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SOME PROPERTIES OF BANACH-VALUED SEQUENCE SPACES $\ell_p[X]$

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ABSTRACT. We discuss some properties of the Banach-valued sequence space $\ell_p[X]$ ($1 \le p < \infty$), the space of weakly p-summable sequences on a Banach space X. For example, we characterize the reflexivity of $\ell_p[X]$, convergent sequences on $\ell_p[X]$, and compact subsets of $\ell_p[X]$.

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1. Introduction. It is known that the general theory of scalar-valued sequence spaces (SVSS) plays an important role in the theory of topological vector spaces (cf. [11, 21]). Furthermore, the theory of generalized sequence spaces, or vector-valued sequence spaces (VVSS) (cf. [10, 18, 24]) which has emerged as an outgrowth of the development of SVSS also plays an important role in the theory of locally convex spaces, especially in the investigation of nuclear spaces through λ-summing operators (cf. [3, 4, 5, 19, 20]). For example, Pietsch [19, 20] characterized a nuclear locally convex space X in terms of the absolutely p-summable sequence space $\ell_p(X)$ and the weakly p-summable sequence space $\ell_p[X]$. Because of Pietsch's work, people began to be interested in the properties of the spaces $\ell_p(X)$ and $\ell_p[X]$. Some results about the space $\ell_p(X)$ are presented (cf. [1, 2, 6, 7, 8, 12, 14, 15, 16, 17]). But few results about the space $\ell_p[X]$ have been presented for a long time. Recently, Wu and Bu [26] have shown a representation of the Köthe dual of the space $\ell_p[X]$ and a characterization that $\ell_p[X]$ is a Grothendieck space. Gupta and Bu [9] have characterized GAK-ness of the space $\ell_p[X]$ in terms of the (q)-property of a Banach space X for $1 < p < \infty$, and in terms of non-containment of c_0 of X for $p = 1$ (a locally convex space containing no copy of c_0 has a number of nice properties, cf. [13, 25]).

In Section 3, by the definitions and basic results given in [9, 26], we characterize the GAK-ness of the space $\ell_p[X]$ in terms of the Köthe dual $\ell_p[X]^\times$ and the topological dual $\ell_p[X]^*$ of $\ell_p[X]$. Then by using this result, we characterize the reflexivity of the space $\ell_p[X]$ in terms of the reflexivity and (q)-property of a Banach space X. This result shows that the characterization of the reflexivity of the space $\ell_p[X]$ is different from the characterization of the reflexivity of the space $\ell_p(X)$ (the reflexivity of $\ell_p(X)$ is discussed in [12]).

In Section 4, by using the characterization that $\ell_p[X]$ is a GAK-space (cf. [9]) and some properties about Köthe dual of $\ell_p[X]$ (cf. [26]), we characterize convergent sequences of $\ell_p[X]$ with respect to the norm topology and with respect to the weak topology.

In Section 5, by using the results in Section 4, we characterize compact subsets of $\ell_p[X]$ and relatively weakly sequentially compact subsets of $\ell_p[X]$.

2. Concepts and basic results. Throughout this paper, we denote a Banach space by X and its topological dual by X^*. Then (X,X^*) forms a dual pair (cf. [23]). We denote the strong topology and the weak topology with respect to the dual pair (X,X^*) by $\beta(X,X^*)$ and $\sigma(X,X^*)$, respectively. B_X stands for the closed unit ball of X. For $1 \le p < \infty$, we introduce the Banach-valued sequence space $\ell_p[X]$, the space of weakly p-summable sequences on X, that is,

$$\ell_p[X] = \left\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i\ge 1} |f(x_i)|^p < \infty \ \forall f \in X^*\right\} \tag{2.1}$$

and introduce a norm $\|\cdot\|_p$ on $\ell_p[X]$, that is,

$$\|\bar{x}\|_p = \sup\left\{\left(\sum_{i\ge 1} |f(x_i)|^p\right)^{1/p} : f \in B_{X^*}\right\}. \tag{2.2}$$

Then $(\ell_p[X], \|\cdot\|_p)$ is a Banach space (cf. [9, 26]). By the results in [9], we have another form about $\ell_p[X]$ and $\|\cdot\|_p$ as follows.

Proposition 2.1. *The space of weekly p-summable sequences on X has the form*

$$\ell_p[X] = \left\{\bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i\ge 1} t_i x_i \text{ converges } \forall (t_i)_i \in \ell_q\right\}. \tag{2.3}$$

and for each $\bar{x} \in \ell_p[X]$,

$$\|\bar{x}\|_p = \sup\left\{\left\|\sum_{i\ge 1} t_i x_i\right\| : (t_i)_i \in B_{\ell_q}\right\}. \tag{2.4}$$

Here the space ℓ_q, denotes the dual space of ℓ_p for $1 < p < \infty$ (i.e., $1/p + 1/q = 1$), and denotes the space c_0 for $p = 1$.

It is easy to see that the coordinate projections

$$P_i : \ell_p[X] \longrightarrow X, \quad P_i(\bar{x}) = x_i \quad \text{for } i = 1, 2, \ldots \tag{2.5}$$

are continuous. For $\bar{x} \in X^{\mathbb{N}}$, we introduce the symbols

$$\bar{x}(i \le n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots); \qquad \bar{x}(i > n) = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots). \tag{2.6}$$

Definition 2.2. The Banach-valued sequence space $\ell_p[X]$ is called a GAK-space if for all $\bar{x} \in \ell_p[X]$, $\lim_n \|\bar{x}(i > n)\|_p = 0$ (cf. [9, 10, 26]).

