}\frac{2y}{t^2+(2y)^2}d\mu(t)<\infty. \tag{3.39}$$

From the monotonicity of $\sigma(t)$ for $t>0$ and (3.36), it follows that

$$\int_0^{+\infty}d\mu(t)\sigma\left(\frac{t}{2}\right)\le 2C\sigma(1)\int_0^1\frac{d\mu(t)}{t}+\sigma\left(\frac{1}{2}\right)\int_0^{+\infty}d\mu(t)<\infty. \tag{3.40}$$

The last inequality is due to (3.6) and (3.4). Further, as $(t/2)y \le ((t/2)^2+y^2)/2$ we have

$$\sigma(1)\int_0^{+\infty}\frac{y(t/2)}{(t/2)^2+y^2}d\mu(t)\le\frac{\sigma(1)}{2}\int_0^{+\infty}d\mu(t)<\infty. \tag{3.41}$$

Finally, we obtain

$$\int_{\mathbb{R}} v(x+iy)\sigma(x)dx \le C \tag{3.42}$$

uniformly for $y>0$. From this, by Chebyshev's inequality, we get

$$\sigma-\text{mes}\{x : v(x+iy)>a\}\le\frac{1}{a}\int_{\mathbb{R}}v(x+iy)\sigma(x)dx\le\frac{C}{a}. \tag{3.43}$$

It is obvious that for $a>4$

$$\left\{x>0 : |1+u(x+iy)|>\frac{a}{2}\right\}\subseteq\left\{x>0 : |u(x+iy)|>\frac{a}{4}\right\}. \tag{3.44}$$

At the same time

$$\begin{aligned}\sigma-\mathrm{mes}\,\{x>0:|f^{-1}(x+iy)|>a\}&\\ \le \sigma-\mathrm{mes}\left\{x>0:|\operatorname{Re} f^{-1}(x+iy)|>\frac{a}{2}\right\}&\\ +\sigma-\mathrm{mes}\left\{x>0:|\operatorname{Im} f^{-1}(x+iy)|>\frac{a}{2}\right\},&\end{aligned} \tag{3.45}$$

that is,

$$\begin{aligned}\sigma-\mathrm{mes}\,\{x>0:|f^{-1}(x+iy)|>a\}&\\ \le \sigma-\mathrm{mes}\left\{x>0:|u(x+iy)|>\frac{a}{4}\right\}&\\ +\sigma-\mathrm{mes}\left\{x>0:|v(x+iy)|>\frac{a}{2}\right\}.&\end{aligned} \tag{3.46}$$

However, according to (3.32) and (3.42),

$$\sigma-\mathrm{mes}\,\{x>0:|u(x+iy)|>a\}\le \frac{1}{a}\int_{\mathbb{R}} v(x+iy)\sigma(x)dx\le \frac{C}{a}. \tag{3.47}$$

As a result we obtain

$$\sigma-\mathrm{mes}\,\{x>0:|f^{-1}(x+iy)|>a\}\le \frac{C}{a}. \tag{3.48}$$

In view of (3.26), this completes the proof. □

Being the function with positive imaginary part in the upper half-plane, $f(x+iy)$ has nontangential limits a.e. in the interval $(0,+\infty)$. Let $f(x):=\lim_{y\downarrow 0} f(x+iy)$. The following theorem shows that the estimate (3.25) is also valid for the limit function $f(x)$. Namely,

$$\sigma-\mathrm{mes}\,\{x>0:|f(x)|<d\}\le Cd. \tag{3.49}$$

Theorem 3.3. *Let* $(\mathcal{U},\Sigma,\rho)$ *be a measure space, and let* $\{\varphi_n\}$ *be a sequence of measurable functions defined on a set* $\mathcal{E}\in\Sigma$. *Suppose that for all sufficiently small* $d>0$

$$\rho\{x\in\mathcal{E}:\varphi_n(x)<d\}\le Cd \tag{3.50}$$

with the constant $C>0$ *independent of* n.

