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ALGEBRAIC AND CATEGORICAL PROPERTIES OF r-IDEAL SYSTEMS

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ABSTRACT. The structures (G,r), where r is a system of ideals defined on a directed group G, play an important role in solving arithmetical problems. In this paper, we investigate how some properties of these systems are transferred in their cartesian products and their substructures. The results we obtain find an application in the study of categorical properties of these structures.

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1. Introduction and preliminaries. The theory of r-ideal systems defined on directed groups was firstly investigated by Lorenzen in 1939 (cf. [4]). Jaffard, in 1960 (cf. [1]), made a systematic study of these systems, which covers a large part of their properties, although the terminology he used was quite difficult, thus some of his results have been later rediscovered. These systems are important since a lot of arithmetical problems, such as the embedding of an integral domain into a greatest common divisor integral domain, the embedding of a po-group into a lattice-group, the investigation of Prüfer groups or Bezout domains, can be solved using their properties.

By an r-system of ideals in a directed po-group G we mean a map $X \mapsto X_r$ (X_r is called the r-ideal generated by X) from the set $B(G)$ of all lower bounded subsets X of G into the power set of G, which satisfies the following conditions:

(1) $X \subseteq X_r$,
(2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$,
(3) $\{a\}_r = a \cdot G^+ = (a)$ for all $a \in G$,
(4) $a \cdot X_r = (a \cdot X)_r$ for all $a \in G$.

An r-ideal is said to be finite if it is finitely generated, and said to be principal if it can be generated by one element. The set $\mathcal{I}_r(G)$ of the r-ideals of G, endowed with the multiplication

$$X_r \times_r Y_r = (X \cdot Y)_r = \left(X_r \cdot Y_r\right)_r, \tag{1.1}$$

is a commutative monoid, which contains the structure $(\mathcal{I}_r^f(G), \times_r)$, where $\mathcal{I}_r^f(G)$ is the set of finite r-ideals, as a submonoid. In the following, a directed group G endowed with an r-system of ideals will be denoted by (G,r). The structure (G,r) has the following properties:

(1) r-α total (respectively, finite) property if any (respectively, finite) r-ideal of G is principal.

(2) r-β total (respectively, finite) property if $(\mathcal{I}_r(G), \times_r)$ (respectively, $(\mathcal{I}_r^f(G), \times_r)$) is a group.

(3) r-γ total (respectively, finite) property if $(\mathcal{I}_r(G),\times_r)$ (respectively, $(\mathcal{I}_r^f(G),\times_r)$) is a cancellative monoid.

(4) r-δ total (respectively, finite) property if for any (respectively, finite) r-ideal X_r of G, the transporter $X_r : X_r = \{x \in G \mid x\cdot X_r \subseteq X_r\}$ is contained into G^+.

We mention that (G,r) has the r-δ total (respectively, finite) property if and only if for every $x,y\in G$ and $Z_r \in \mathcal{I}_r(G)$ (respectively, $Z_r \in \mathcal{I}_r^f(G)$) such that $x\cdot Z_r \subseteq y\cdot Z_r$, it follows that $y \le x$. Among all the r-systems defined on G, there exist two special ones, called the v-system and the t-system defined, respectively, by

$$X_v = \bigcap_{X\subseteq(x)} (x), \qquad X_t = \bigcup_{\substack{Y\subseteq X\\ Y \text{ finite}}} Y_v \tag{1.2}$$

for any $X \in B(G)$.

In the next section, we study how the above-mentioned properties of the structures (G_1,r_1) and (G_2,r_2) can be transferred into the cartesian product $G_1\times G_2$ and vice versa, considering that the directed group $G_1\times G_2$ is endowed with a system of ideals denoted by $r_1\otimes r_2$, (cf. [2]), where

$$X_{r_1\otimes r_2} = \left(p_1(X)\right)_{r_1} \times \left(p_2(X)\right)_{r_2} \tag{1.3}$$

for any $X \in B(G_1\times G_2)$.

