IJMMS 25:6 (2001) 411–415
PII. S0161171201001818
http://ijmms.hindawi.com

COMMON FIXED POINTS OF SET-VALUED MAPPINGS

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(Received 27 August 1996 and in revised form 8 July 1998)

Dedicated to late P. V. Lakshmaiah

ABSTRACT. The main purpose of this paper is to obtain a common fixed point for a pair of set-valued mappings of Greguš type condition. Our theorem extend Diviccaro et al. (1987), Guay et al. (1982), and Negoescu (1989).

2000 Mathematics Subject Classification. Primary 54H25, 47H10.

1. Introduction.

Greguš [4] proved the following result.

THEOREM 1.1. *Let C be a closed convex subset of a Banach space X. If T is a mapping of C into itself satisfying the inequality*

$$\|Tx-Ty\| \le a\|x-y\|+b\|x-Tx\|+c\|y-Ty\| \tag{1.1}$$

for all x,y in C, where $0<a<1$, $0\le c$, $0\le b$, and $a+b+c=1$, then T has a unique fixed point in C.

Mappings satisfying the inequality (1.1) with $a=1$ and $b=c=0$ is called nonexpansive and it was considered by Kirk [6], whereas the mapping with $a=0$, $b=c=1/2$ by Wong [13]. Recently, Fisher et al. [3], Diviccaro et al. [2], Mukherjee et al. [9], and Murthy et al. [10] generalized Theorem 1.1 in many ways. In this context, we prove a common fixed point theorem for set-valued mappings using Greguš type condition. Before presenting our main theorem we need the following definitions and lemma for our main theorem.

Let (X,d) be a metric space and $CB(X)$ be the class of nonempty closed bounded subsets of X. For any nonempty subsets A,B of X we define

$$\begin{gathered} D(A,B)=\inf\{d(a,b):a\in A,\ b\in B\},\\ H(A,B)=\max\{\sup\{D(a,B):a\in A\},\ \sup\{D(A,b):b\in B\}\}. \end{gathered} \tag{1.2}$$

The space $CB(X)$ is a metric space with respect to the above defined distance function H (see Kuratowski [7, page 214] and Berge [1, page 126]). Nadler [11] has defined the contraction mapping for set-valued mappings. A set-valued mapping $F:X\to CB(X)$ is said to be contraction if there exists a real number k, $0\le k<1$ such that $H(Fx,Fy)\le k.\ d(x,y)$, for all $x,y\in X$.

Throughout this paper $C(X)$ stands for a class of nonempty compact subset of X, $D(A,B)$ is the distance between two sets A and B.

The following Definitions 1.2, 1.3, 1.4, and 1.5 are given in [5].

Definition 1.2. An orbit for a set-valued mapping $F : X \to CB(X)$ at a point x_0 is a sequence $\{x_n\}$, where $x_n \in Fx_{n-1}$ for all n.

Definition 1.3. For two set-valued mappings S and $T : X \to CB(X)$, we define an orbit at a point $x_0 \in X$, if there exists a sequence $\{x_n\}$ where $x_n \in Sx_{n-1}$ or $x_n \in Tx_{n-1}$ depending on whether n is even or odd.

Definition 1.4. The metric space X is said to be x_0-jointly orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X.

Definition 1.5. Let $F : X \to CB(X)$ be continuous. Then the mapping $x \to d(x,Fx)$ is continuous for all $x \in X$.

Definition 1.6 [11]**.** If $A,B \in C(X)$ then for all $a \in A$, there exists a point $b \in B$ such that $d(a,b) \le H(A,B)$.

Lemma 1.7 [8]**.** *Suppose that ϕ is a mapping of $[0,\infty)$ into itself, which is nondecreasing, upper-semicontinuous and $\phi(t) < t$ for all $\phi(t) > 0$. Then $\lim_{n\to\infty} \phi^n(t) = 0$, where ϕ^n is the composition of ϕn times.*

2. Main result

Theorem 2.1. *Let S and T be mappings of a metric space X into $C(X)$ and let X be x_0-jointly orbitally complete for some $x_0 \in X$. Suppose that $p > 0$ and for all $x,y \in X$ satisfying:*

$$H^p(Sx,Ty) \le \phi\left(ad^p(x,y) + (1-a)\max\{D^p(x,Sx),D^p(y,Ty)\}\right), \tag{2.1}$$

where $a \in (0,1)$ and $\phi : [0,\infty) \to [0,\infty)$ is nondecreasing, upper-semicontinuous and $\phi(t) < t$ for all $t > 0$. Then S and T have a common fixed point in X.

Proof. Let $x_0 \in X$. For any $x_1 \in Sx_0$, then by Definition 1.6, there exists a point $x_2 \in Tx_1$ such that $d(x_1,x_2) \le H(Sx_0,Tx_1)$. The choice of the sequence $\{x_n\}$ in X guarantees that

$$x_n \in Sx_{n-1} \quad \text{if } n \text{ is even}, \qquad x_n \in Tx_{n-1} \quad \text{if } n \text{ is odd}. \tag{2.2}$$

Now, we claim that $d(x_1,x_2) \le d(x_0,x_1)$. Suppose $d(x_1,x_2) > d(x_0,x_1)$ and $\varepsilon = d(x_1,x_2)$. Then by using (2.1) it follows that

$$\begin{aligned} \varepsilon = d(x_1,x_2) &\le H(Sx_0,Tx_1) \\ &\le \left[\phi\left(ad^p(x_0,x_1) + (1-a)\max\{D^p(x_0,Sx_0),D^p(x_1,Tx_1)\}\right)\right]^{1/p} \\ &\le \left[\phi\left(a\varepsilon^p + (1-a)\varepsilon^p\right)\right]^{1/p} \\ &\le \left[\phi(\varepsilon^p)\right]^{1/p} < \varepsilon, \quad \text{a contradiction.} \end{aligned} \tag{2.3}$$

Therefore $d(x_1,x_2) \le d(x_0,x_1)$ and

$$\begin{aligned} d^p(x_1,x_2) &\le H^p(Sx_0,Tx_1) \\ &\le \phi\left(ad^p(x_0,x_1) + (1-a)\max\{D^p(x_0,Sx_0),D^p(x_1,Tx_1)\}\right) \\ &\le \phi\left(d^p(x_0,x_1)\right). \end{aligned} \tag{2.4}$$

Similarly, we have $d^p(x_2,x_3) \le \phi(d^p(x_1,x_2)) \le \phi^2(d^p(x_0,x_1))$.