If for a.e. $x\in\mathcal{E}$ *with respect to* ρ *there exists* $\lim_{n\to+\infty}\varphi_n(x)=:\varphi(x)$, *then the analogous inequality is also valid for the limit function* $\varphi(x)$. *Namely,*

$$\rho\{x\in\mathcal{E}:\varphi(x)<d\}\le Cd \tag{3.51}$$

with the same constant $C>0$.

Proof. If $\chi_{\{\varphi<d\}}(t)$ is the indicator function of the set $\{\varphi<d\}\equiv\{x\in\mathcal{E}:\varphi(x)<d\}$, then

$$\rho\{\varphi<d\}=\int_{\mathcal{E}}\chi_{\{\varphi<d\}}(t)d\rho(t). \tag{3.52}$$

Suppose that for a certain $t_0 \in \mathscr{E}$ the function $\chi_{\{\varphi<d\}}(t_0) = 1$, that is, $\varphi(t_0) < d$. If $\varphi(t_0) = \lim_{n\to+\infty} \varphi_n(t_0)$, then $\varphi_n(t_0) < d$ for all n large enough. Thus, $\chi_{\{\varphi_n<d\}}(t_0) = 1$ for these values of n. Therefore,

$$\chi_{\{\varphi_n<d\}}(t_0) \longrightarrow \chi_{\{\varphi<d\}}(t_0) \quad \text{as } n \longrightarrow +\infty. \tag{3.53}$$

Hence, a.e. in $\mathscr{E}$ with respect to ρ

$$\chi_{\{\varphi<d\}}(t) \le \liminf_{n\to\infty} \chi_{\{\varphi_n<d\}}(t). \tag{3.54}$$

It now follows that

$$\rho\{\varphi < d\} = \int_{\mathscr{E}} \chi_{\{\varphi<d\}}(t)d\rho(t) \le \int_{\mathscr{E}} \liminf_{n\to\infty} \chi_{\{\varphi_n<d\}}(t)d\rho(t). \tag{3.55}$$

By Fatou's lemma

$$\begin{aligned}\int_{\mathscr{E}} \liminf_{n\to\infty} \chi_{\{\varphi_n<d\}}(t)d\rho(t) &\le \liminf_{n\to\infty} \int_{\mathscr{E}} \chi_{\{\varphi_n<d\}}(t)d\rho(t)\\ &\equiv \liminf_{n\to\infty} \rho\{\varphi_n < d\} \le Cd,\end{aligned} \tag{3.56}$$

and the proof is complete. □

Corollary 3.4. *The estimate (3.49) holds.*

Proof. Let the sequence $\varphi_n(x) := |f(x+iy_n)|$, where $y_n \downarrow 0$. By the absolute continuity of the measure $d\rho(t) := \sigma(t)dt$, the limit $\lim_{n\to\infty} \varphi_n(x) = |f(x)|$ also exists a.e. in $(0,+\infty)$ with respect to ρ. □

It is clear that this theorem imposes some restrictions on the decrease character of such analytic functions in a neighborhood of their real roots and therefore on the structure of the set of these roots, too.

A first uniqueness theorem of this type was obtained by Pavlov [11]. Then Naboko proved some theorems of this kind for operator-valued functions (see [8, 9]). One can apply these theorems in our case, but the structure of the zero set in the neighborhood of the singular point $t = 0$ cannot be described precisely. This is due to some special restriction on the weight function $\sigma(t)$: uniqueness theorems proved earlier allowed to use only Lebesgue measure, that is, to consider only the following weight function $\sigma(t) = 1$. Our theorem gives an opportunity to consider different measures: in this paper we use the function $\sigma(t) = 1/t^q$, where $q \in [0,1)$. This permits us to obtain sharp results concerning the structure of the roots set N.

4. Structure of the singular spectrum in a neighborhood of the origin. In order to apply the uniqueness theorem (Theorem 3.2) proved above for the description of the structure of the set N near the singular point zero, we need to know the behavior of $M(\lambda)$ near its roots. In what follows, we restrict our consideration to the case where the function φ belongs to the class Lipα, $\alpha \in (0,1/2)$, in other words, for a certain $\alpha \in (0,1/2)$ the following inequality holds:

$$\left|\varphi(x+h) - \varphi(x)\right| \le C|h|^{\alpha}, \quad |h| < 1. \tag{4.1}$$

If $\alpha \ge 1/2$, then the roots set N, as it has been shown in [15], is empty near zero and consists of at most finitely many eigenvalues of finite multiplicity.