We denote the GAK-subspace of $\ell_p[X]$ by $\ell_p[X]_G$, that is,

$$\ell_p[X]_G = \left\{\bar{x} \in \ell_p[X] : \lim_n \|\bar{x}(i > n)\|_p = 0\right\}. \tag{2.7}$$

The Köthe dual of $\ell_p[X]$ with respect to the dual pair (X,X^*) is denoted by $\ell_p[X]^\times|_{(X,X^*)}$ (cf. [10, 26]), that is,

$$\ell_p[X]^\times|_{(X,X^*)} = \left\{ \bar{f} = (f_i)_i \in X^{*\mathbb{N}} : \sum_{i\ge 1} |f_i(x_i)| < \infty \ \forall \bar{x} \in \ell_p[X] \right\}. \tag{2.8}$$

We denote $\ell_p[X]^\times|_{(X,X^*)}$ simply by $\ell_p[X]^\times$ if the meaning is clear from the context.

Proposition 2.3. *The Köthe dual of $\ell_p[X]^\times$ is*

$$\left(\ell_p[X]^\times|_{(X,X^*)}\right)^\times|_{(X^*,X^{**})} = \ell_p[X^{**}]. \tag{2.9}$$

Proof. See [26]. □

For $\bar{x} \in \ell_p[X]$ and $\bar{f} \in \ell_p[X]^\times$, define

$$\left\langle \bar{x}, \bar{f} \right\rangle = \sum_{i\ge 1} f_i(x_i). \tag{2.10}$$

Then $(\ell_p[X], \ell_p[X]^\times)$ forms a dual pair with respect to the bilinear functional $\langle\cdot,\cdot\rangle$ defined in (2.10). And for each $\bar{f} \in \ell_p[X]^\times, \langle\cdot,\bar{f}\rangle$ is a linear functional on $\ell_p[X]$. Furthermore, we have the following proposition.

Proposition 2.4. *For each $\bar{f} \in \ell_p[X]^\times$, $\langle\cdot,\bar{f}\rangle$ is a continuous linear functional on $(\ell_p[X], \|\cdot\|_p)$, that is,*

$$\left\langle \cdot, \bar{f} \right\rangle \in \left(\ell_p[X], \|\cdot\|_p\right)^* := \ell_p[X]^*. \tag{2.11}$$

Proof. For $\bar{f} \in \ell_p[X]^\times$, $n \in \mathbb{N}$, define the linear functionals F and F_n on $\ell_p[X]$ by

$$F(\bar{x}) = \sum_{i\ge 1} f_i(x_i), \qquad F_n(\bar{x}) = \sum_{i=1}^{n} f_i(x_i). \tag{2.12}$$

Then it is easy to see that $F_n \in \ell_p[X]^*$ and for each $\bar{x} \in \ell_p[X]$, $\lim_n F_n(\bar{x}) = F(\bar{x})$. So the Banach-Steinhaus theorem (cf. [23, page 137]) implies that $F \in \ell_p[X]^*$. □

By Proposition 2.4, the space $\ell_p[X]^\times$ can be considered as a subspace of $\ell_p[X]^*$, that is,

$$\ell_p[X]^\times \subseteq \ell_p[X]^*. \tag{2.13}$$

Now denote the dual norm of $\|\cdot\|_p$ on the dual space $\ell_p[X]^*$ by $\|\cdot\|_p^*$. Then we have the following proposition belonging to [26].

Proposition 2.5. *For $1 < p < \infty$, $(\ell_p[X]^\times, \|\cdot\|_p^*)$ is a GAK-space.*

By [26, Lemma 1] and [10, Corollary 4.9], we have the following two propositions.

Proposition 2.6. *For* $1 \le p < \infty$,

$$\ell_p[X]^\times = (\ell_p[X]_G)^\times = (\ell_p[X]_G, \|\cdot\|_p)^* := (\ell_p[X]_G)^*. \tag{2.14}$$

Proposition 2.7. *If a Banach-valued sequence space* $(\lambda[X], \|\cdot\|)$ *is both a Banach space and a GAK-space, then* $(\lambda[X], \|\cdot\|)^* = \lambda[X]^\times|_{(X,X^*)}$.

Definition 2.8. For $1 < q \le \infty$, a Banach space X is said to have the (q)-property if the following two statements about a sequence $\{x_i\}_1^\infty$ in X are equivalent:

(i) $\sum_{i\ge1} t_i x_i$ converges for each $(t_i)_i \in \ell_q$;

(ii) $\sum_{i\ge1} t_i x_i$ converges uniformly for all $(t_i)_i \in B_{\ell_q}$.

Swartz [22] has proved that every Banach space has the (∞)-property. More results about (q)-property can be seen in [9]. For $1 < p < \infty$, let q be its conjugate, that is, $1/p + 1/q = 1$. Then we have the following proposition which belongs to [9].

Proposition 2.9. *The space* $\ell_p[X]$ $(1 < p < \infty)$ *is a GAK-space if and only if* X *has the* (q)*-property.*

3. Reflexivity

Lemma 3.1. *For a sequence* $\bar{x} \in \ell_p[X]$, *define a linear map*

$$I_{\bar{x}} : \ell_p[X]^\times \longrightarrow \ell_1, \qquad I_{\bar{x}}(\bar{f}) = (f_i(x_i))_i. \tag{3.1}$$

Then $I_{\bar{x}}$ *is* $\sigma(\ell_p[X]^\times, \ell_p[X])$-$\sigma(\ell_1, \ell_\infty)$ *continuous.*

Proof. The proof can be easily completed and so is omitted. □

Lemma 3.2. *Let* $\bar{f}^{(n)} = (f_i^{(n)})_i \in \ell_p[X]^\times$ *such that* $\{\bar{f}^{(n)}\}_1^\infty$ *is* $\sigma(\ell_p[X]^\times, \ell_p[X])$*-bounded. If for each* $i \in \mathbb{N}$, *there exists* f_i *in* X^* *such that* $\sigma(X^*, X)$-$\lim_n f_i^{(n)} = f_i$, *then* $\bar{f} = (f_i)_i \in \ell_p[X]^\times$.

