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ON MINIMAL ARTINIAN MODULES AND MINIMAL ARTINIAN LINEAR GROUPS

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Abstract. The paper is devoted to the study of some important types of minimal artinian linear groups. The authors prove that in such classes of groups as hypercentral groups (so also, nilpotent and abelian groups) and FC-groups, minimal artinian linear groups have precisely the same structure as the corresponding irreducible linear groups.

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Let F be a field, A a vector space over F. The group $\mathrm{GL}(F,A)$ of all automorphisms of A and its distinct subgroups are the oldest subjects of investigation in Group Theory. For the case when A has a finite dimension over F, every element of $\mathrm{GL}(F,A)$ defines some nonsingular $n\times n$-matrix over F, where $n = \dim_F A$. Thus, for the finite-dimensional case, the theory of linear groups is exactly the theory of matrix groups. That is why the theory of finite-dimensional linear groups is one of the best developed in algebra. However, for the case when $\dim_F A$ is infinite, the situation is totally different. The study of this case always requires some essential additional restrictions. Thus, the transition from the study of finite groups to the study of infinite groups generated the finiteness conditions. It is natural to apply these finiteness conditions to the study of infinite-dimensional linear groups. The study of finitary linear groups (the linear analogies of FC-groups) shows the effectiveness of such approach (cf. a survey of Phillips [6]).

The groups having a finite composition series were one of the first generalization of the finite groups. Let $G \le \mathrm{GL}(F,A)$, then we can consider A as an FG-module. We say that A has a finite composition length if A has a finite series $\langle 0\rangle = B_0 \le B_1 \le \cdots \le B_n = A$ of FG-submodules, every factor of which is a simple FG-module. We can consider $G/C_G(B_{i+1}/B_i)$ as an irreducible linear group, $0 \le i \le n-1$. Let $T = \bigcap_{0\le i\le n-1} C_G(B_{i+1}/B_i)$; then $G/T \le \mathsf{X}_{0\le i\le n-1} G/C_G(B_{i+1}/B_i)$, and T is a nilpotent bounded p-subgroup whenever $\operatorname{char} F = p$, or T is a nilpotent divisible torsion-free subgroup whenever $\operatorname{char} F = 0$. Thus, the case of irreducible linear groups is basic. Irreducible linear groups as the automorphism groups of abelian chief factors, play a crucial role in Group Theory, and their investigation is very useful for the solution of many group theoretical problems. For the infinite-dimensional case, the irreducible groups under some natural restrictions have been studied by Hartley and McDougall [2], Zaĭcev [13], Robinson and Zhang [9], Franciosi, de Giovanni, and Kurdachenko [1], and Kurdachenko and Subbotin [5].

The minimal and the maximal conditions were the very next classical finiteness conditions that have appeared in algebra. Note that every FG-module of finite composition length is artinian (i.e., it satisfies the minimal condition on FG-submodules) and noetherian (i.e., it satisfies the maximal condition on FG-submodules).

Let R be a ring, A an artinian R-module. Put

$$\mathfrak{S}_{\text{icl}}(A) = \{B \mid B \text{ is an } R\text{-submodule of } A \text{ and has no finite composition series}\}. \quad (1)$$

If A has no finite composition series, then $\mathfrak{S}_{\text{icl}}(A) \neq \varnothing$. Since A is artinian, $\mathfrak{S}_{\text{icl}}(A)$ has a minimal element M. Thus, if U is a proper R-submodule of M, then U has a finite composition length.

An R-module M is said to be a minimal artinian, if M has no finite composition series, but each of its proper submodule has a finite composition length.

Thus every artinian module includes a minimal artinian submodule. On the other hand, the structure of artinian modules depends on the structure of its minimal artinian submodules, therefore, the study of minimal artinian modules is one of the important steps for the study of artinian modules.