In addition, we make a similar research for the structures (G,r) and (H,r'), where H is a directed subgroup of G and $X_{r'} = X_r \cap H$, for any $X \in B(H)$. The system r' will be mentioned as the restriction of r. Moreover, the results we derive find an application in the investigation of categorical properties. We recall some notions in order to specify the categorical approach we attempt.

A map $f : (G_1,r_1) \to (G_2,r_2)$ is called (r_1,r_2)-morphism if it is a group homomorphism and $f(X_{r_1}) \subseteq (f(X))_{r_2}$, for every $X \in B(G_1)$. The map

$$f^* : \mathcal{I}_{r_1}(G_1) \longrightarrow \mathcal{I}_{r_2}(G_2), \qquad f^*(X_{r_1}) = \left(f(X)\right)_{r_2}, \tag{1.4}$$

is a semigroup homomorphism and it will be mentioned as the map induced by f. We denote by K the category with objects (G,r) and morphisms the (r_1,r_2)-morphisms and by L the category with objects $(\mathcal{I}_r(G),\times_r)$ and morphisms the semigroup homomorphisms. In [2], we have studied limits in the category K and we have proved that the map $\mathcal{J} : K \to L$, with

$$\mathcal{J}(G,r) = (\mathcal{I}_r(G),\times_r), \qquad \mathcal{J}f = f^*, \tag{1.5}$$

for every object (G,r) and every morphism f of K, is a functor which preserves the products.

In Section 3, we continue the study of the categories K,L and of the functor $\mathcal{J}$, in what concerns the existence of limits and the ability of $\mathcal{J}$ to preserve or reflect them. Moreover, we define a proper subcategory L^* of L, which is equivalent to K. We finish by defining subcategories of K and L according to the properties their objects have and we investigate limits in them as well as their relation via the above-mentioned functor.

2. Special structures with r-ideal systems. This section is devoted to the investigation of the properties of ideal systems. We denote by $R(G)$ the set of all the r-systems defined on G and by $R_j(G)$ (respectively, $R_j^f(G)$), $j=\alpha,\beta,\gamma,\delta$, the subset of $R(G)$ which contains the r-systems having the $r-j$ total (respectively, finite) property, $j=\alpha,\beta,\gamma,\delta$. In the following, whenever we refer to a cartesian product $G=G_1\times G_2$, we consider it endowed with the $r_1\otimes r_2$-system, where $r_i\in R(G_i)$, $i=1,2$, and we denote by $p_i:G\to G_i$, $i=1,2$, the usual projection maps. Especially, we prove that the properties a cartesian product $G_1\times G_2$ possesses are determined by the properties its factors have and vice versa.

Proposition 2.1 (see [3]). *Consider the structures (G_1,r_1) and (G_2,r_2). If G is the cartesian product $G_1\times G_2$, then*

$$\begin{aligned}(\mathcal{I}_{r_1}(G_1),\times_{r_1})\times(\mathcal{I}_{r_2}(G_2),\times_{r_2})&\cong(\mathcal{I}_{r_1\otimes r_2}(G),\times_{r_1\otimes r_2}),\\(\mathcal{I}^f_{r_1}(G_1),\times_{r_1})\times(\mathcal{I}^f_{r_2}(G_2),\times_{r_2})&\cong(\mathcal{I}^f_{r_1\otimes r_2}(G),\times_{r_1\otimes r_2}).\end{aligned}\tag{2.1}$$

Proof. Since [3] has not yet been published, we mention that the isomorphism needed in the first congruence is defined by $f((X_1)_{r_1},(X_2)_{r_2})=(X_1\times X_2)_{r_1\otimes r_2}$, for every $X_i\in B(G_i)$, $i=1,2$, while the one needed in the second congruence is its restriction into $\mathcal{I}^f_{r_1}(G_1)\times\mathcal{I}^f_{r_2}(G_2)$. □

