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ISHIKAWA ITERATION PROCESS WITH ERRORS FOR NONEXPANSIVE MAPPINGS

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Abstract. We study the construction and the convergence of the Ishikawa iterative process with errors for nonexpansive mappings in uniformly convex Banach spaces. Some recent corresponding results are generalized.

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1. Introduction.

Let C be a closed convex subset of a Banach space X and $T: C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all x, y in C). Recently, Deng and Li [1] introduced an Ishikawa iteration sequence with errors as follows: for any given $x_0 \in C$

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n, \quad n \ge 0. \end{aligned} \tag{1.1}$$

Here $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in C, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$, and $\{\hat{\gamma}_n\}$ are six sequences in $[0,1]$ satisfying the conditions

$$\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1 \quad \forall n \ge 0. \tag{1.2}$$

Remark 1.1. Note that the Ishikawa iteration processes [2] is a special case of the Ishikawa iteration processes with errors.

Deng and Li [1] obtained the following result. Let C be a closed convex subset of a uniformly convex Banach space X. If for any initial guess $x_0 \in C$, $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n \beta_n \hat{\beta}_n < \infty$, $\sum_{n=0}^{\infty} \gamma_n < \infty$, and $\sum_{n=0}^{\infty} \hat{\gamma}_n < \infty$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. So Deng and Li extended the result of Tan and Xu [6].

In this paper, we first extend and unify [1, Theorem 1] and [6, Lemma 3]. Then, we generalize [1, Theorems 2, 3, and 4] and [6, Theorems 1, 2, and 3].

2. Lemmas

Lemma 2.1 (see [6]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \le a_n + b_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.2 (see [1]). *Let C be a closed convex subset of a Banach space X, $T: C \to C$ a nonexpansive mapping. Then for any initial guess x_0 in C, $\{x_n\}$ defined by (1.1),*

$$||x_{n+1} - p|| \le ||x_n - p|| + \gamma_n ||u_n - p|| + \beta_n \hat{\gamma}_n ||v_n - p|| \tag{2.1}$$

for all $n \ge 1$ and for all $p \in F(T)$, where $F(T)$ denotes the set of fixed point of T.

Remark 2.3. Since the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, so the sequences $\{\|u_n - p\|\}$ and $\{\|v_n - p\|\}$ are bounded too, then $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 2.1.

Lemma 2.4 (see [7]). *Let C be a bounded closed convex subset of a uniformly convex Banach space X. Suppose that $T: C \to C$ is a nonexpansive mapping. If $y_n \to y$ weakly (y_n, $y \in C$, $n = 1, 2, \ldots$), then there exists a strictly increasing convex function $g: \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$g(\|y - Ty\|) \le \liminf_{n\to\infty} ||y_n - Ty_n||. \tag{2.2}$$

3. Main results

Theorem 3.1. *Let C be a closed convex subset of a uniformly convex Banach space X, $T: C \to C$ a nonexpansive mapping with a fixed point. If for any initial guess x_0 in C, $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \hat{\gamma}_n < \infty$, and there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} = \infty$, $\sum_{k=0}^{\infty} \alpha_{n_k} \beta_{n_k} \hat{\beta}_{n_k} < \infty$. Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Since T has a fixed point, and by Lemma 2.2, we may set

$$M = \sup_{n\ge 0} \{||Tx_n - u_n||, ||x_n - u_n||, ||Ty_n - v_n||, ||y_n - u_n||, ||x_n - v_n||\}. \tag{3.1}$$

If $\liminf_{n\to\infty} \|x_n - Tx_n\| > 0$, we may assume that $\liminf_{n\to\infty} \|x_n - p\| > 0$, where $p \in F(T)$. Since $\|Ty_n - p\| \le \|x_n - p\| + \hat{\gamma}_n M$, we obtain

$$\begin{aligned} ||x_{n+1} - p|| &\le ||\alpha_n (x_n - p) + \beta_n (Ty_n - p)|| + \gamma_n M \\ &= (\alpha_n + \beta_n) \left|\left| \frac{\alpha_n}{\alpha_n + \beta_n} (x_n - p) + \frac{\beta_n}{\alpha_n + \beta_n} (Ty_n - p) \right|\right| + \gamma_n M \\ &\le \left[1 - 2\frac{\alpha_n \beta_n}{(\alpha_n + \beta_n)^2} \delta_X \left(\frac{||x_n - Ty_n||}{||x_n - p|| + \hat{\gamma}_n M} \right)\right] (||x_n - p|| + \hat{\gamma}_n M) + \gamma_n M \\ &\le \left[1 - 2\alpha_n \beta_n \delta_X \left(\frac{||x_n - Ty_n||}{||x_n - p|| + \hat{\gamma}_n M} \right)\right] ||x_n - p|| + (\hat{\gamma}_n + \gamma_n) M, \end{aligned} \tag{3.2}$$

where δ_X is the modulus of convexity of the uniformly convex Banach space X. Setting

$$D_n = 1 - 2\alpha_n \beta_n \delta_X \left(\frac{||x_n - Ty_n||}{||x_n - p|| + \hat{\gamma}_n M} \right). \tag{3.3}$$