Proof. Let $\bar{x} \in \ell_p[X]$. By Lemma 3.1, $\{(f_i^{(n)}(x_i))_i\}_{n=1}^\infty$ is a $\sigma(\ell_1, \ell_\infty)$-bounded subset of ℓ_1 and hence, is $\beta(\ell_1, \ell_\infty)$-bounded. Thus

$$M = \sup_{n\ge1} \sum_{i\ge1} \left| f_i^{(n)}(x_i) \right| < \infty. \tag{3.2}$$

Fix $m \in \mathbb{N}$. Since $\sigma(X^*, X)$-$\lim_n f_i^{(n)} = f_i$ for each $i \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that

$$\left| f_i^{(n)}(x_i) - f_i(x_i) \right| < \frac{1}{m}, \quad i = 1, 2, \ldots, m. \tag{3.3}$$

So

$$\sum_{i=1}^m |f_i(x_i)| \le \sum_{i=1}^m \left| f_i^{(n)}(x_i) - f_i(x_i) \right| + \sum_{i=1}^m \left| f_i^{(n)}(x_i) \right| < 1 + M. \tag{3.4}$$

Since m is arbitrary in $\mathbb{N}$, $\sum_{i\ge1} |f_i(x_i)| < \infty$. Thus we have proved that $\bar{f} \in \ell_p[X]^\times$. □

Theorem 3.3. *For* $1 \le p < \infty$, *the following hold:*

(i) $\ell_p[X]^\times$ *is a closed subspace of* $\ell_p[X]^*$;

(ii) *for each* $\bar{x} \in \ell_p[X]$,

$$\|\bar{x}\|_p = \sup\left\{\left|\left\langle \bar{x}, \bar{f}\right\rangle\right| : \bar{f} \in \ell_p[X]^\times, \left\|\bar{f}\right\|_p^* \leq 1\right\}; \tag{3.5}$$

(iii) $\ell_p[X]^\times = \ell_p[X]^*$ *if and only if* $\ell_p[X]$ *is a GAK-space.*

Proof. (i) To prove that $\ell_p[X]^\times$ is a closed subspace of $\ell_p[X]^*$, we only need to prove that $\ell_p[X]^\times$ is complete with respect to the norm $\|\cdot\|_p^*$.

Let $\{\bar{f}^{(n)}\}_1^\infty$ be a Cauchy sequence in $(\ell_p[X]^\times, \|\cdot\|_p^*)$. By the continuity of the coordinate projections from $(\ell_p[X]^\times, \|\cdot\|_p^*)$ to X^*, $\{f_i^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in X^* for each $i \in \mathbb{N}$. Thus the completeness of X^* implies that there exist f_i in X^* such that $\lim_n f_i^{(n)} = f_i$ for each $i \in \mathbb{N}$. So $\bar{f} = (f_i)_i \in \ell_p[X]^\times$ by Lemma 3.2. Next we will prove that $\lim_n \bar{f}^{(n)} = \bar{f}$.

Fix $\bar{x} \in \ell_p[X]$. By Lemma 3.1, $\{(f_i^{(n)}(x_i))_i\}_{n=1}^\infty$ is a $\sigma(\ell_1, \ell_\infty)$-Cauchy sequence in ℓ_1 and hence, is $\beta(\ell_1, \ell_\infty)$-Cauchy. So for any given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\sum_{i\geq 1}\left|f_i^{(n)}(x_i) - f_i^{(n_0)}(x_i)\right| < \frac{\varepsilon}{4}. \tag{3.6}$$

Since $\bar{f}, \bar{f}^{(n_0)} \in \ell_p[X]^\times$, there exists an $i_0 \in \mathbb{N}$ such that

$$\sum_{i>i_0}\left|f_i(x_i)\right| < \frac{\varepsilon}{4}, \qquad \sum_{i>i_0}\left|f_i^{(n_0)}(x_i)\right| < \frac{\varepsilon}{4}. \tag{3.7}$$

And furthermore, since $\lim_n f_i^{(n)} = f_i$ for each $i \in \mathbb{N}$, there exists an $n_1 \in \mathbb{N}$ with $n_1 > n_0$ such that for $n > n_1$,

$$\left|f_i^{(n)}(x_i) - f_i(x_i)\right| < \frac{\varepsilon}{4i_0}, \quad i = 1, 2, \ldots, i_0. \tag{3.8}$$

So for $n > n_1$, we have

$$\begin{aligned}\left|\left\langle \bar{x}, \bar{f}^{(n)} - \bar{f}\right\rangle\right| \leq& \sum_{i=1}^{i_0}\left|f_i^{(n)}(x_i) - f_i(x_i)\right| + \sum_{i>i_0}\left|f_i^{(n)}(x_i) - f_i^{(n_0)}(x_i)\right| \\ &+ \sum_{i>i_0}\left|f_i^{(n_0)}(x_i)\right| + \sum_{i>i_0}\left|f_i(x_i)\right| < \varepsilon.\end{aligned} \tag{3.9}$$

It follows that *weak**-$\lim_n \bar{f}^{(n)} = \bar{f}$. Note that $\{\bar{f}^{(n)}\}_1^\infty$ is Cauchy. So $\lim_n \bar{f}^{(n)} = \bar{f}$. Therefore, we have proved that $(\ell_p[X]^\times, \|\cdot\|_p^*)$ is complete and (i) follows.