Proposition 2.2. *Consider the structures (G_1,r_1), (G_2,r_2), and $(G,r_1\otimes r_2)$, where $G=G_1\times G_2$. Then, $r_1\in R_j(G_1)$ and $r_2\in R_j(G_2)$ if and only if $r_1\otimes r_2\in R_j(G)$ for $j=\alpha,\beta,\gamma,\delta$, respectively.*

Proof. We distinguish the following cases:

(i) If $r_1\in R_\alpha(G_1)$ and $r_2\in R_\alpha(G_2)$, then for every $X\in B(G)$ the following hold

$$(p_1(X))_{r_1}=\{a_1\}_{r_1},\qquad(p_2(X))_{r_2}=\{a_2\}_{r_2},\tag{2.2}$$

where $a_i\in G_i$, $i=1,2$, since $p_i(X)\in B(G_i)$, $i=1,2$. Put $a=(a_1,a_2)$. Obviously, $X_{r_1\otimes r_2}=\{a\}_{r_1\otimes r_2}$, which means that $r_1\otimes r_2\in R_\alpha(G)$. Conversely, if $r_1\otimes r_2\in R_\alpha(G)$, then for every lower bounded subset X_1 of G_1, the set $X=X_1\times\{1_{G_2}\}$ is a lower bounded subset of G and there exists $(x_1,x_2)\in G$ such that

$$X_{r_1\otimes r_2}=(X_1)_{r_1}\times\{1_{G_2}\}_{r_2}=\{(x_1,x_2)\}_{r_1\otimes r_2}=\{x_1\}_{r_1}\times\{x_2\}_{r_2}.\tag{2.3}$$

Hence, $(X_1)_{r_1}=\{x_1\}_{r_1}$, $x_1\in G_1$, thus, the structure (G_1,r_1) has the r_1-α total property. In the same way, we prove that $r_2\in R_\alpha(G_2)$.

(ii) It results directly from Proposition 2.1 that $r_1\in R_j(G_1)$ and $r_2\in R_j(G_2)$ if and only if $r_1\otimes r_2\in R_j(G)$ for $j=\beta,\gamma$, respectively.

(iii) Suppose that $r_1\in R_\delta(G_1)$ and $r_2\in R_\delta(G_2)$. Let $Z\in B(G)$ and $x,y\in G$, $x=(x_1,x_2)$, $y=(y_1,y_2)$, with $x\cdot Z_{r_1\otimes r_2}\subseteq y\cdot Z_{r_1\otimes r_2}$. Then,

$$x_i\cdot(p_i(Z))_{r_i}\subseteq y_i\cdot(p_i(Z))_{r_i},\tag{2.4}$$

thus $y_i\le x_i$, for $i=1,2$. Hence, $y\le x$, which means that $r_1\otimes r_2\in R_\delta(G)$. Conversely, if $r_1\otimes r_2\in R_\delta(G)$, then for every $Z_1\in B(G_1)$ and $x_1,y_1\in G_1$, with $x_1\cdot(Z_1)_{r_1}\subseteq y_1\cdot(Z_1)_{r_1}$, we consider the lower bounded subset $Z = Z_1 \times \{1_{G_2}\}$ of G and we put $x = (x_1, 1_{G_2})$, $y = (y_1, 1_{G_2})$. Then,

$$\begin{aligned} x \cdot Z_{r_1 \otimes r_2} &= x \cdot \left((Z_1)_{r_1} \times \{1_{G_2}\}_{r_2}\right) = (x_1 \cdot Z_1)_{r_1} \times \{1_{G_2}\}_{r_2} \\ &\subseteq (y_1 \cdot Z_1)_{r_1} \times \{1_{G_2}\}_{r_2} = y \cdot Z_{r_1 \otimes r_2}. \end{aligned} \tag{2.5}$$

Hence, $y \le x$, thus $y_1 \le x_1$, that is, $r_1 \in R_\delta(G_1)$. In the same way, we prove that (G_2, r_2) has the r_2-δ total property. □