Let again F be a field, A a vector space over F, $G \le \mathrm{GL}(F,A)$. We want to consider the situation when A is a minimal artinian FG-module. This consideration will lead us to the fact that the group G is lying in the class $\mathfrak{X}$ such that all irreducible $\mathfrak{X}$-groups have been described. So we may set that if an FG-module B has finite composition series, then the structure of G is defined.

Let F be a field, A a vector space over F, $G \le \mathrm{GL}(A)$. A group G is called a minimal artinian if the following conditions hold:

(MA1) A has no finite composition series;

(MA2) if B is a proper FG-submodule of A, then B has a finite composition length.

The study of minimal artinian FG-modules (as any FG-module) consists of two parts: the study of internal structure of the module and the study of the group $G/C_G(A)$. The last group is imbedded in $\mathrm{GL}(F,A)$, that is, it is a linear minimal artinian group. Our paper is devoted to the study of some important types of minimal artinian linear groups. The main results of this paper show that in such classes of groups as hypercentral groups (so also, nilpotent and abelian groups) or FC-groups the minimal artinian linear groups have precisely the same structure as the corresponding irreducible linear groups have.

Now we mention some needed results on hypercentral irreducible groups. The irreducible $\mathbb{Z}G$-modules have been studied in [5]. These results can be extended almost without changes on the case of irreducible subgroups of $\mathrm{GL}(F,A)$, where A is a vector space over a field F.

Lemma 1. *Let F be a field, G a group, A a simple FG-module, $I = \mathrm{Ann}_{FG}(A)$. If C/I is a center of FG/I, then C/I is an integral domain. In particular, the periodic part of $\zeta(G/C_G(A))$ is a locally cyclic p'-subgroup where $p = \mathrm{char}\,F$.*

As usual, $0'$ denotes the set of all primes.

This statement is an immediate corollary of the known theorem of I. Schur.

A group G is said to have finite 0-rank $r_0(G) = r$ (or finite torsion-free rank) if G has a finite subnormal series with exactly r infinite cyclic factors being the others periodic.

We note that every refinement of each of these series has only r infinite cyclic factors. Since every two subnormal series have the isomorphic refinements, 0-rank is independent of the choice of the subnormal series.

Note also that if G is a locally nilpotent group of finite 0-rank, then the factor-group $G/t(G)$ by the periodic part $t(G)$ has a finite special rank.

Lemma 2. *Let G be a hypercentral group of finite 0-rank, F a locally finite field, A a simple FG-module. Then $\zeta(G/C_G(A))$ is periodic.*

This lemma follows from [4, Theorem 2].

Lemma 3. *Let F be a field, $p = \operatorname{char} F$, G an abelian group of finite 0-rank.*

(1) *If the field F is locally finite, and G is a locally cyclic p'-group, then there exists a simple FG-module A such that $C_G(A) = \langle 1 \rangle$.*

(2) *If F is not locally finite, and $t(G)$ is a locally cyclic p'-group, then there exists a simple FG-module A such that $C_G(A) = \langle 1 \rangle$.*

This construction is contained in [2].

Lemma 4. *Let F be a field, $p = \operatorname{char} F$, G an abelian group of infinite 0-rank. If $t(G)$ is a locally cyclic p'-group, then there exists a simple FG-module A such that $C_G(A) = \langle 1 \rangle$.*

This assertion has been proved in [9] for the case of finite field, however it is valid also for an arbitrary field.

Lemma 5. *Let F be a field, $p = \operatorname{char} F$, G a hypercentral group of finite 0-rank, $C = \zeta(G)$, $T = t(C)$.*

(1) *If the field F is locally finite, and $C = T$ is a locally cyclic p'-group, then there exists a simple FG-module A such that $C_G(A) = \langle 1 \rangle$.*

(2) *If F is not locally finite and T is a locally cyclic p'-group, then there exists a simple FG-module A such that $C_G(A) = \langle 1 \rangle$.*

Lemma 6. *Let F be a field, $p = \operatorname{char} F$, G a hypercentral group of infinite 0-rank, $C = \zeta(G)$, $T = t(C)$. If T is a locally cyclic p'-group, then there exists a simple FG-module A such that $C_G(A) = \langle 1 \rangle$.*

The proof of both these assertions is similar to the proof of the respective results of [5].