(ii) Let $\bar{x} \in \ell_p[X]$ and q the conjugate of p, that is, $1/p + 1/q = 1$. Then

$$\begin{aligned}\|\bar{x}\|_p &= \sup\left\{\left(\sum_{i\geq 1}\left|f(x_i)\right|^p\right)^{1/p} : f \in B_{X^*}\right\} \\ &= \sup\left\{\left|\sum_{i\geq 1} t_i f(x_i)\right| : f \in B_{X^*}, \ (t_i)_i \in B_{\ell_q}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sup\left\{\left|\left\langle \bar{x}, (t_i f)_i\right\rangle\right| : f \in B_{X^*},\ (t_i)_i \in B_{\ell_q}\right\} \\
&\le \sup\left\{\left|\left\langle \bar{x}, \bar{f}\right\rangle\right| : \bar{f} \in \ell_p[X]^{\times},\ \left\|\bar{f}\right\|_p^* \le 1\right\}
\end{aligned}
\tag{3.10}
$$

since $(t_i f)_i \in \ell_p[X]^{\times}$ and $\|(t_i f)_i\|_p^* \le 1$ for each $(t_i)_i \in B_{\ell_q}$ and for each $f \in B_{X^*}$. On the other hand,

$$
\begin{aligned}
\|\bar{x}\|_p &= \sup\left\{|\langle \bar{x}, F\rangle| : F \in \ell_p[X]^*,\ \|F\|_p^* \le 1\right\} \\
&\ge \sup\left\{\left|\left\langle \bar{x}, \bar{f}\right\rangle\right| : \bar{f} \in \ell_p[X]^{\times},\ \left\|\bar{f}\right\|_p^* \le 1\right\}.
\end{aligned}
\tag{3.11}
$$

So (ii) follows.

(iii) If $\ell_p[X]$ is a GAK-space, it follows from Proposition 2.6 that $\ell_p[X]^{\times} = \ell_p[X]^*$. On the other hand, suppose that $\ell_p[X]^{\times} = \ell_p[X]^*$. Fix $\bar{x} \in \ell_p[X]$. By Lemma 3.1 and the Banach-Alaoglu theorem (cf. [23, page 130]),

$$
\left\{I_{\bar{x}}\left(\bar{f}\right) : \bar{f} \in \ell_p[X]^{\times},\ \left\|\bar{f}\right\|_p^* \le 1\right\}
\tag{3.12}
$$

is a $\sigma(\ell_1, \ell_\infty)$-compact subset of ℓ_1. So by [11, Proposition 6.11, page 108],

$$
\lim_n \sup\left\{\left|\sum_{i>n} f_i(x_i)\right| : \bar{f} \in \ell_p[X]^{\times},\ \left\|\bar{f}\right\|_p^* \le 1\right\} = 0.
\tag{3.13}
$$

It follows from (ii) that $\lim_n \|\bar{x}(i > n)\|_p = 0$. Thus we have proved that $\ell_p[X]$ is a GAK-space and hence, (iii) follows. □

Lemma 3.4. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and Y a closed subspace of X. If $(Y, \|\cdot\|)^* = (X, \|\cdot\|)^*$, then $Y = X$.*

Proof. The proof can be easily completed and so is omitted. □

Theorem 3.5. *The Banach space $\ell_p[X]$ $(1 < p < \infty)$ is a reflexive space if and only if*

(i) *X is a reflexive space; and*

(ii) *$\ell_p[X]$ is a GAK-space.*

Proof. Suppose that (i) and (ii) hold. By Theorem 3.3, $\ell_p[X]^{\times} = \ell_p[X]^*$. Moreover, by Propositions 2.5 and 2.7,

$$
\left(\ell_p[X]^{\times}\right)^{\times}\big|_{(X^*, X^{**})} = \left(\ell_p[X]^{\times}, \|\cdot\|_p^*\right)^* = \left(\ell_p[X]^*, \|\cdot\|_p^*\right)^* := \ell_p[X]^{**}.
\tag{3.14}
$$

So by Proposition 2.3, $\ell_p[X^{**}] = \ell_p[X]^{**}$. It follows from (i) that $\ell_p[X] = \ell_p[X]^{**}$. So $\ell_p[X]$ is a reflexive space.

On the other hand, suppose that $\ell_p[X]$ is a reflexive space. Since X is isometrically isomorphic to a subspace of $\ell_p[X]$, (i) holds.

Now, by Propositions 2.3, 2.5, and 2.7,

$$
\left(\ell_p[X]^{\times}, \|\cdot\|_p^*\right)^* = \left(\ell_p[X]^{\times}\right)^{\times}\big|_{(X^*, X^{**})} = \ell_p[X^{**}].
\tag{3.15}
$$

Since X and $\ell_p[X]$ are reflexive,

$$(\ell_p[X]^\times, \|\cdot\|_p^*)^* = \ell_p[X^{**}] = \ell_p[X] = (\ell_p[X]^*, \|\cdot\|_p^*)^*. \tag{3.16}$$

It follows from Lemma 3.4 that $\ell_p[X]^\times = \ell_p[X]^*$. Thus (ii) holds by Theorem 3.3. □

Now by Proposition 2.9, we have the following theorem.