Proposition 2.3. *Consider the structures $(G_1, r_1), (G_2, r_2)$, and $(G, r_1 \otimes r_2)$, where $G = G_1 \times G_2$. Then, $r_1 \in R_j^f(G_1)$ and $r_2 \in R_j^f(G_2)$ if and only if $r_1 \otimes r_2 \in R_j^f(G)$ for $j = \alpha, \beta, \gamma, \delta$, respectively.*

Proof. We observe that if X is a finite subset of G, then $p_i(X)$ is a finite subset of G_i, $i = 1, 2$ and vice versa; if X_i is a finite subset of G_i, then we can always construct a finite subset X of G such that $p_i(X) = X_i$, for $i = 1, 2$. The result follows by arguing as in Proposition 2.2. □

We can now prove proportionate results concerning subgroups of a directed group.

Proposition 2.4. *Consider the structures $(G_1, r_1), (G_2, r_2)$, and $f, g : G_1 \to G_2$ are two (r_1, r_2)-morphisms. Put $E = \{a \in G_1 \mid f(a) = g(a)\}$ and r_1' the restriction of the r_1-system on E. If $r_1 \in R_\alpha(G_1)$ (respectively, $r_1 \in R_\alpha^f(G_1)$), then $r_1' \in R_\alpha(E)$ (respectively, $r_1' \in R_\alpha^f(E)$).*

Proof. Obviously, the set E is a directed subgroup of G_1, so the system r_1' is well defined. Let X be a lower bounded (respectively, finite) subset of E, that is, $f(X) = g(X)$ and $X_{r_1'} = X_{r_1} \cap E$, where $X_{r_1} = \{a\}_{r_1}$, $a \in G_1$. Then,

$$(f(X))_{r_2} = (g(X))_{r_2} \Longrightarrow f^*(X_{r_1}) = g^*(X_{r_1}) \Longrightarrow \{f(a)\}_{r_2} = \{g(a)\}_{r_2} \Longrightarrow f(a) = g(a), \tag{2.6}$$

thus $a \in E$ and $X_{r_1'} = \{a\}_{r_1'}$. Hence, $r_1' \in R_\alpha(E)$ (respectively, $r_1' \in R_\alpha^f(E)$). □

Proposition 2.5. *Consider the structure (G, r), H a directed subgroup of G and r' the restriction of r into H. If $r \in R_j(G)$, (respectively, $r \in R_j^f(G)$), then $r' \in R_j(H)$, (respectively, $r' \in R_j^f(H)$), for $j = \gamma, \delta$, respectively.*

Proof. We denote by $i : H \to G$ the injection map, which is obviously an (r', r)-morphism and let $i^* : \mathcal{I}_{r'}(H) \to \mathcal{I}_r(G)$ be the induced semigroup homomorphism. If (G, r) has the r-γ total property, then for every $X, Y, Z \in B(H)$ with $X_{r'} \times_{r'} Z_{r'} = Y_{r'} \times_{r'} Z_{r'}$, it follows that

$$i^*\left((XZ)_{r'}\right) = i^*\left((YZ)_{r'}\right) \Longrightarrow (XZ)_r = (YZ)_r \Longrightarrow X_r = Y_r \Longrightarrow X_{r'} = Y_{r'}. \tag{2.7}$$

Thus, the monoid $\mathcal{I}_{r'}(H)$ is cancellative. Suppose now that (G, r) has the r-δ total property and let $X \in B(H)$ and $a \in X_{r'} \mid X_{r'}$. Then

$$a \cdot X \subseteq a \cdot X_{r'} \subseteq X_{r'} \subseteq X_r, \tag{2.8}$$

and therefore, $(a \cdot X)_r \subseteq X_r$, that is, $a \in X_r \mid X_r \subseteq G^+$. Thus, $a \in H^+$, hence, $X_{r'} \mid X_{r'} \subseteq H^+$, which means that $r' \in R_\delta(H)$.