Lemma 7. *Let R be a ring, A a minimal artinian R-module. Then A does not decompose into a direct sum of two proper R-submodules.*

The lemma is obvious.

If A is an R-module, then let $\operatorname{Soc}_R(A)$ denotes the sum of all minimal R-submodules whenever A includes such submodules, and $\operatorname{Soc}_R(A) = \langle 0 \rangle$ otherwise.

Clearly, $\operatorname{Soc}_R(A)$ is a direct sum of some minimal R-submodules (if it is nonzero). If A is an artinian R-module, then $\operatorname{Soc}_R(A) \neq \langle 0 \rangle$ and $\operatorname{Soc}_R(A)$ is a direct sum of finitely many minimal R-submodules. So we come to the following lemma.

Lemma 8. *Let R be a ring, A a minimal artinian R-module. Then $\operatorname{Soc}_R(A)$ is a nonzero proper submodule of A.*

Lemma 9. *Let F be a field, G a group, H a normal subgroup having a finite index in G, A an FG-module. If A has finite composition length as an FG-module, then A has finite composition length as an FH-module.*

Proof. Let

$$\langle 0\rangle = B_0 \leq B_1 \leq \cdots \leq B_n = A \tag{2}$$

be a series of FG-submodules with simple FG-factors. Then B_{i+1}/B_i is a direct sum of finitely many simple FH-submodules [12], $0 \leq i \leq n-1$. Thus A has a finite series of FH-submodules with simple factors. □

Proposition 10. *Let F be a field, G a group, A a minimal artinian FG-module such that $C_G(A) = \langle 1\rangle$, H a normal subgroup having in G finite index, X a transversal to H in G. Then*

(1) *A includes a minimal artinian FH-submodule B;*
(2) *$A = \sum_{x\in X} Bx$;*
(3) *$\bigcap_{x\in X} x^{-1}C_H(B)x = \langle 1\rangle$, in particular, $H \leq \mathsf{X}_{x\in X} H/(x^{-1}C_H(B)x)$.*

Proof. By Wilson's theorem [11] A is an artinian FH-module. Since A has no finite composition series as FG-module, then A has no finite composition series as FH-module by Lemma 9. Let

$$\mathfrak{S} = \{U \mid U \text{ is an } FH\text{-submodule of } A \text{ and has no finite composition series}\}. \tag{3}$$

Since $A \in \mathfrak{S}$, $\mathfrak{S} \neq \emptyset$. Then $\mathfrak{S}$ has a minimal element B. This means that B is minimal artinian FH-submodule. The sum $C = \sum_{x\in X} Bx$ is an FG-submodule. If we suppose that C is a proper FG-submodule of A, then it has a finite composition length. By Lemma 9 it has also a finite composition length as an FH-module, which contradicts the choice of B. This contradiction proves the equality $A = \sum_{x\in X} Bx$. Since $C_H(Bx) = x^{-1}C_H(B)x$, then it follows that $\bigcap_{x\in X} x^{-1}C_H(B)x \leq C_H(A) = \langle 1\rangle$. By Remak's theorem, $H \leq \mathsf{X}_{x\in X} H/(x^{-1}C_H(B)x)$. □

Lemma 11. *Let F be a field, G a group, A a minimal artinian FG-module such that $C_G(A) = \langle 1\rangle$. If $1 \neq x \in \zeta(G)$, then $A = A(x-1)$.*

Proof. The mapping $\varphi : a \to a(x-1)$, $a \in A$, is an FG-endomorphism of A. In particular, $\operatorname{Im}\varphi = A(x-1)$ and $\operatorname{Ker}\varphi = C_A(x)$ are the FG-submodules of A. Since $x \in C_G(A)$, then $C_A(x) \neq A$. By $A(x-1) \cong A/C_A(x)$, we obtain that $A(x-1)$ has no finite composition length. It follows that $A(x-1) = A$. □