We need the next refinement of the Pavlov and Petras lemma [12] (see also [9, 13, 14]).

Lemma 4.1 (on smoothness of $M(\lambda)$). *Let the function φ belong to $L_2(\mathbb{R})\cap \operatorname{Lip}\alpha$, $\alpha\in(0,1/2)$, and the point $\lambda_0\in N$. Then the following estimate holds in an ε-neighborhood of λ_0 with $0\le\varepsilon\le\lambda_0/4$*

$$|M(\lambda)| = |M(\lambda)-M(\lambda_0)| \le C\frac{|\lambda-\lambda_0|^{2\alpha}}{\lambda_0^{1/2+\alpha}}. \tag{4.2}$$

Proof. From the representation (3.1) by Sohockiy's formulas we find that

$$M(\lambda)=1+\text{v.p.}\int_0^{+\infty}\frac{|\varphi^2(\sqrt{\tau})|+|\varphi^2(-\sqrt{\tau})|}{\tau-\lambda}\frac{d\tau}{2\sqrt{\tau}}+i\pi\frac{|\varphi^2(\sqrt{\lambda})|+|\varphi^2(-\sqrt{\lambda})|}{2\sqrt{\lambda}}. \tag{4.3}$$

Hence, from the equality $M(\lambda_0)=0$ it follows that $\varphi(\sqrt{\lambda_0})=\varphi(-\sqrt{\lambda_0})=0$. Obviously, it suffices to check the estimate (4.2) for the function

$$f(\lambda):=\text{v.p.}\int_{-\infty}^{+\infty}\frac{\eta(t)}{t-\lambda}dt+i\eta(\lambda), \tag{4.4}$$

where the function $\eta(t)$ is defined as follows:

$$\eta(t):=\begin{cases}0, & t\le 0,\\ \dfrac{|\varphi^2(\sqrt{t})|}{\sqrt{t}}, & t>0.\end{cases} \tag{4.5}$$

It is easy to check that $\eta(t)$ satisfies a local Lipschitz condition in $(0,+\infty)$, therefore, understanding the integral $\int_{-\infty}^{+\infty}(\cdot)dt$ as $\lim_{N\to+\infty}\int_{-N}^{N}(\cdot)dt$, we can write

$$f(\lambda)-f(\lambda_0)=\int_{-\infty}^{+\infty}\frac{\eta(t)-\eta(\lambda)}{t-\lambda}dt-\int_{-\infty}^{+\infty}\frac{\eta(t)-\eta(\lambda_0)}{t-\lambda_0}dt+i\eta(\lambda). \tag{4.6}$$

Letting $\delta:=|\lambda-\lambda_0|$, define the interval $S:=(\lambda_0-2\delta,\lambda_0+2\delta)$. Then, since $\eta(\lambda_0)=0$, the difference $f(\lambda)-f(\lambda_0)$ can be rewritten in the form, [12],

$$\begin{aligned}f(\lambda)-f(\lambda_0)&=\int_S\frac{\eta(t)-\eta(\lambda)}{t-\lambda}dt+\int_{\mathbb{R}\setminus S}\frac{\eta(t)-\eta(\lambda_0)}{t-\lambda}dt-\int_{\mathbb{R}\setminus S}\frac{\eta(\lambda)}{t-\lambda}dt\\&\quad-\int_S\frac{\eta(t)-\eta(\lambda_0)}{t-\lambda_0}dt-\int_{\mathbb{R}\setminus S}\frac{\eta(t)-\eta(\lambda_0)}{t-\lambda_0}dt+i\eta(\lambda).\end{aligned} \tag{4.7}$$

Combining the second integral with the last one and calculating the third integral, we find that

$$\begin{aligned}f(\lambda)-f(\lambda_0)&=\int_S\frac{\eta(t)-\eta(\lambda)}{t-\lambda}dt-\int_S\frac{\eta(t)-\eta(\lambda_0)}{t-\lambda_0}dt\\&\quad+\int_{\mathbb{R}\setminus S}(\lambda-\lambda_0)\frac{\eta(t)-\eta(\lambda_0)}{(t-\lambda)(t-\lambda_0)}dt+(i+\operatorname{sgn}(t-\lambda_0)\ln 3)\eta(\lambda)\\&\equiv I_1+I_2+I_3+I_4,\end{aligned} \tag{4.8}$$

and we estimate each summand separately.