Similarly, we can prove that if $r \in R_j^f(G)$, then $r' \in R_j^f(H)$, for $j = \gamma, \delta$. □

In the previous propositions, we have used the notion of a semigroup homomorphism induced by an (r_1,r_2)-morphism. We shall prove that this kind of map does not include all semigroup homomorphisms from $\mathcal{I}_{r_1}(G_1)$ to $\mathcal{I}_{r_2}(G_2)$.

Proposition 2.6. *Consider the structures (G_1,r_1) and (G_2,r_2). There exist semigroup homomorphisms from $\mathcal{I}_{r_1}(G_1)$ to $\mathcal{I}_{r_2}(G_2)$, which are not induced by any (r_1,r_2)-morphism $f:G_1\to G_2$.*

Proof. We prove this proposition by giving an example. Let $(\mathbb{Z},+,\le)$ be the additive group of the integers endowed with the usual ordering. Consider the cartesian product $\mathbb{Z}\times\mathbb{Z}$, which becomes a partially ordered group with the componentwise ordering and addition. Put $G_1=(\mathbb{Z},+,\le)$, $G_2=(\mathbb{Z}\times\mathbb{Z},+,\le)$ and consider the structures $(G_1,t_1),(G_2,t_2)$, where t_1,t_2 are the t-systems defined on G_1,G_2, respectively. Then, from the definition of the t-system, it follows that $(X_i)_{t_i}=\{\wedge_{G_i}X_i\}_{t_i}$, for $X_i\in B(G_i)$, where $\wedge_{G_i}X_i$ is the infimum of X_i, $i=1,2$. Put

$$\bar{f}:\mathcal{I}_{t_2}(G_2)\longrightarrow\mathcal{I}_{t_1}(G_1),\qquad \bar{f}(X_{t_2})=\big(p_1(X)+p_2(X)\big)_{t_1},\tag{2.9}$$

where $p_i:G_2\to G_1$, $i=1,2$ are the usual projection maps. Let $X,Y\in B(G_2)$, with $X_{t_2}=Y_{t_2}$. Then, $\wedge_{G_2}X=\wedge_{G_2}Y=(x_1,x_2)$, where $x_i=\wedge_{G_1}p_i(X)=\wedge_{G_1}p_i(Y)$, $i=1,2$. Thus, $\wedge_{G_1}(p_1(X)+p_2(X))=\wedge_{G_1}p_1(X)+\wedge_{G_1}p_2(X)=x_1+x_2$ and similarly, $\wedge_{G_1}(p_1(Y)+p_2(Y))=x_1+x_2$. Hence,

$$\bar{f}(X_{t_2})=\big\{\wedge_{G_1}\big(p_1(X)+p_2(X)\big)\big\}_{t_1}=\{x_1+x_2\}_{t_1}=\big(p_1(Y)+p_2(Y)\big)_{t_1}=\bar{f}(Y_{t_2})\tag{2.10}$$

which means that the map $\bar{f}$ is well defined. Moreover, this map is a semigroup homomorphism, since for $X_{t_2},Y_{t_2}\in\mathcal{I}_{r_2}(G_2)$, there holds

$$\begin{aligned}\bar{f}(X_{t_2}\times_{t_2}Y_{t_2})&=\bar{f}\big((X+Y)_{t_2}\big)=\big(p_1(X+Y)+p_2(X+Y)\big)_{t_1}\\&=\big(p_1(X)+p_2(X)\big)_{t_1}\times_{t_1}\big(p_1(Y)+p_2(Y)\big)_{t_1}=\bar{f}(X_{t_2})\times_{t_1}\bar{f}(Y_{t_2}).\end{aligned}\tag{2.11}$$