Corollary 12. *Let F be a field, A a vector space over F, G a minimal artinian subgroup of $\mathrm{GL}(F,A)$. Suppose that G is hypercentral. If $\operatorname{char} F = p > 0$, then G does not contain p-elements.*

Proof. Denote by P the Sylow p-subgroup of G, and suppose that $P \neq \langle 1\rangle$. Since G is a hypercentral group, $P \cap \zeta(G) \neq \langle 1\rangle$. Let $1 \neq z \in \zeta(G) \cap P$. Since the additive group of A is an elementary abelian p-group, a natural semidirect product $B \leftthreetimes \langle z\rangle$ is a nilpotent group (cf. [8, Lemma 6.34]). Therefore $[A\langle z\rangle, A\langle z\rangle] = A(z-1) \neq A$, which contradicts Lemma 11. This contradiction shows that $P = \langle 1\rangle$. □

Let G be a group. Put

$$FC(G) = \{x \in G \mid x^G = \{g^{-1}xg \mid g \in G\} \text{ is finite}\}. \tag{4}$$

That is, $FC(G)$ is a characteristic subgroup of G. This subgroup is called the FC-center of G.

Furthermore, the set T of all elements of finite order is a (characteristic) subgroup of $FC(G)$ and $FC(G)/T$ is an abelian torsion-free group (cf. [7, Theorem 4.32]).

Let G be a group, π a set of primes. Denote by $O_\pi(G)$ the maximal normal π-subgroup of G. In particular, if p is prime, then $O_p(G)$ denotes the maximal normal p-subgroup of G, and $O_{p'}(G)$ denotes the maximal periodic subgroup, which does not contain the p-elements.

Corollary 13. *Let F be a field, A a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$. *If* $\operatorname{char} F = p > 0$, *then* $O_p(FC(G)) = \langle 1\rangle$.

Proof. Suppose the contrary, let $1 \neq y \in O_p(FC(G))$. Put $Y = \langle y\rangle^G$. By Ditsmann's lemma (cf. [7, Corollary 2 to Lemma 2.14]), Y is a finite normal subgroup of G. Since Y is a finite p-subgroup, $\zeta(Y) = Z \neq \langle 1\rangle$. Let $H = C_G(Z)$, then H is a normal subgroup of finite index, and $Z \leq \zeta(H)$. By Proposition 10 A includes a minimal artinian FH-submodule B. Since the additive group of B is an elementary abelian p-group, the natural semidirect product $B \leftthreetimes \langle z\rangle$ is a nilpotent group for each element $z \in Z$ (cf. [8, Lemma 6.34]). Therefore $[B\langle z\rangle, B\langle z\rangle] = B(z-1) \neq B$. Corollary 12 yields that $z \in C_G(B)$. It is valid for every element $z \in Z$, therefore $Z \leq C_G(B)$. In turn $Z = x^{-1}Zx \leq x^{-1}C_G(B)x = C_G(Bx)$ for an arbitrary element $x \in G$. Since it is true for every element $x \in G$, $Z \leq \bigcap_{x\in G} C_G(Bx) = C_G(A)$, because $A = \sum_{x\in X} Bx$. But $C_G(A) = \langle 1\rangle$. This contradiction proves that $O_p(FC(G)) = \langle 1\rangle$. □

Corollary 14. *Let F be a field, A a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$. *If* $\operatorname{char} F = p > 0$, *then the locally soluble radical of $FC(G)$ has no p-elements.*

Lemma 15. *Let F be a field,* $\operatorname{char} F = p$, *$A$ a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$. *If H is a nonidentity finite normal p'-subgroup of G, then* $\mathrm{Soc}_{FH}(A) = A$.

Proof. For every element $0 \neq a \in A$, an FH-submodule aFH is finite-dimensional. In particular, it includes a simple FH-submodule. This means that $\mathrm{Soc}_{FH}(A) \neq \langle 0\rangle$. By Maschke's theorem (cf. [10, Theorem 1.5]), $\mathrm{Soc}_{FH}(A) = A$. □

If R is a ring, G a group, then ωRG denotes the augmentation ideal of the group ring RG.