For the function $\eta(t)$ the following estimate holds

$$|\eta(t)| \le C\frac{|t-\lambda_0|^{2\alpha}}{\sqrt{t}\cdot\lambda_0^{\alpha}}, \tag{4.9}$$

the constant C is independent of $\lambda_0 \in N$. Indeed, according to (4.1), for $|\sqrt{t}-\sqrt{\lambda_0}| < 1$ we have

$$|\varphi(\sqrt{t})| = |\varphi(\sqrt{t}) - \varphi(\sqrt{\lambda_0})| \le C|\sqrt{t}-\sqrt{\lambda_0}|^{\alpha}. \tag{4.10}$$

The relation $\varphi \in L_2(\mathbb{R})$ together with (4.1) means that $\varphi(t) \to 0$ as $t \to +\infty$. Therefore, substituting the constant C in inequality (4.10) for $(\max_{\mathbb{R}}|\varphi(\tau)|+C)$, we see that this inequality is also valid for $|\sqrt{t}-\sqrt{\lambda_0}| \ge 1$. Consequently,

$$\begin{aligned} |\eta(t)| &= \frac{|\varphi(\sqrt{t})-\varphi(\sqrt{\lambda_0})|^2}{\sqrt{t}} \le C\frac{|\sqrt{t}-\sqrt{\lambda_0}|^{2\alpha}}{\sqrt{t}} \\ &= C\frac{|t-\lambda_0|^{2\alpha}}{\sqrt{t}(\sqrt{t}+\sqrt{\lambda_0})^{2\alpha}} \le C\frac{|t-\lambda_0|^{2\alpha}}{\sqrt{t}\cdot\lambda_0^{\alpha}}. \end{aligned} \tag{4.11}$$

Clearly, if $t \in S$, then $1/t \le 2/\lambda_0$, therefore,

$$|\eta(t)-\eta(\lambda_0)| = |\eta(t)| \le C\frac{|t-\lambda_0|^{2\alpha}}{\lambda_0^{1/2+\alpha}}, \quad t \in S. \tag{4.12}$$

Now, we immediately deduce that

$$\begin{aligned} |I_2| &\le \frac{C}{\lambda_0^{1/2+\alpha}}\int_0^{2\delta}\frac{t^{2\alpha}}{t}dt \le C\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}, \\ |I_4| &\le C|\eta(\lambda)| = C|\eta(t)|\big|_{t=\lambda} \le C\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}. \end{aligned} \tag{4.13}$$

Since $1/|t-\lambda| \le 2/|t-\lambda_0|$ for $t \notin S$, by (4.9), we obtain

$$\begin{aligned} |I_3| &\le C\Big(\int_0^{\lambda_0/2} + \int_{\lambda_0/2}^{\lambda_0-2\delta} + \int_{\lambda_0+2\delta}^{+\infty}\Big)\delta\cdot\frac{|t-\lambda_0|^{2\alpha-2}}{\sqrt{t}\cdot\lambda_0^{\alpha}}dt \\ &\le C\delta\Big(\lambda_0^{\alpha-2}\int_0^{\lambda_0/2}\frac{dt}{\sqrt{t}} + \int_{\lambda_0/2}^{\lambda_0-2\delta}\frac{|t-\lambda_0|^{2\alpha-2}}{\lambda_0^{1/2+\alpha}}dt + \int_{2\delta}^{+\infty}\frac{t^{2\alpha-2}}{\lambda_0^{1/2+\alpha}}dt\Big). \end{aligned} \tag{4.14}$$

Hence,

$$|I_3| \le C\Big(\frac{\delta}{\lambda_0^{3/2-\alpha}} + \frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}\Big). \tag{4.15}$$