Theorem 3.6. *The Banach space $\ell_p[X]$ is a reflexive space if and only if the Banach space X is both a reflexive space and has the (q)-property* $(1 < p < \infty,\ 1/p + 1/q = 1)$.

4. Convergent sequences

Lemma 4.1. *Let $\bar{x} \in \ell_p[X]$ $(1 \le p < \infty)$. Then* $\lim_n \|\bar{x}(i \le n)\|_p = \|\bar{x}\|_p$.

Proof. By Proposition 2.1,

$$\begin{gathered}\|\bar{x}\|_p = \sup\left\{\left\|\sum_{i\ge 1} t_i x_i\right\| : (t_i)_i \in B_{\ell_q}\right\},\\ \|\bar{x}(i \le n)\|_p = \sup\left\{\left\|\sum_{i=1}^{n} t_i x_i\right\| : (t_i)_i \in B_{\ell_q}\right\}.\end{gathered} \tag{4.1}$$

It is easy to see that $\|\bar{x}(i \le n)\|_p \le \|\bar{x}(i \le n+1)\|_p$. So $\{\|\bar{x}(i \le n)\|_p\}_{n=1}^{\infty}$ is an increasing sequence. Note that $\sup_{n\ge 1} \|\bar{x}(i \le n)\|_p = \|\bar{x}\|_p$. Hence $\lim_n \|\bar{x}(i \le n)\|_p = \|\bar{x}\|_p$. □

Theorem 4.2. *Let $\bar{x}^{(n)}, \bar{x} \in \ell_p[X]$ $(1 \le p < \infty)$. Then*

$$\lim_n \bar{x}^{(n)} = \bar{x} \tag{4.2}$$

is equivalent to

(i) $\lim_n x_i^{(n)} = x_i$ *for each* $i \in \mathbb{N}$; *and*

(ii) $\lim_n \|\bar{x}^{(n)}(i > m)\|_p = \|\bar{x}(i > m)\|_p$ *for each* $m \in \mathbb{N}$, *if and only if* $\ell_p[X]$ *is a GAK-space.*

Proof. $(\Leftarrow)$ Suppose that $\lim_n \bar{x}^{(n)} = \bar{x}$. Then (i) and (ii) hold obviously from the continuity of the coordinate projections P_i and the norm $\|\cdot\|_p$. On the other hand, suppose that (i) and (ii) hold. We will prove that $\lim_n \bar{x}^{(n)} = \bar{x}$.

Fix $\varepsilon > 0$. Since $\ell_p[X]$ is a GAK-space, there exists an $m_0 \in \mathbb{N}$ such that

$$\|\bar{x}(i > m_0)\|_p < \frac{\varepsilon}{4}. \tag{4.3}$$

By (i) and (ii), there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\begin{gathered}\left\|x_i^{(n)} - x_i\right\| < \frac{\varepsilon}{4m_0}, \quad \text{for } i = 1, 2, \ldots, m_0,\\ \left|\,\|\bar{x}^{(n)}(i > m_0)\|_p - \|\bar{x}(i > m_0)\|_p\right| < \frac{\varepsilon}{4}.\end{gathered} \tag{4.4}$$

So for $n > n_0$, we have

$$\begin{aligned} \|\bar{x}^{(n)} - \bar{x}\|_p &= \|\bar{x}^{(n)}(i \le m_0) + \bar{x}^{(n)}(i > m_0) - \bar{x}(i \le m_0) - \bar{x}(i > m_0)\|_p \\ &\le \|\bar{x}^{(n)}(i \le m_0) - \bar{x}(i \le m_0)\|_p + \|\bar{x}^{(n)}(i > m_0)\|_p + \|\bar{x}(i > m_0)\|_p \\ &\le \sum_{i=1}^{m_0} \left\|x_i^{(n)} - x_i\right\| + \left| \|\bar{x}^{(n)}(i > m_0)\|_p - \|\bar{x}(i > m_0)\|_p \right| + 2\|\bar{x}(i > m_0)\|_p \\ &< \varepsilon. \end{aligned} \tag{4.5}$$

Thus we have proved that $\lim_n \bar{x}^{(n)} = \bar{x}$ and the sufficiency is proved.

($\Rightarrow$) For any given $\bar{x} \in \ell_p[X]$, we want to show that

$$\lim_n \|\bar{x}(i > n)\|_p = 0. \tag{4.6}$$

Let $\bar{x}^{(n)} = \bar{x}(i \le n)$. Then for each $i \in \mathbb{N}$, $x_i^{(n)} = x_i$ if $n > i$. So $\bar{x}^{(n)}$ and $\bar{x}$ satisfy condition (i). Now for each $m \in \mathbb{N}$, by Lemma 4.1, $\lim_n \|\bar{x}^{(n)}(i > m)\|_p = \lim_n \|\bar{x}(m < i \le n)\|_p = \|\bar{x}(i > m)\|_p$. So $\bar{x}^{(n)}$ and $\bar{x}$ satisfy condition (ii). Thus (i) and (ii) imply that $\lim_n \|\bar{x}^{(n)} - \bar{x}\|_p = 0$. Hence

$$\lim_n \|\bar{x}(i > n)\|_p = \lim_n \|\bar{x}^{(n)} - \bar{x}\|_p = 0. \tag{4.7}$$

Therefore, we have proved that $\ell_p[X]$ is a GAK-space and the necessity is proved. □

Corollary 4.3. *Let $\bar{x}^{(n)}, \bar{x} \in \ell_p[X]_G$ $(1 \le p < \infty)$. Then*

$$\lim_n \bar{x}^{(n)} = \bar{x} \tag{4.8}$$

if and only if

(i) $\lim_n x_i^{(n)} = x_i$ *for each* $i \in \mathbb{N}$*; and*

(ii) $\lim_n \|\bar{x}^{(n)}(i > m)\|_p = \|\bar{x}(i > m)\|_p$ *for each* $m \in \mathbb{N}$.

