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POWER SUBGROUPS OF HECKE GROUPS $H(\sqrt{n})$

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ABSTRACT. Results in discrete group theory are applied to some Hecke groups to determine the group theoretical structure of power subgroups.

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1. Introduction. Hecke groups $H(\lambda)$ have been introduced by E. Hecke (see [2]). They are subgroups of $\mathrm{PSL}(2,\mathbb{R})$ generated by $R(z) = -1/z$ and $T(z) = z+\lambda$. Hecke asked the question, "For what values of λ these groups are discrete?" In answering this question he proved that

$$F_\lambda = \left\{ z \in U : |z| > 1, |\mathrm{Re}\, z| < \frac{\lambda}{2} \right\} \tag{1.1}$$

is a fundamental region for $H(\lambda)$ if and only if $\lambda \geq 2$ and real or $\lambda = \lambda_q = 2\cos(\pi/q)$, $q \in \mathbb{N}$, $q \geq 3$. Therefore, $H(\lambda)$ is discrete only for these values of λ. The most important and interesting Hecke group is the modular group $H(\lambda_3) = \mathrm{PSL}(2,\mathbb{Z})$. Next two interesting Hecke groups are obtained for $q = 4$ and $q = 6$. As $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$, $H(\sqrt{2})$ and $H(\sqrt{3})$ denote the Hecke groups corresponding to λ_4 and λ_6, respectively. One of the main reasons for $H(\sqrt{2})$ and $H(\sqrt{3})$ to be two of the most important Hecke groups is that apart from modular group, they are the only Hecke groups $H(\lambda_q)$ whose elements can be completely described. Here we deal with the cases $H(\sqrt{n})$, n square-free integer. $H(\sqrt{n})$ consists of the set of all matrices of the following types:

(i) $\begin{pmatrix} a & b\sqrt{n} \\ c\sqrt{n} & d \end{pmatrix}$; $a,b,c,d \in \mathbb{Z}$, $ad - nbc = 1$,

(ii) $\begin{pmatrix} a\sqrt{n} & b \\ c & d\sqrt{n} \end{pmatrix}$; $a,b,c,d \in \mathbb{Z}$, $nad - bc = 1$.

Those of type (i) are called even while those of type (ii) are called odd. Even elements form a subgroup of index 2 called the even subgroup [1].

Let $S = RT$ so that $S(z) = -1/(z+\lambda)$. In the cases $H(\sqrt{n})$, $n = 2,3$, S is an element of order $q = 2n$. Thus $R^2 = S^q = I$ and $RS = T$ is parabolic. It is known that $H(\sqrt{n})$ is isomorphic to the free product $C_2 * C_q$. Therefore $H(\sqrt{n})$ has the signature $(0;2,q,\infty)$, [1]. In the case $n > 3$ square-free integer, S is an element of infinite order and $H(\sqrt{n})$ is isomorphic to the free product $C_2 * \mathbb{Z}$, [6]. The signature of $H(\sqrt{n})$ is $(0;2,\infty;1)$. That is, all the groups $H(\sqrt{n})$, n square-free integer, are triangle groups containing a parabolic element. It is well known that a triangle group $(2,m,n)$ acts on the sphere, Euclidean plane or hyperbolic plane according to $1/m+1/n > 1/2$, $1/m+1/n = 1/2$, and $1/m+1/n < 1/2$, respectively, [3].

The purpose of this paper is to determine the structure of the groups $H^m(\sqrt{n})$ of the Hecke groups $H(\sqrt{n})$, n is a square-free integer. The groups $H^m(\sqrt{n})$ are defined to be the subgroups generated by the mth powers of all the elements of $H(\sqrt{n})$, for some positive integer m. $H^m(\sqrt{n})$ is called the mth power subgroup of $H(\sqrt{n})$. As fully invariant subgroups, they are normal in $H(\sqrt{n})$.

From the definition, one can easily deduce that

$$H^m(\sqrt{n}) > H^{mk}(\sqrt{n}), \tag{1.2}$$

and that

$$(H^m(\sqrt{n}))^k > H^{mk}(\sqrt{n}). \tag{1.3}$$

Using (1.2), it is easy to deduce that

$$H^m(\sqrt{n}) \cdot H^k(\sqrt{n}) = H^{(m,k)}(\sqrt{n}). \tag{1.4}$$

Here (m,k) denotes the greatest common divisor of m and k.

2. Structure of power subgroups.

We now discuss the group theoretical structure of these subgroups. First we have the following theorem.

Theorem 2.1. (i) *Let $n = 2$ or 3. The normal subgroup $H^2(\sqrt{n})$ is isomorphic to the free product of infinite cyclic group $\mathbb{Z}$ and two finite cyclic groups of order n. Also*

$$\begin{gathered} H(\sqrt{n})/H^2(\sqrt{n}) \cong C_2 \times C_2, \\ H(\sqrt{n}) = H^2(\sqrt{n}) \cup RH^2(\sqrt{n}) \cup SH^2(\sqrt{n}) \cup RSH^2(\sqrt{n}), \\ H^2(\sqrt{n}) = \langle S^2 \rangle * \langle RS^2R \rangle * \langle RSRS^{2n-1} \rangle. \end{gathered} \tag{2.1}$$

The elements of $H^2(\sqrt{n})$ are characterized by the property that the sums of the exponents of R and S are both even.