For estimating I_1 we need to consider the difference $\eta(t)-\eta(\lambda)$ for $t \in S$.

$$|\eta(t)-\eta(\lambda)| \le \frac{\big||\varphi^2(\sqrt{t})|-|\varphi^2(\sqrt{\lambda})|\big|}{\sqrt{t}} + \frac{|\varphi(\sqrt{\lambda})-\varphi(\sqrt{\lambda_0})|^2}{\sqrt{t}\sqrt{\lambda}}\cdot|\sqrt{t}-\sqrt{\lambda}|. \tag{4.16}$$

Using (4.1) and (4.9), we find

$$\begin{aligned}\frac{\left|\,|\varphi^2(\sqrt{t})|-|\varphi^2(\sqrt{\lambda})|\,\right|}{\sqrt{t}}&\\ \leq C\frac{|\varphi(\sqrt{t})-\varphi(\sqrt{\lambda})|}{\sqrt{\lambda_0}}&\left(\left|\varphi(\sqrt{t})-\varphi\left(\sqrt{\lambda_0}\right)\right|+\left|\varphi(\sqrt{\lambda})-\varphi\left(\sqrt{\lambda_0}\right)\right|\right)\\ \leq C\frac{|t-\lambda|^{\alpha}}{\lambda_0^{1/2+\alpha/2}}\cdot\frac{|\lambda-\lambda_0|^{\alpha}}{\lambda_0^{\alpha/2}},&\end{aligned}\tag{4.17}$$

$$\begin{aligned}\frac{|\varphi(\sqrt{\lambda})-\varphi(\sqrt{\lambda_0})|^2}{\sqrt{t}\sqrt{\lambda}}\cdot|\sqrt{t}-\sqrt{\lambda}|&\leq C\frac{|(\sqrt{\lambda})-(\sqrt{\lambda_0})|^{2\alpha}}{\lambda_0}\cdot\sqrt{|t-\lambda|}\\ &\leq C\frac{|\lambda-\lambda_0|^{\alpha}}{\lambda_0}\cdot|t-\lambda|^{1/2}.\end{aligned}$$

Thus we have

$$\begin{aligned}|I_1|&\leq C\int_0^{2\delta}\frac{dt}{t}\left(t^{\alpha}\cdot\frac{|\lambda-\lambda_0|^{\alpha}}{\lambda_0^{1/2+\alpha}}+t^{1/2}\cdot\frac{|\lambda-\lambda_0|^{\alpha}}{\lambda_0}\right)\\ &\leq C\left(\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}+\frac{\delta^{1/2+\alpha}}{\lambda_0}\right).\end{aligned}\tag{4.18}$$

Finally, for $\lambda_0\in N$ and $\delta\leq\lambda_0/4$ we obtain

$$|f(\lambda)-f(\lambda_0)|\leq C\left(\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}+\frac{\delta}{\lambda_0^{3/2-\alpha}}+\frac{\delta^{1/2+\alpha}}{\lambda_0}\right)\leq 3C\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}.\tag{4.19}$$

This completes the proof of Lemma 4.1. □

For $\gamma>1$ we define the metric ρ_γ on the positive half of the real axis

$$\rho_\gamma(x,y):=\left|\int_x^y\frac{du}{u^\gamma}\right|,\quad x,y\in(0,+\infty).\tag{4.20}$$

Let $B_\delta(x):=\{y>0:\rho_\gamma(y,x)<\delta\}$ be the ball of radius δ with the center at the point x. The following lemma gives us a certain information on the structure of the set $B_\delta(x)$ from the point of view of the Euclidean metric.

Lemma 4.2. *If $\varepsilon_x=2\delta x^\gamma$, $x>0$, then for each x from any finite interval $(0,a)$, $a>0$, for all sufficiently small δ (depending on γ and a but independent of x) there exists the inclusion*

$$\left(x-\frac{\varepsilon_x}{3},x+\frac{\varepsilon_x}{3}\right)\subset B_\delta(x)\subset(x-\varepsilon_x,x+\varepsilon_x).\tag{4.21}$$

Proof. Obviously, for checking the inclusion $B_\delta(x)\subseteq(x-\varepsilon_x,x+\varepsilon_x)$, it suffices to show that for all sufficiently small δ uniformly for $x\in(0,a)$ the inequality

$$\delta>\int_x^{x+\varepsilon}\frac{du}{u^\gamma}\tag{4.22}$$

implies $\varepsilon<\varepsilon_x$.