Theorem 4.4. *Let $\bar{x}^{(n)}, \bar{x} \in \ell_p[X]$ $(1 < p < \infty)$. Then*

$$\sigma(\ell_p[X], \ell_p[X]^\times)\text{-}\lim_n \bar{x}^{(n)} = \bar{x} \tag{4.9}$$

if and only if

(i) $\sigma(X, X^*)\text{-}\lim_n x_i^{(n)} = x_i$ *for each* $i \in \mathbb{N}$*; and*

(ii) $\sup_{n \ge 1} \|\bar{x}^{(n)}\|_p < \infty$.

Proof. It is easy to see that $\sigma(\ell_p[X], \ell_p[X]^\times)\text{-}\lim_n \bar{x}^{(n)} = \bar{x}$ implies that (i) and (ii) hold. Conversely, suppose that (i) and (ii) hold. We will prove that

$$\sigma(\ell_p[X], \ell_p[X]^\times)\text{-}\lim_n \bar{x}^{(n)} = \bar{x}. \tag{4.10}$$

Fix any given $\bar{f} \in \ell_p[X]^\times$ and any given $\varepsilon > 0$. Let $M = \sup_{n \ge 1} \|\bar{x}^{(n)}\|_p$. By Proposition 2.5, $\ell_p[X]^\times$ is a GAK-space. So there exists a $k \in \mathbb{N}$ such that

$$\|\bar{f}(i > k)\|_p^* < \frac{\varepsilon}{2(M + \|\bar{x}\|_p)}. \tag{4.11}$$

By (i), there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\left| f_i\left(x_i^{(n)}\right) - f_i(x_i) \right| < \frac{\varepsilon}{2k}, \quad i = 1,2,\dots,k. \tag{4.12}$$

So for $n > n_0$, we have

$$\begin{aligned} \left| \left\langle \bar{x}^{(n)} - \bar{x}, \bar{f} \right\rangle \right| &\le \sum_{i=1}^{k} \left| f_i\left(x_i^{(n)}\right) - f_i(x_i) \right| + \left| \sum_{i>k} f_i\left(x_i^{(n)} - x_i\right) \right| \\ &= \sum_{i=1}^{k} \left| f_i\left(x_i^{(n)}\right) - f_i(x_i) \right| + \left| \langle \bar{x}^{(n)} - \bar{x}, \bar{f}(i>k) \rangle \right| \\ &\le \frac{\varepsilon}{2} + \|\bar{x}^{(n)} - \bar{x}\|_p \left\| \bar{f}(i>k) \right\|_p^* \\ &\le \frac{\varepsilon}{2} + (M + \|\bar{x}\|_p) \|\bar{f}(i>k)\|_p^* < \varepsilon. \end{aligned} \tag{4.13}$$

Thus we have proved that

$$\sigma(\ell_p[X], \ell_p[X]^\times)\text{-}\lim_n \bar{x}^{(n)} = \bar{x}. \tag{4.14}$$

This completes the proof. □

Remark 4.5. Theorem 4.4 is not valid for the space $\ell_1[X]$. For example, pick $x_0 \in X$ and $f_0 \in X^*$ such that $f_0(x_0) = 1$. Let

$$\bar{f} = (f_0, f_0, \dots), \qquad \bar{x}^{(n)} = \Big(\overbrace{0,\dots,0}^{n}, \frac{1}{2}x_0, \frac{1}{2^2}x_0, \frac{1}{2^3}x_0, \dots \Big), \tag{4.15}$$

where $(1/2^k)x_0$ is at the $(n+k)$th place, $k = 1,2,\dots$. Then $\bar{f} \in \ell_1[X]^\times$, $\bar{x}^{(n)} \in \ell_1[X]$ and

$$\|\bar{x}^{(n)}\|_1 = \sup\left\{ \sum_{k\ge1} \left| \frac{1}{2^k} f(x_0) \right| : f \in B_{X^*} \right\} = \|x_0\|, \quad n = 1,2,\dots. \tag{4.16}$$

So $\bar{x}^{(n)}$ and 0 satisfy conditions (i) and (ii) in Theorem 4.4. But

$$\left\langle \bar{x}^{(n)}, \bar{f} \right\rangle = \sum_{k\ge1} \frac{1}{2^k} f_0(x_0) = 1. \tag{4.17}$$

Hence

$$\sigma(\ell_1[X], \ell_1[X]^\times)\text{-}\lim_n \bar{x}^{(n)} \ne 0. \tag{4.18}$$

Thus we have shown that Theorem 4.4 is not valid for the space $\ell_1[X]$.

5. Compact sets. A subset B of $\ell_p[X]$ is called normal if $(x_i)_i \in B$ and $(t_i)_i \in \ell_\infty$ imply that $(t_i x_i)_i \in B$.

Theorem 5.1. *A normal subset B of $\ell_p[X]$ $(1 \le p < \infty)$ is compact if and only if*

(i) *B is closed;*

(ii) $\lim_n \bar{x}(i > n) = 0$ *uniformly for all* $\bar{x} \in B$;
(iii) $P_i(B)$ *is a compact subset of* X *for each* $i \in \mathbb{N}$.