Now suppose that the map $\bar{f}$ is induced by a (t_2,t_1)-morphism $f:G_2\to G_1$. Then, for every $x=(x_1,x_2)\in G_2$, there holds $\{f(x)\}_{t_1}=\bar{f}(\{x\}_{t_2})=\{x_1+x_2\}_{t_1}$. Thus, $f(x)=x_1+x_2$. It is obvious that the map f is a group homomorphism. In order to prove that it is not a (t_2,t_1)-morphism, it would be enough to find a lower bounded subset X of G_2, such that $f(X_{t_2})\not\subseteq(f(X))_{t_1}$. Put $X=\{(3,-2),(-3,2)\}$. Then, $X\in B(G_2)$ and

$$X_{t_2}=\{\wedge_{G_2}X\}_{t_2}=\{(-3,-2)\}_{t_2}=\{(a,b)\in G_2\mid a\ge -3,\ b\ge -2\}.\tag{2.12}$$

Moreover, $f(X)=\{1,-1\}$ and $(f(X))_{t_1}=\{\wedge_{G_1}f(X)\}_{t_1}=\{-1\}_{t_1}$. Since, $(-3,-2)$ belongs to X_{t_2}, it results that $-5\in f(X_{t_2})$. But $-5\notin(f(X))_{t_1}$, which means that $f(X_{t_2})\not\subseteq(f(X))_{t_1}$. Hence, the map f is not a (t_2,t_1)-morphism. □

3. Categorical properties. In [2], we have proved the existence of finite products and equalizers in the category K. More specifically, the product of the objects (G_1,r_1) and (G_2,r_2) is $((G_1\times G_2,r_1\otimes r_2),p_1,p_2)$, where p_1,p_2 are the projection maps and the equalizer of the (r_1,r_2)-morphisms $f,g:G_1\to G_2$ is $((E,r_1'),l)$, where $E=\{a\in G_1\mid f(a)=g(a)\}$, r_1' is the restriction of the r_1-system into E and $l:E\to G_1$ is the injection map.

Proposition 3.1. *The category K is complete.*

Proof. We can easily generalize the construction of finite products (cf. [2]) in order to verify that the product of an arbitrary family $(G_i, r_i)_{i\in I}$ of objects of K is the pair $((\prod_{i\in I} G_i, r), (p_i)_{i\in I})$, where $X_r = \prod_{i\in I}(p_i(X))_{r_i}$ and p_i, $i \in I$, the projection maps. $\square$

Corollary 3.2. *Consider the objects (G_i, r_i), $i = 1,2,3$, of K and the morphisms $f : G_1 \to G_3$, $g : G_2 \to G_3$. Their pullback is $((H, r'), p_1, p_2)$, where $H = \{(x_1, x_2) \in G_1 \times G_2 \mid f(x_1) = g(x_2)\}$, r' is the restriction of the $r_1 \otimes r_2$-system into H and $p_i : H \to G_i$, $i = 1,2$, the projection maps.*

Proof. The proof is obvious, from the form the products and the equalizers have in the category K. $\square$

Proposition 3.3. *The inclusion functor $L \to$ Sem does not reflect equalizers.*

Proof. Consider the objects $(G_1, t_1), (G_2, t_2)$ of the category K, as they have been defined in Proposition 2.6, the corresponding objects $(\mathcal{I}_{t_1}(G_1), \times_{t_1}), (\mathcal{I}_{t_2}(G_2), \times_{t_2})$ of L and the semigroup homomorphisms $\bar{f}, p_1^*$ from $(\mathcal{I}_{t_2}(G_2), \times_{t_2})$ to $(\mathcal{I}_{t_1}(G_1), \times_{t_1})$, with

$$\bar{f}(X_{t_2}) = (p_1(X) + p_2(X))_{t_1}, \qquad p_1^*(X_{t_2}) = (p_1(X))_{t_1}, \tag{3.1}$$

where p_i, $i = 1,2$, are the projection maps from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. Let (E, l) be the equalizer of $\bar{f}$ and p_1^* in the category Sem, that is,

$$E = \left\{X_{t_2} \in \mathcal{I}_{t_2}(G_2) \mid (p_1(X) + p_2(X))_{t_1} = (p_1(X))_{t_1}\right\}. \tag{3.2}$$