From (4.22) we see that $\varepsilon \to 0$ as $\delta \to 0$. After making the change of variables $u = xt$, we have

$$\int_x^{x+\varepsilon} \frac{du}{u^\gamma} = \frac{1}{x^{\gamma-1}} \int_1^{1+\varepsilon/x} \frac{dt}{t^\gamma}. \tag{4.23}$$

Therefore,

$$\delta > \frac{1}{a^{\gamma-1}} \int_1^{1+\varepsilon/x} \frac{dt}{t^\gamma}. \tag{4.24}$$

Hence, ε/x tends to 0 with δ uniformly for $x \in (0,a)$. Now,

$$\begin{aligned}\delta > \int_x^{x+\varepsilon} \frac{du}{u^\gamma} &= \frac{1}{(\gamma-1)x^{\gamma-1}} \left[1 - \left(1 - \frac{\varepsilon}{x}\right)^{1-\gamma}\right] \\ &= \frac{1}{(\gamma-1)x^{\gamma-1}} \left[(\gamma-1)\frac{\varepsilon}{x} + o\left(\frac{\varepsilon}{x}\right)\right].\end{aligned} \tag{4.25}$$

For sufficiently small δ uniformly for $x \in (0,a)$, we have

$$\left| o\left(\frac{\varepsilon}{x}\right) \right| < \frac{1}{2}(\gamma-1)\frac{\varepsilon}{x}. \tag{4.26}$$

Consequently,

$$\delta > \frac{\varepsilon}{2x^\gamma} \qquad \text{or} \qquad \varepsilon < 2\delta x^\gamma \equiv \varepsilon_x. \tag{4.27}$$

Further,

$$\int_{x-\varepsilon_x/3}^{x} \frac{du}{u^\gamma} = \frac{1}{(\gamma-1)x^{\gamma-1}} \left[(\gamma-1)\frac{\varepsilon_x}{3x} + o\left(\frac{\varepsilon_x}{x}\right)\right], \tag{4.28}$$

where $\varepsilon_x/x = 2\delta x^{\gamma-1} \le 2\delta a^{\gamma-1}$ tends to 0 with δ uniformly for $x \in (0,a)$. Therefore for δ small enough uniformly for $x \in (0,a)$

$$\left| o\left(\frac{\varepsilon_x}{x}\right) \right| < \frac{1}{2}(\gamma-1)\frac{\varepsilon_x}{3x}. \tag{4.29}$$

Thus,

$$\int_{x-\varepsilon_x/3}^{x} \frac{du}{u^\gamma} < \frac{\varepsilon_x}{2x^\gamma} = \delta, \tag{4.30}$$

and the proof is complete. □

Now, by combining a uniqueness theorem with Lemma 4.1 on smoothness of $M(\lambda)$, we can prove the main theorem of this section.

Theorem 4.3. *Let the function φ belong to $L_2(\mathbb{R}) \cap \operatorname{Lip}\alpha$, $\alpha \in (0,1/2)$. Let the measure $\sigma(x)dx$ with the positive weight function $\sigma(x)$ satisfy conditions (3.22) and (3.24). Then for all sufficiently small $\delta > 0$ the estimate*

$$\sigma - \operatorname{mes}\{\lambda > 0 : \rho_{(1/2+\alpha)/2\alpha}(\lambda, N) < \delta\} \le C\delta^{2\alpha} \tag{4.31}$$

holds with the constant C depending only on the weight function $\sigma(x)$.