Thus for all $n \geq 0$, $0 \leq D_n \leq 1$. From (3.2), for all $k \geq 0$, we have

$$\begin{aligned}
&\|x_{n_{k+1}} - p\| \\
&\quad \leq D_{n_{k+1}-1}\|x_{n_{k+1}-1} - p\| + (\hat{\gamma}_{n_{k+1}-1} + \gamma_{n_{k+1}-1})M \\
&\quad \leq D_{n_{k+1}-1}D_{n_{k+1}-2}\cdots D_{n_k+1}D_{n_k}\|x_{n_k} - p\| + \sum_{i=1}^{n_{k+1}-n_k}(\hat{\gamma}_{n_{k+1}-i} + \gamma_{n_{k+1}-i})M \\
&\quad \leq D_{n_k}\|x_{n_k} - p\| + \sum_{i=1}^{n_{k+1}-n_k}(\hat{\gamma}_{n_{k+1}-i} + \gamma_{n_{k+1}-i})M \\
&\quad \leq \|x_{n_k} - p\|\left[1 - 2\alpha_{n_k}\beta_{n_k}\delta_X\left(\frac{\|x_{n_k} - Ty_{n_k}\|}{\|x_{n_k} - p\| + \hat{\gamma}_n M}\right)\right] + \sum_{i=1}^{n_{k+1}-n_k}(\hat{\gamma}_{n_{k+1}-i} + \gamma_{n_{k+1}-i})M.
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
&\sum_{i=0}^{k} 2\alpha_{n_i}\beta_{n_i}\delta_X\left(\frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{\gamma}_{n_i}M}\right)\|x_{n_i} - p\| \\
&\qquad \leq \|x_{n_0} - p\| - \|x_{n_{k+1}} - p\| + \sum_{i=0}^{n_{k+1}-1}(\hat{\gamma}_i + \gamma_i)M.
\end{aligned} \tag{3.5}$$

It follows that

$$\sum_{i=0}^{\infty} \alpha_{n_i}\beta_{n_i}\delta_X\left(\frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{\gamma}_{n_i}M}\right) < +\infty. \tag{3.6}$$

By condition $\sum_{i=0}^{\infty} \alpha_{n_i}\beta_{n_i}\hat{\beta}_{n_i} < +\infty$, we have

$$\sum_{i=0}^{\infty} \alpha_{n_i}\beta_{n_i}\left[\delta_X\left(\frac{\|x_{n_i} - Ty_{n_i}\|}{\|x_{n_i} - p\| + \hat{\gamma}_{n_i}M}\right) + \hat{\beta}_{n_i}\right] < +\infty. \tag{3.7}$$

It follows that

$$\liminf_{k\to\infty}\left[\delta_X\left(\frac{\|x_{n_k} - Ty_{n_k}\|}{\|x_{n_k} - p\| + \hat{\gamma}_{n_k}M}\right) + \hat{\beta}_{n_k}\right] = 0 \tag{3.8}$$

since $\sum_{k=0}^{\infty} \alpha_{n_k}\beta_{n_k} = \infty$. Hence, there is a sequence $\{n_{k_i}\} \subset \{n_k\}$ such that

$$\lim_{i\to\infty}\left\|x_{n_{k_i}} - Ty_{n_{k_i}}\right\| = 0, \quad \lim_{i\to\infty}\hat{\beta}_{n_{k_i}} = 0. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}
\left\|x_{n_{k_i}} - Tx_{n_{k_i}}\right\| &\leq \left\|x_{n_{k_i}} - Ty_{n_{k_i}}\right\| + \left\|Tx_{n_{k_i}} - Ty_{n_{k_i}}\right\| \\
&\leq \left\|x_{n_{k_i}} - Ty_{n_{k_i}}\right\| + \hat{\beta}_{n_{k_i}}\left\|x_{n_{k_i}} - Tx_{n_{k_i}}\right\| + \hat{\gamma}_{n_{k_i}}M.
\end{aligned} \tag{3.10}$$

Setting $i \to \infty$ in (3.10), it follows from (3.9) that

$$\lim_{i\to\infty}\left\|x_{n_{k_i}} - Tx_{n_{k_i}}\right\| = 0. \tag{3.11}$$

Thus,

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.12}$$

This completes the proof. □

Recall that a Banach space X is said to satisfy Opial's condition [4] if the condition $x_n \to x_0$ weakly implies

$$\limsup_{n\to\infty} \|x_n - x_0\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \neq x_0. \tag{3.13}$$

A mapping $T : C \to C$ with a nonempty fixed points set $F(T)$ in C will be said to satisfy Condition A in [5] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$, such that $\|x - Tx\| \geq f(d(x,F(T)))$ for all $x \in C$, where $d(x,F(T)) = \inf\{\|x - z\| : z \in F(T)\}$.