Proof. First, let B be a normal compact subset of $\ell_p[X]$. Then (i) holds obviously. By the continuity of P_i, (iii) holds. Now suppose (ii) does not hold. Then there exist $\varepsilon_0 > 0$, $\bar{x}^{(k)} \in B$, and $n_1 < n_2 < \cdots$ such that

$$\|\bar{x}^{(k)}(i > n_k)\|_p \geq \varepsilon_0, \quad k = 1, 2, \ldots. \tag{5.1}$$

Let

$$\bar{y}^{(k)} = \left(0, \ldots, 0, x^{(k)}_{n_k+1}, x^{(k)}_{n_k+2}, \ldots\right). \tag{5.2}$$

Since B is normal, $\bar{y}^{(k)} \in B$ for $k = 1, 2, \ldots$. Noticing that B is compact and $\bar{y}^{(k)}$ converges to 0 coordinate-wise as $k \to \infty$, $\lim_k \bar{y}^{(k)} = 0$ which contradicts (5.1). This contradiction shows that (ii) holds.

Now, suppose (i), (ii), and (iii) hold. We want to show that B is compact. By (i), B is complete. By [23, page 88], it suffices to show that B is totally bounded.

Fix $\varepsilon > 0$, by (ii), there exists $n_0 \in \mathbb{N}$ such that

$$\sup\left\{\|\bar{x}(i > n_0)\|_p : \bar{x} \in B\right\} < \frac{\varepsilon}{2}. \tag{5.3}$$

By (iii), $A =: \bigcup_{i=1}^{n_0} P_i(B)$ is a totally bounded subset of X. So there exists a finite subset F of X such that for each $x \in A$, there exists $y \in F$ satisfying $\|x - y\| < \varepsilon/2n_0$. Now let

$$D =: \{\bar{y} = (y_i)_i \in X^{\mathbb{N}} : y_i \in F \text{ for } 1 \leq i \leq n_0,\ y_i = 0 \text{ for } i > n_0\}. \tag{5.4}$$

Then D is a finite subset of $\ell_p[X]$. Now for each $\bar{x} = (x_i)_i \in B$, since $x_i \in A$ for $i = 1, 2, \ldots, n_0$, there exists $y_i \in F$ such that

$$\|x_i - y_i\| < \frac{\varepsilon}{2n_0}, \quad i = 1, 2, \ldots, n_0. \tag{5.5}$$

Let $\bar{y} = (y_1, \ldots, y_{n_0}, 0, 0, \ldots)$. Then $\bar{y} \in D$ and

$$\begin{aligned} \|\bar{x} - \bar{y}\|_p &\leq \|\bar{x}(i \leq n_0) - \bar{y}(i \leq n_0)\|_p + \|\bar{x}(i > n_0)\|_p \\ &\leq \sup\left\{\left\|\sum_{i=1}^{n_0} t_i(x_i - y_i)\right\| : (t_i)_i \in B_{\ell_q}\right\} + \frac{\varepsilon}{2} < \varepsilon, \end{aligned} \tag{5.6}$$

where $1/p + 1/q = 1$. So we have showed that B is totally bounded. This completes the proof. □

Lemma 5.2. *Let* $\bar{x}^{(n)} = (x_i^{(n)})_i \in \ell_p[X]$ *such that* $\{\bar{x}^{(n)}\}_1^\infty$ *is* $\sigma(\ell_p[X], \ell_p[X]^\times)$*-bounded. If for each* $i \in \mathbb{N}$*, there exists* $x_i \in X$ *such that* $\sigma(X, X^*)$*-*$\lim_n x_i^{(n)} = x_i$*, then* $\bar{x} = (x_i)_i \in \ell_p[X]$.

Proof. As in the proof of Lemma 3.2, we can show that $\sum_{i\geq 1} |f_i(x_i)| < \infty$ for each $\bar{f} = (f_i)_i \in \ell_p[X]^\times$. So $\bar{x} \in (\ell_p[X]^\times|_{(X,X^*)})^\times|_{(X^*,X)}$. By Proposition 2.3,

$$\left(\ell_p[X]^\times|_{(X,X^*)}\right)^\times\Big|_{(X^*,X)} = \ell_p[X]. \tag{5.7}$$

Thus $\bar{x} \in \ell_p[X]$ and the proof is completed. □

Theorem 5.3. *A subset B of $\ell_p[X]$ $(1 < p < \infty)$ is a relatively $\sigma(\ell_p[X], \ell_p[X]^\times)$-sequentially compact if and only if*

(i) *B is bounded;*

(ii) *$P_i(B)$ is a relatively $\sigma(X, X^*)$-sequentially compact subset of X for each $i \in \mathbb{N}$.*

Proof. Suppose that B is a relatively $\sigma(\ell_p[X], \ell_p[X]^\times)$-sequentially compact subset of $\ell_p[X]$. Then B is $\sigma(\ell_p[X], \ell_p[X]^\times)$-bounded. By Theorem 3.3, B is bounded and (i) holds. Since P_i is continuous and hence, $\sigma(\ell_p[X], \ell_p[X]^\times)$-$\sigma(X, X^*)$ continuous for each $i \in \mathbb{N}$, (ii) holds.