Proof. Consider the set $N_{\rho_\gamma}^\delta := \{\lambda > 0 : \rho_\gamma(\lambda, N) < \delta\}$, which is the δ-neighborhood of the set N in the metric ρ_γ. It is clear that $N_{\rho_\gamma}^\delta = \cup_{x\in N} B_\delta(x)$. The set N is bounded, hence, according to Lemma 4.2, the set $\cup_{x\in N} B_\delta(x)$ embeds into $\cup_{x\in N}(x-\varepsilon_x, x+\varepsilon_x)$ with $\varepsilon_x = 2\delta x^\gamma$. At the same time, by Lemma 4.1, for $x \in N$ in the interval $(x-\varepsilon_x, x+\varepsilon_x)$ the following inequality holds

$$|M(\lambda)| \le C\frac{|\lambda - x|^{2\alpha}}{x^{1/2+\alpha}} \le C\frac{\varepsilon_x^{2\alpha}}{x^{1/2+\alpha}} \le C\delta^{2\alpha}x^{2\alpha\gamma-(1/2+\alpha)}. \tag{4.32}$$

With $\gamma = (1/2+\alpha)/2\alpha$ this gives the uniform estimate $|M(\lambda)| \le C\delta^{2\alpha}$. So for this value of γ the following inclusion holds

$$N_{\rho_\gamma}^\delta \subset \{\lambda > 0 : |M(\lambda)| < C\delta^{2\alpha}\}. \tag{4.33}$$

Hence, by (3.49), we get

$$\sigma - \text{mes}\, N_{\rho_\gamma}^\delta \le \sigma - \text{mes}\,\{\lambda > 0 : |M(\lambda)| < C\delta^{2\alpha}\} \le C\delta^{2\alpha}. \tag{4.34}$$

The theorem is proved. □

Thus, the σ-measure of the δ-neighborhood in the metric ρ_γ with $\gamma = (1/2+\alpha)/2\alpha$ of the roots set N is $O(\delta^{2\alpha})$ as $\delta \to 0$. It is evident that the estimate (4.31) imposes some restrictions on the possible structure of the set N, and hence, on the structure of $\sigma_{\text{sing}}(A_2) \subset N \cup \{0\}$, too.

Corollary 4.4. *If the function φ belongs to $L_2(\mathbb{R}) \cap \text{Lip}\,\alpha$, $\alpha \in (0, 1/2)$, and the sequence $\{\lambda_k\}_{k=1}^\infty$ of eigenvalues of the operator $A_2 = t^2 \cdot + (\cdot, \varphi)\varphi$ decreases to zero in a power scale, that is, $\lambda_k = 1/k^\beta$, then it follows from the estimate (4.31) that the index*

$$\beta \ge \frac{4\alpha}{1-2\alpha}. \tag{4.35}$$

Proof. Suppose that

$$\beta < \frac{4\alpha}{1-2\alpha}. \tag{4.36}$$

Then the intervals $I_k := (\lambda_k - \varepsilon_{\lambda_k}/3, \lambda_k + \varepsilon_{\lambda_k}/3)$ will be overlapping for all k large enough. In fact, if $\lambda_k = 1/k^\beta$, then $\Delta\lambda_k := \lambda_k - \lambda_{k+1} \le C/k^{\beta+1}$. The intervals I_k and I_{k+1} will intersect provided $\Delta\lambda_k \le \varepsilon_{\lambda_k}/3$. Since $\varepsilon_{\lambda_k} = 2\delta\lambda_k^\gamma$ with $\gamma = (1/2+\alpha)/2\alpha$, the last inequality is necessarily fulfilled if

$$\frac{C}{k^{\beta+1}} \le \frac{2\delta}{3k^{\beta\gamma}}. \tag{4.37}$$

Hence, for $k \ge C(1/\delta)^{1/(1-\beta(\gamma-1))} =: m$ the $\varepsilon_{\lambda_k}/3$-neighborhoods of the points λ_k will be overlapping, and therefore, $(0, \lambda_m) \subseteq \cup_{k=1}^{+\infty}(\lambda_k - \varepsilon_{\lambda_k}/3, \lambda_k + \varepsilon_{\lambda_k}/3)$. (Note that $1 - \beta(\gamma - 1) > 0$ if and only if $\beta < 4\alpha/(1-2\alpha)$.) By (4.21) and (4.31), we obtain

$$\begin{aligned} C\delta^{2\alpha} &\ge \sigma - \text{mes}\,\{\lambda > 0 : \rho_\gamma(\lambda, N) < \delta\} \\ &\ge \sigma - \text{mes}\left(\cup_{k=1}^{+\infty}\left(\lambda_k - \frac{\varepsilon_{\lambda_k}}{3}, \lambda_k + \frac{\varepsilon_{\lambda_k}}{3}\right)\right) \ge \int_0^{\lambda_m}\sigma(t)\,dt. \end{aligned} \tag{4.38}$$