Theorem 3.2. *Let C be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition or whose norm is Fréchet differentiable. Let $T : C \to C$ a nonexpansive mapping with a fixed point, and $\{x_n\}$ defined by (1.1), with the restrictions that $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \hat{\gamma}_n < \infty$, and for any subsequence $\{n_k\}$ of $\{n\}$, $\sum_{k=0}^{\infty} \alpha_{n_k}\beta_{n_k} = \infty$, $\sum_{k=0}^{\infty} \alpha_{n_k}\beta_{n_k}\hat{\beta}_{n_k} < \infty$, converges weakly to a fixed point of T.*

By Theorem 3.1 and Lemma 2.4, we can prove Theorem 3.2 easily. The proof is similar to that of [7, Theorem 3.1], so the details are omitted.

Let X, C, T, and $\{x_n\}$ be as in Theorem 3.1. Then we have the following theorem.

Theorem 3.3. *If the range of C under T is contained in a compact subset of X, then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 3.4. *Let C be a bounded closed convex subset of a uniformly convex Banach space X. If T satisfies Condition A, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since C is a bounded closed convex subset of a uniformly convex Banach space X, then T has a fixed point [3]. So $F(T)$ is nonempty. It follows from Theorem 3.1 and Condition A, that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} f(d(x_{n_k},F(T))) = 0$, therefore we have $\lim_{k\to\infty} d(x_{n_k},F(T)) = 0$. So we can choose a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ and some sequence $\{p_i\}$ in $F(T)$ such that $\|x_{n_{k_i}} - p_i\| < 2^{-i}$ for all integers $k \geq 0$.

We denote $\sup_n\{\|u_n - p\|, \|v_n - p\|\}$ by M and $(\gamma_{n_{k_i}} + \beta_{n_{k_i}}\hat{\gamma}_{n_{k_i}})M$ by $\lambda_{n_{k_i}}$. By Lemma 2.1 we have

$$\begin{aligned}
\|p_{i+1} - p_i\| &\leq \left\|x_{n_{k_{i+1}}} - p_{i+1}\right\| + \left\|x_{n_{k_{i+1}}} - p_i\right\| \\
&\leq 2^{-(i+1)} + \left\|x_{n_{k_{i+1}}-1} - p_i\right\| + \lambda_{n_{k_{i+1}}-1} \\
&\leq 2^{-(i+1)} + \left\|x_{n_{k_{i+1}}-2} - p_i\right\| + \lambda_{n_{k_{i+1}}-2} + \lambda_{n_{k_{i+1}}-1}
\end{aligned}$$

$$\le 2^{-(i+1)} + \left\| x_{n_{k_i}} - p_i \right\| + \sum_{j=n_{k_i}}^{n_{k_{i+1}}-1} \lambda_j$$
$$\le 2^{-i+1} + \sum_{j=n_{k_i}}^{n_{k_{i+1}}-1} \lambda_j. \tag{3.14}$$

It follows, from (3.14) and $\sum_j \lambda_j$ is convergent, that $\{p_i\}$ is a Cauchy sequence therefore converges strongly to a point $p \in F(T)$, since $F(T)$ is closed. We have seen that $\{x_{n_{k_i}}\}$ converges strongly to p, so does $\{x_n\}$ by the Remark 2.3. This completes the proof. □

Remark 3.5. The above three theorems generalize [6, Theorems 1, 2, and 3] and [1, Theorems 2, 3, and 4], respectively.

References

[1] L. Deng and S. Li, *Ishikawa iteration process with errors for nonexpansive mappings in uniformly convex Banach spaces*, Int. J. Math. Math. Sci. **24** (2000), no. 1, 49–53. CMP 1 773 969. Zbl 958.47029.

[2] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150. MR 49#1243. Zbl 286.47036.

[3] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006. MR 32#6436. Zbl 141.32402.

[4] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597. MR 35#2183. Zbl 179.19902.

[5] H. F. Senter and W. G. Dotson, Jr., *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), 375–380. MR 49#11333. Zbl 299.47032.

[6] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), no. 2, 301–308. MR 94g:47076. Zbl 895.47048.

[7] L.-C. Zeng, *A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **226** (1998), no. 1, 245–250. MR 99g:47138. Zbl 916.47047.

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