If the inclusion functor $L \to$ Sem reflects equalizers, then there exists an object $(G, r) \in K$, such that $E = (\mathcal{I}_r(G), \times_r)$. The group G is a subset of G_2, since for $g \in G$, it is $\{g\}_r \in \mathcal{I}_r(G) \subseteq \mathcal{I}_{t_2}(G_2)$. Moreover,

$$\{p_1(g)\}_{t_1} = p_1^*(\{g\}_{t_2}) = \bar{f}(\{g\}_{t_2}) = \{p_1(g) + p_2(g)\}_{t_1}, \tag{3.3}$$

thus, $p_1(g) = p_1(g) + p_2(g)$. Hence,

$$G \subseteq \{(x, y) \in G_2 \mid x = x + y\} = \{(x, 0) \in G_2 \mid x \in G_1\}. \tag{3.4}$$

Put $X = \{(-3,2), (4,0)\}$. Obviously, $X \in B(G_2)$, $p_1(X) + p_2(X) = \{-1, -3, 6, 4\}$ and $p_1(X) = \{-3, 4\}$. Then,

$$\wedge_{G_1}(p_1(X)) = \wedge_{G_1}(p_1(X) + p_2(X)) = -3, \tag{3.5}$$

which means that $\bar{f}(X_{t_2}) = p_1^*(X_{t_2})$. Thus, $X_{t_2} \in E$. Consider now $Y \in B(G)$, such that $Y_r = X_{t_2}$. Then $(-3,2) \in G$, since $(-3,2)$ belongs to X_{t_2}, which is absurd. So, the equalizer (E, l) is not an object of the category L and this completes the proof. $\square$

The previous proposition shows that in the category L equalizers do not exist, in general. It is then natural for one to define a proper subcategory of L, which should have more properties. We put L^* the subcategory of L, which has the same objects and as morphisms the semigroup homomorphisms induced by morphisms of the category K.

Proposition 3.4. *The functor $\mathcal{J} : K \to L^*$, with $\mathcal{J}(G,r) = (\mathcal{I}_r(G), \times_r)$ and $\mathcal{J}f = f^*$, is an equivalence.*

Proof. It is obvious that this functor is well defined. In order to prove the equivalence of the categories K and L^*, it is enough to observe that for every $f,g \in \mathrm{Hom}_K((G_1,r_1),(G_2,r_2))$ with $\mathcal{J}f = \mathcal{J}g$, it follows that

$$\mathcal{J}f(\{x\}_{r_1}) = \mathcal{J}g(\{x\}_{r_1}) \Longrightarrow \{f(x)\}_{r_2} = \{g(x)\}_{r_2} \Longrightarrow f(x) = g(x) \tag{3.6}$$

for any $x \in G_1$. □

Corollary 3.5. *The category L^* is complete and the functor $\mathcal{J} : K \to L^*$ preserves and reflects limits.*

In the following, we denote by K_j (respectively, K_j^f), the subcategory of K with objects (G,r) which have the $r-j$ total (respectively, finite) property and by L_j (respectively, L_j^f), the corresponding subcategories of L, for $j = \alpha,\beta,\gamma,\delta$. To avoid confusion, we symbolize the restriction of the functor $\mathcal{J} : K \to L$, into the subcategories K_j and K_j^f, by the same letter. We investigate the existence of limits in K_j and K_j^f as well as the proportionate results for the categories L_j and L_j^f.

Proposition 3.6. (1) *The categories K_j and K_j^f, $j = \alpha,\gamma,\delta$, are complete.*

(2) *The categories K_β and K_β^f have products.*

Proof. Since all these categories are subcategories of K, it is enough to check whether the limits existing in K are reflected in them by the inclusion functor or not. The answer is obvious from Propositions 2.2, 2.3, 2.4, and 2.5. □

It is obvious that the restriction of the functor $\mathcal{J} : K \to L$ into K_j and K_j^f, $j = \alpha,\beta,\gamma,\delta$, preserves the products. We prove that the functor $\mathcal{J} : K_\alpha \to L_\alpha$ also preserves equalizers and pullbacks.