Consequently, for $\sigma(t) = 1/|t|^q$, $q \in [0,1)$, we have

$$C\delta^{2\alpha} \geq \int_0^{\lambda_m} \frac{dt}{t^q} \tag{4.39}$$

with the constant C independent of δ (but possibly dependent on q).

$$\int_0^{\lambda_m} \frac{dt}{t^q} = \frac{1}{1-q} \cdot \frac{1}{m^{\beta(1-q)}} \geq C\delta^{\beta(1-q)/(1-\beta(\gamma-1))}. \tag{4.40}$$

Thus for all sufficiently small $\delta > 0$ there must be fulfilled the following inequality:

$$\delta^{\beta(1-q)/(1-\beta(\gamma-1))} \leq C\delta^{2\alpha}. \tag{4.41}$$

It follows that $\beta(1-q)/(1-\beta(\gamma-1)) \geq 2\alpha$ for all $q \in [0,1)$. As $\gamma = (1/2+\alpha)/2\alpha$ this implies that

$$\beta \geq \frac{4\alpha}{3-2q-2\alpha} \quad \forall q \in [0,1). \tag{4.42}$$

Letting $q \to 1^-$ yields $\beta \geq 4\alpha/(1-2\alpha)$, contrary to (4.36). □

The index β makes sense of the convergence speed of λ_k to zero. The estimate (4.35) implies that the points of N, in particular, the eigenvalues of the operator A_2 cannot tend to zero too slowly. The slower accumulation corresponds to a greater density of N and hence to a greater value of its measure. As the function $4\alpha/(1-2\alpha)$ is increasing for $\alpha \in (0,1/2)$ a better smoothness of the perturbation operator $V = (\cdot,\varphi)\varphi$ corresponds to a greater lower bound of the admissible values of β, that is, to a greater rarefaction of the roots set N. Further, the index $\beta \uparrow +\infty$ as $\alpha \uparrow 1/2$, that is, the smoothness $\alpha = 1/2$ is critical. This fact is consistent with the finiteness of the roots set N for $\alpha \geq 1/2$ (see [15]).

Theorem 4.3 can also be used for describing the structure of N outside of any neighborhood of zero, that is, of the set $N_b := N \cap [b,+\infty)$ for any $b > 0$. In this case (4.31) coincides with the result of [12] (we already noted in Section 1 that the structure of the roots set of the operator $S_1 = t \cdot +(\cdot,\varphi)\varphi$ is identical with that of N_b). In fact, the set N is bounded, in every finite interval bounded away from zero $\varepsilon_x \geq c\delta$, and the measures dt/t^q are equivalent for different q. Putting $q = 0$, we obtain the following estimate of Lebesgue measure of the δ-neighborhood of the set N_b

$$\operatorname{mes}\{\lambda > 0 : \operatorname{dist}(\lambda, N_b) < \delta\} \leq C\delta^{2\alpha}. \tag{4.43}$$

For the eigenvalues (roots) $\lambda_k = \lambda_0 + 1/k^\beta$, $\lambda_0 > 0$, of the operator A_2 the estimate (4.43) leads to the restriction $\beta \geq 2\alpha/(1-2\alpha)$. It follows therefore from (4.35) that we observe the duplication of the admissible speed of the eigenvalues convergence to the limit point $\lambda_0 = 0$.

Acknowledgements. The author expresses a deep gratitude to S. N. Naboko for the formulation of the problem and for the usefull disscussions and thanks J. Boman and P. Kurasov for their attention to the work. This work was partially supported by a grant from the Swedish Royal Academy of Sciences. The author thanks the Stockholm University for hospitality.

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