Corollary 16. *Let F be a field,* $\operatorname{char} F = p$, *$A$ a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$. *If H is a nonidentity finite normal p'-subgroup of G, then $C_A(H) = \langle 0\rangle$, $A(\omega FH) = A$.*

Proof. By Lemma 15, $A = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where M_λ is a simple FH-submodule, $\lambda \in \Lambda$. Since $M_\lambda(\omega FH)$ is an FH-submodule of M_λ, then either $M_\lambda(\omega FH) = M_\lambda$ or $M_\lambda(\omega FH) = \langle 0\rangle$. It implies the equality $A = C_A(H) \oplus A(\omega FH)$. Since H is a normal subgroup of

G, both $C_A(H)$ and $A(\omega FH)$ are FG-submodules. Lemma 7 yields that $C_A(H) = \langle 0\rangle$ and $A = A(\omega FH)$. □

Corollary 17. *Let F be a field,* $\operatorname{char} F = p$*, A a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$*. If H is a nonidentity finite normal p'-subgroup of G and B is a nonzero FG-submodule of A, then $C_H(B) = \langle 1\rangle$.*

Proof. In fact, if $H_1 = C_H(B) \neq \langle 1\rangle$, then H_1 is a nonidentity finite normal p'-subgroup of G. Since $B \le C_A(H_1)$, we obtain a contradiction with Corollary 12. □

Corollary 18. *Let F be a field,* $\operatorname{char} F = p$*, A a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$*. Furthermore, let H be a nonidentity normal p'-subgroup having an ascending series of G-invariant subgroups*

$$\langle 1\rangle = H_0 \le H_1 \le \cdots \le H_\alpha \le H_{\alpha+1} \le \cdots \le H_\gamma = H \tag{5}$$

with finite factors. If B is a nonzero proper FG-submodule of A, then $C_H(B) = \langle 1\rangle$.

Proof. We use induction on α. If $\alpha = 1$, then the assertion follows from Corollary 13. Let $\alpha > 1$, and we have already proved that $C_{H_\beta}(B) = \langle 1\rangle$ for all $\beta < \alpha$.

Let $C_\alpha = C_{H_\alpha}(B)$. If α is a limit ordinal, then $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, and therefore $C_\alpha = \bigcup_{\beta<\alpha}(C_\alpha \cap H_\beta)$. But $C_\alpha \cap H_\beta = C_{H_\beta}(B) = \langle 1\rangle$. Thus $C_\alpha = \langle 1\rangle$.

Suppose now that α is not a limit, and put $L = H_{\alpha-1}$. Assume that $C_\alpha \neq \langle 1\rangle$. Then $C_\alpha \cap L = C_L(B) = \langle 1\rangle$, so that $C_\alpha \cong C_\alpha/(C_\alpha \cap L) \cong C_\alpha L/L \le H_\alpha/L$. It follows that C_α is a finite normal subgroup of G. And we obtain a contradiction with Corollary 12 because $B \le C_A(C_\alpha)$. Hence $C_{H_\alpha}(B) = \langle 1\rangle$. For $\alpha = \gamma$ we obtain that $C_H(B) = \langle 1\rangle$. □

Let G be a group. A normal subgroup H is called the hyperfinite radical of G if H satisfies the following conditions:

(1) H possesses an ascending series of G-invariant subgroups

$$\langle 1\rangle = H_0 \le H_1 \le \cdots \le H_\alpha \le H_{\alpha+1} \le \cdots \le H_\gamma = H, \tag{6}$$

every factor of which is finite;

(2) G/H has no nonidentity finite normal subgroups.

We will denote the hyperfinite radical of G by $HF(G)$.