Proposition 3.7. *The functor $\mathcal{J} : K_\alpha \to L_\alpha$ preserves limits.*

Proof. It is enough to prove that this functor preserves equalizers. Let $((E,r_1'),l)$ be the equalizer of $f,g \in \mathrm{Hom}_{K_\alpha}((G_1,r_1),(G_2,r_2))$. We prove that the equalizer of $\mathcal{J}f,\mathcal{J}g \in \mathrm{Hom}_{L_\alpha}((\mathcal{I}_{r_1}(G_1),\times_{r_1}),(\mathcal{I}_{r_2}(G_2),\times_{r_2}))$ is $((\mathcal{I}_{r_1'}(E),\times_{r_1'}),\mathcal{J}l)$. There holds $\mathcal{J}f \circ \mathcal{J}l = \mathcal{J}g \circ \mathcal{J}l$, since $\mathcal{J}$ is a functor. Let $(\mathcal{I}_r(G),\times_r)$ be another object of L_α and $h : (\mathcal{I}_r(G),\times_r) \to (\mathcal{I}_{r_1}(G_1),\times_{r_1})$ a morphism, such that $\mathcal{J}f \circ h = \mathcal{J}g \circ h$. Put

$$k : (\mathcal{I}_r(G),\times_r) \longrightarrow (\mathcal{I}_{r_1'}(E),\times_{r_1'}), \qquad k(\{x\}_r) = \{a\}_{r_1'}, \tag{3.7}$$

where $h(\{x\}_r) = \{a\}_{r_1}$. The map k is well defined, since for $\{x\}_r \in \mathcal{I}_r(G)$, there holds $(\mathcal{J}f \circ h)(\{x\}_r) = (\mathcal{J}g \circ h)(\{x\}_r)$, so $\{f(a)\}_{r_2} = \{g(a)\}_{r_2}$, and therefore, $f(a) = g(a)$. If $\{x\}_r,\{y\}_r \in \mathcal{I}_r(G)$, then $k(\{x\}_r) = \{a\}_{r_1'}$ and $k(\{y\}_r) = \{b\}_{r_1'}$, with $h(\{x\}_r) = \{a\}_{r_1}$ and $h(\{y\}_r) = \{b\}_{r_1}$. Thus,

$$k(\{x\}_r) \times_{r_1'} k(\{y\}_r) = \{a \cdot b\}_{r_1'}, \qquad k(\{x\}_r \times_r \{y\}_r) = k(\{x \cdot y\}_r) = \{c\}_{r_1'}, \tag{3.8}$$

where

$$\{c\}_{r_1} = h(\{x \cdot y\}_r) = h(\{x\}_r) \times_{r_1} h(\{y\}_r) = \{a \cdot b\}_{r_1}. \tag{3.9}$$

Hence, $\{a \cdot b\}_{r_1'} = \{c\}_{r_1} \cap E = \{c\}_{r_1'}$, which means that the map k is a semigroup homomorphism. Obviously, $\mathcal{J}l \circ k = h$. Moreover, the map k is unique, because if $m : (\mathcal{I}_r(G), \times_r) \to (\mathcal{I}_{r_1'}(E), \times_{r_1'})$ is another morphism such that $\mathcal{J}l \circ m = h$, then for $\{x\}_r \in \mathcal{I}_r(G)$, with $m(\{x\}_r) = \{b\}_{r_1'}$ and $k(\{x\}_r) = \{a\}_{r_1'}$, we have $\mathcal{J}l(\{b\}_{r_1'}) = \mathcal{J}l(\{a\}_{r_1'})$, that is $\{b\}_{r_1} = \{a\}_{r_1}$. Hence, $\{a\}_{r_1'} = \{b\}_{r_1'}$ and finally $m = k$. □

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