Proceeding in this way, we have

$$d^p(x_n,x_{n+1}) \le \phi^n(d^p(x_0,x_1)) \quad \text{for } n = 0,1,2,\ldots. \tag{2.5}$$

By Lemma 1.7, it follows that $\lim_{n\to\infty} d^p(x_n,x_{n+1}) = 0$, that is,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.6}$$

In order to prove that $\{x_n\}$ is a Cauchy sequence, it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for a sequence of even integers $\{n(k)\}$ defined inductively with $n(1) = 2$ and $n(k+1)$ is the smallest even integer greater than $n(k)$ such that

$$d(x_{n(k+1)},x_{n(k)}) > \varepsilon. \tag{2.7}$$

So that

$$d(x_{n(k+1)-2},x_{n(k)}) \le \varepsilon. \tag{2.8}$$

It follows that

$$\begin{aligned} \varepsilon &< d(x_{n(k+1)},x_{n(k)}) \\ &\le d(x_{n(k+1)},x_{n(k+1)-1}) + d(x_{n(k+1)-1},x_{n(k+1)-2}) + d(x_{n(k+1)-2},x_{n(k)}) \end{aligned} \tag{2.9}$$

for $k = 1,2,3,\ldots$. Using (2.6) and (2.8) it follows that

$$\lim_{k\to\infty} d(x_{n(k+1)},x_{n(k)}) = \varepsilon. \tag{2.10}$$

By the triangle inequality, we have

$$\begin{aligned} \left| d(x_{n(k+1)},x_{n(k)}) - d(x_{n(k)},x_{n(k+1)-1}) \right| &\le d(x_{n(k+1)},x_{n(k+1)-1}), \\ \left| d(x_{n(k+1)-1},x_{n(k)+1}) - d(x_{n(k+1)},x_{n(k)}) \right| &\le d(x_{n(k+1)},x_{n(k+1)-1}). \end{aligned} \tag{2.11}$$

It follows from (2.6) and (2.10) that

$$\lim_{k\to\infty} d(x_{n(k)},x_{n(k+1)-1}) = \lim_{k\to\infty} d(x_{n(k+1)-1},x_{n(k)+1}) = \varepsilon. \tag{2.12}$$

Using (2.6), we have

$$\begin{aligned} D(x_{n(k+1)},x_{n(k)}) &\le d(x_{n(k+1)},x_{n(k)+1}) + d(x_{n(k)+1},x_{n(k)}) \\ &\le H(Sx_{n(k+1)-1},Tx_{n(k)}) + d(x_{n(k)+1},x_{n(k)}) \end{aligned} \tag{2.13}$$

and using (2.1), we have

$$\begin{aligned} &H^p(Sx_{n(k+1)-1},Tx_{n(k)}) \\ &\le \phi(ad^p(x_{n(k+1)-1},x_{n(k)}) + (1-a)\max\{D^p(x_{n(k+1)-1},Sx_{n(k+1)-1}),D^p(x_{n(k)},Tx_{n(k)})\}). \end{aligned} \tag{2.14}$$

Using (2.8), (2.10), (2.13), (2.14), and upper semi-continuity of ϕ it follows by letting $k \to \infty$ that

$$\varepsilon \le [\phi(a\varepsilon^p)]^{1/p} \le [\phi(\varepsilon^p)]^{1/p} < \varepsilon, \tag{2.15}$$

a contradiction. Therefore, $\{x_{2n}\}$ is a Cauchy sequence in X and since X is x_0-jointly orbitally complete metric space, so the sequence $\{x_n\}$ of each orbit at x_0 is convergent in X. Therefore there exists a point $z \in X$ such that $x_0 \to z$.

Then again using (2.1), we have

$$D^p(x_{2n-1}, Tz) \le H^p(Sx_{2n-2}, Tz) \le \phi\left(ad^p(x_{2n-2}, z) + (1-a)\max\{D^p(x_{2n-2}, Sx_{2n-2}), D^p(z, Tz)\}\right) \tag{2.16}$$

or equivalent to

$$D^p(x_{2n-1}, Tz) \le \phi\left(ad^p(x_{2n-2}, z) + (1-a)\max\{D^p(x_{2n-2}, Sx_{2n-2}), D^p(z, Tz)\}\right). \tag{2.17}$$

Now taking $n \to \infty$ in (2.17), then we have $D^p(z, Tz) \le \phi((1-a)D^p(z, Tz))$ if $z \notin Tz$, a contradiction. Thus $z \in Tz$.

Similarly, we show that $z \in Sz$. Hence, $z \in Sz \cap Tz$. This completes the proof. □

Open problem. What further restrictions are necessary for the convergence of the sequence $\{x_n\}$ if ϕ is dropped from (2.1)?

Acknowledgement. The authors would like to express their deep gratitude to the referee for helpful comments and suggestions to present the original paper into this form.

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