Let $\mathrm{Soc}(G) = \mathsf{X}_{\lambda\in\Lambda} S_\lambda$, where S_α is a minimal normal subgroup of G, $\lambda \in \Lambda$. Put

$$\Lambda_{ab} = \{\lambda \in \Lambda \mid S_\lambda \text{ is abelian}\}, \qquad \mathrm{Soc}_{ab}(G) = \mathsf{X}_{\lambda\in\Lambda_{ab}} S_\lambda. \tag{7}$$

Corollary 19. *Let F be a field,* $\operatorname{char} F = p$*, A a vector space over F, G a minimal artinian subgroup of* $\mathrm{GL}(F,A)$*. Let $S = \mathrm{Soc}_{ab}(G) \cap HF(G)$. Then S is a p'-subgroup including a subgroup Q such that S/Q is a locally cyclic group and* $\mathrm{Core}_G(Q) = \langle 1\rangle$*.*

Proof. Clearly S is a subgroup of the locally soluble radical of $FC(G)$. By Corollary 17 of Lemma 11, S is a p'-subgroup. Let B be a minimal FG-submodule of A. By Corollary 18, $C_S(B) = \langle 1\rangle$. In other words, S is imbedded in an irreducible subgroup of $\mathrm{GL}(F,B)$. And now we can use [1, Lemma 8.2]. □

Now we can expose the main results.

Theorem 20. *Let F be a field, A a vector space over F, G a minimal artinian subgroup of $\mathrm{GL}(F,A)$. If G is an FC-group, then $\mathrm{Soc}_{ab}(G)$ is a p'-subgroup including a subgroup Q such that $\mathrm{Soc}_{ab}(G)/Q$ is a locally cyclic group and $\mathrm{Core}_G(Q) = \langle 1\rangle$, where $p = \mathrm{char}\, F$.*

Proof. Let T be the periodic part of G, S the locally soluble radical of G. For every element $x \in T$, the subgroup $\langle x\rangle^G$ is finite by Ditsmann's lemma (cf. [7, Corollary 2 to Lemma 2.14]). This implies the inclusion $T \le HF(G)$. In particular, $\mathrm{Soc}_{ab}(G) \le HF(G)$. Now we can use Corollary 19 of Lemma 15. □

The results of [3] imply that for the group G having the structure, described in Theorem 20, there is a simple FG-module A such that $C_G(A) = \langle 1\rangle$. This means that this theorem cannot be strengthened. Thus, minimal artinian linear FC-groups have the same structure as irreducible linear FC-groups.

Theorem 21. *Let F be a field, A a vector space over F, G a minimal artinian subgroup of $\mathrm{GL}(F,A)$. If G is a hypercentral, then $t(\zeta(G))$ is a locally cyclic p'-subgroup, where $p = \mathrm{char}\, F$.*

Proof. By Corollary 12 of Lemma 11, the periodic part T of the group G is a p'-subgroup. Since G is a hypercentral group, $T = HF(G)$. Choose a minimal FG-submodule B of A. By Corollary 18 of Lemma 15, $T \cap C_G(B) = \langle 1\rangle$, that is, $T \cong TC_G(B)/C_G(B)$. In other words, T is imbedded in an irreducible subgroup of $\mathrm{GL}(F,B)$. Now we can use Lemma 1. □

Corollary 22. *Let F be a field, A a vector space over F, G a minimal artinian subgroup of $\mathrm{GL}(F,A)$. If G is abelian, then $t(G)$ is a locally cyclic p'-subgroup, where $p = \mathrm{char}\, F$.*

Lemmas 3, 4, 5, and 6 show that, for the group G having the structure described in Theorem 21 (and in its corollary), there is a simple FG-module A such that $C_G(A) = \langle 1\rangle$. This means that this theorem (and its corollary) cannot be strengthened. Thus, minimal artinian linear hypercentral (and abelian) groups have the same structure as irreducible linear hypercentral (abelian) groups.

In connection with Lemma 2 and Theorem 21, there arises the following question: let F be a locally finite field, G a hypercentral group of finite 0-rank. Let G be a minimal artinian subgroup of $\mathrm{GL}(F,A)$. Can we claim $\zeta(G)$ to be periodic? The following simple example gives a negative answer to it.