On the other hand, suppose (i) and (ii) hold. Let $\{\bar{x}^{(n)}\}_1^\infty \subseteq B$. Using the diagonal method, by (ii), there exist a subsequence $\{\bar{x}^{(n_k)}\}_{k\geq 1}$ of $\{\bar{x}^{(n)}\}_{n\geq 1}$ and $x_i \in X$ such that

$$\sigma(X, X^*)\text{-}\lim_k x_i^{(n_k)} = x_i, \quad i = 1, 2, \ldots. \tag{5.8}$$

By (i) and Lemma 5.2, $\bar{x} = (x_i)_i \in \ell_p[X]$. By Theorem 4.4, $\sigma(\ell_p[X], \ell_p[X]^\times)$-$\lim_k \bar{x}^{(n_k)} = \bar{x}$. So we have proved that B is relatively $\sigma(\ell_p[X], \ell_p[X]^\times)$-sequentially compact. This completes the proof. □

References

[1] P. Cembranos, *Some properties of the Banach space $c_0(E)$*, Mathematics Today, Gauthier-Villars, Paris, 1982, pp. 333–336. MR 84a:46045. Zbl 493.46021.

[2] ______, *The hereditary Dunford-Pettis property for $\ell_1(E)$*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 947–950. MR 90i:46019. Zbl 833.46007.

[3] N. K. De Grande-De, *Generalized sequence spaces*, Bull. Soc. Math. Belg. **23** (1971), 123–166. MR 46 #9688. Zbl 232.46017.

[4] ______, *Criteria for nuclearity in terms of generalized sequence spaces*, Arch. Math. (Basel) **28** (1977), no. 6, 644–651. MR 57 #1070. Zbl 359.46011.

[5] ______, *Operators factoring through a generalized sequence space; applications*, Math. Nachr. **95** (1980), 79–88. MR 84b:46011. Zbl 0453.46008.

[6] M. A. Fugarolas, *On Besselian Schauder bases in $\ell_p(E)$*, Monatsh. Math. **97** (1984), no. 2, 99–105. MR 85h:46024. Zbl 531.46009.

[7] W. Govaerts, *Bornological spaces of type $c_0(E)$*, Portugal. Math. **41** (1982), no. 1-4, 51–55. MR 86f:46003. Zbl 545.46026.

[8] M. Gupta, *The generalized spaces $\ell^1(X)$ and $m_0(X)$*, J. Math. Anal. Appl. **78** (1980), no. 2, 357–366. MR 82c:46010. Zbl 453.46012.

[9] M. Gupta and Q. Bu, *On Banach-valued GAK-sequence spaces $\ell^p[X]$*, J. Anal. **2** (1994), 103–113. MR 95f:46008. Zbl 818.46023.

[10] M. Gupta, P. K. Kamthan, and J. Patterson, *Duals of generalized sequence spaces*, J. Math. Anal. Appl. **82** (1981), no. 1, 152–168. MR 82j:46006. Zbl 492.46010.

[11] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, New York, 1981. MR 83g:46011. Zbl 447.46002.

[12] I. E. Leonard, *Banach sequence spaces*, J. Math. Anal. Appl. **54** (1976), no. 1, 245–265. MR 54 #8230. Zbl 343.46010.

[13] R. L. Li and Q. Y. Bu, *Locally convex spaces containing no copy of c_0*, J. Math. Anal. Appl. **172** (1993), no. 1, 205–211. MR 94a:46002. Zbl 779.46012.

[14] A. Marquina and J. M. Sanz-Serna, *Barrelledness conditions on $c_0(E)$*, Arch. Math. (Basel) **31** (1978/79), no. 6, 589–596. MR 80i:46010. Zbl 396.46005.

[15] A. Marquina and J. Schmets, *On bornological $c_0(E)$ spaces*, Bull. Soc. Roy. Sci. Liège **51** (1982), no. 5-8, 170–173. MR 84f:46013. Zbl 514.46025.

[16] J. Mendoza, *A barrelledness criterion for $c_0(E)$*, Arch. Math. (Basel) **40** (1983), no. 2, 156–158. MR 84k:46006. Zbl 503.46026.

[17] C. Piñeiro, $\aleph_0$-*quasibarrelledness on* $c_0(E)$, Arch. Math. (Basel) **53** (1989), no. 1, 61–64. MR 90f:46008. Zbl 655.46004.

[18] A. Pietsch, *Verallgemeinerte Vollkommene Folgenräume*, Akademie-Verlag, Berlin, 1962. MR 28 #452. Zbl 105.30701.

[19] ______, *Absolut* p*-summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1967), 333–353. MR 35 #7162. Zbl 0156.37903.

[20] ______, *Nuclear locally convex spaces*, Springer-Verlag, New York, 1972. MR 50#2853. Zbl 236.46001.

[21] W. H. Ruckle, *Sequence Spaces*, Pitman (Advanced Publishing Program), Massachusetts, 1981. MR 83g:46012. Zbl 491.46007.

[22] C. Swartz, *The Schur lemma for bounded multiplier convergent series*, Math. Ann. **263** (1983), no. 3, 283–288. MR 84h:46015. Zbl 506.40006.

[23] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1978. MR 81d:46001. Zbl 395.46001.

[24] C. X. Wu and Q. Y. Bu, *Vector-valued sequence space* $\Lambda[X]$ *and its Köthe dual. I. GAK-property, separability and barrelledness*, Northeast. Math. J. **8** (1992), no. 3, 275–282. MR 94g:46010. Zbl 807.46010.

[25] ______, *Characterizations of* $\mathrm{cmc}(X)$ *which is GAK-space*, J. Harbin Inst. Tech. **25** (1993), no. 1, 93–96. MR 94g:46011.

[26] ______, *Köthe dual of Banach sequence spaces* $\ell_p[X]$ $(1 \le p < \infty)$ *and Grothendieck space*, Comment. Math. Univ. Carolin. **34** (1993), no. 2, 265–273. MR 95e:46022. Zbl 785.46009.

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