Let F be a field, A a vector space over F of countable dimension, $\{a_n \mid n \in \mathbb{N}\}$ a basis of A, $\langle x\rangle$ an infinite cyclic group. Define the action of x on A by the rule

$$a_1x = a_1, \quad a_{n+1}x = a_{n+1} + a_n, \quad \text{or} \quad a_1(x-1) = 0, \quad a_{n+1}(x-1) = a_n, \quad n \in \mathbb{N}. \tag{8}$$

Then we can consider A as an $F\langle x\rangle$-module. It is easy to see that $A = A(x-1)$ and every proper $F\langle x\rangle$-submodule of A coincides with some $a_1F + \cdots + a_nF$, $n \in \mathbb{N}$. In particular, the $F\langle x\rangle$-module A is minimal artinian and $C_{\langle x\rangle}(A) = \langle 1\rangle$.

Also it shows that the question about the internal structure of minimal artinian modules requires separate consideration.

References

[1] S. Franciosi, F. de Giovanni, and L. A. Kurdachenko, *Groups whose proper quotients are FC-groups*, J. Algebra **186** (1996), no. 2, 544–577. MR 97k:20061. Zbl 866.20030.

[2] B. Hartley and D. McDougall, *Injective modules and soluble groups satisfying the minimal condition for normal subgroups*, Bull. Austral. Math. Soc. **4** (1971), 113–135. MR 43#2067. Zbl 206.03101.

[3] L. A. Kurdachenko and J. Otal, *Simple modules over CC-groups and monolithic just non-CC-groups*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **4** (2001), no. 2, 381–390. CMP 1 831 995.

[4] L. A. Kurdachenko, B. V. Petrenko, and I. Y. Subbotin, *On generalized hypercenters in Artinian modules*, Comm. Algebra **25** (1997), no. 4, 1023–1046. MR 98c:20066. Zbl 877.20026.

[5] L. A. Kurdachenko and I. Y. Subbotin, *Groups whose proper quotients are hypercentral*, J. Austral. Math. Soc. Ser. A **65** (1998), no. 2, 224–237. MR 99h:20060. Zbl 924.20030.

[6] R. E. Phillips, *Finitary linear groups: a survey*, Finite and Locally Finite Groups (Istanbul, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 471, Kluwer, Dordrecht, 1995, pp. 111–146. MR 96m:20080. Zbl 840.20048.

[7] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups. Part 1*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 62, Springer-Verlag, New York, 1972. MR 48#11314. Zbl 243.20032.

[8] ———, *Finiteness Conditions and Generalized Soluble Groups. Part 2*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 63, Springer-Verlag, New York, 1972. MR 48#11315. Zbl 243.20033.

[9] D. J. S. Robinson and Z. R. Zhang, *Groups whose proper quotients have finite derived subgroups*, J. Algebra **118** (1988), no. 2, 346–368. MR 89m:20038. Zbl 658.20019.

[10] B. A. F. Wehrfritz, *Infinite Linear Groups. An Account of the Group-Theoretic Properties of Infinite Groups of Matrices*, Ergebnisse der Matematik und ihrer Grenzgebiete, vol. 76, Springer-Verlag, New York, 1973. MR 49#436. Zbl 261.20038.

[11] J. S. Wilson, *Some properties of groups inherited by normal subgroups of finite index*, Math. Z. **114** (1970), 19–21. MR 41#3577. Zbl 185.05303.

[12] D. Ī. Zaĭcev, *Infinitely irreducible normal subgroups*, Structures of Groups and Properties of their Subgroups, Inst. Kibernet. Akad. Nauk Ukrain. SSR, Kiev, 1978, pp. 17–38, 150–151 (Russian). MR 81d:20031.

[13] D. I. Zaĭcev, *Products of abelian groups*, Algebra i Logika **19** (1980), no. 2, 150–172, 250 (Russian). MR 82g:20063.

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