(ii) *Let $n > 3$ square-free integer. The normal subgroup $H^2(\sqrt{n})$ is the free product of three infinite cyclic groups.*

Also

$$\begin{gathered} H(\sqrt{n})/H^2(\sqrt{n}) \cong C_2 \times C_2, \\ H(\sqrt{n}) = H^2(\sqrt{n}) \cup RH^2(\sqrt{n}) \cup SH^2(\sqrt{n}) \cup RSH^2(\sqrt{n}), \\ H^2(\sqrt{n}) = \langle S^2 \rangle * \langle RS^2R \rangle * \langle RSRS^{-1} \rangle. \end{gathered} \tag{2.2}$$

The elements of $H^2(\sqrt{n})$ can be characterized by the requirement that the sums of the exponents of R and S are both even.

Proof. We use the Reidemeister-Schreier process to find a presentation of $H^2(\sqrt{n})$, [5]. We add the relation $X^2 = 1$ to the presentation of $H(\sqrt{n})$. This gives a presentation of $H(\sqrt{n})/H^2(\sqrt{n})$ the order of which is the index. We have

$$H(\sqrt{n})/H^2(\sqrt{n}) = \langle R, S; R^2 = S^2 = (RS)^2 = 1 \rangle = C_2 \times C_2. \tag{2.3}$$

Thus $|H(\sqrt{n}) : H^2(\sqrt{n})| = 4$. Now we choose $\{I, R, S, RS\}$ as a Schreier transversal for $H^2(\sqrt{n})$. Then we can form all possible products

$$\begin{gathered} S_{IR} = IRR^{-1} = I, \qquad S_{IS} = ISS^{-1} = I, \qquad S_{R^2} = RRI = I, \\ S_{RS} = RS(RS)^{-1} = I, \qquad S_{SR} = SR(RS)^{-1} = SRS^{-1}R, \\ S_{S^2} = SSI = S^2, \qquad S_{RSR} = RSR(S)^{-1} = RSRS^{-1}, \qquad S_{RS^2} = RS^2R. \end{gathered} \tag{2.4}$$

Since $(RSRS^{-1}) = SRS^{-1}R$, we get $x_1 = S^2$, $x_2 = RS^2R$, and $x_3 = RSRS^{-1}$ as the generators of $H^2(\sqrt{n})$. Clearly the elements of $H^2(\sqrt{n})$ satisfy the requirements of the theorem, that is, the sums of the exponents of R and S are both even for each element. Note that we have $S^{-1} = S^3$, $S^{-1} = S^5$ for $n = 2$, $n = 3$, respectively. Using the Reidemeister rewriting process, we get the relations

$$\begin{gathered} \tau(IRRI) = \tau(RR) = S_{IR} \cdot S_{R^2} = I, \\ \tau(RRRR) = S_{IR} \cdot S_{R^2} \cdot S_{IR} \cdot S_{R^2} = I, \\ \tau(SRRS^{-1}) = S_{IS} \cdot S_{SR} \cdot S_{RSR} \cdot S_{IS}^{-1} = ISRS^{-1}RRSRS^{-1} = I, \\ \tau(RSRRS^{-1}R) = S_{IR} \cdot S_{RS} \cdot S_{RSR} \cdot S_{SR} \cdot S_{RS}^{-1} \cdot S_{R^2} = IIRSRS^{-1}SRS^{-1}RII = I. \end{gathered} \tag{2.5}$$

Therefore there are no nontrivial relations and $H^2(\sqrt{n})$ is the free product of three infinite cyclic groups generated by x_1, x_2, and x_3. As each of R, S, and T goes to elements of order 2, they have the following permutation representations:

$$R \longrightarrow (1\ 2)(3\ 4), \qquad S \longrightarrow (1\ 3)(2\ 4), \qquad T \longrightarrow (1\ 4)(2\ 3). \tag{2.6}$$

By the permutation method (see [4, 7]), the signature of $H^2(\sqrt{2})$ is $(g; 2, 2, \infty, \infty) = (g; 2^{(2)}, \infty^{(2)})$ and the signature of $H^2(\sqrt{3})$ is $(g; 3^{(2)}, \infty^{(2)})$. Since the signature of all the Hecke groups $H(\sqrt{n})$, $n > 3$ square-free integer, is $(0; 2, \infty; 1)$, we find the signature of $H^2(\sqrt{n})$, $n > 3$ square-free integer, as $(g; \infty^{(2)}; 2)$. Now by the Riemann-Hurwitz formula, we have $g = 0$ in all cases. Hence $H^2(\sqrt{n})$, $n > 3$ square-free integer, is isomorphic to the free product of three $\mathbb{Z}$'s and $H^2(\sqrt{2})$ is isomorphic to the free product of $\mathbb{Z}$ and two finite cyclic groups of order 2 and $H^2(\sqrt{3})$ is isomorphic to the free product of $\mathbb{Z}$ and two finite cyclic groups of order 3. □

Theorem 2.2. *Let m be a positive odd integer. Then $H^m(\sqrt{2}) = H(\sqrt{2})$.*

Proof. Teh proof is clear as the quotient is trivial. □

Theorem 2.3. *Let m be a positive integer such that $m \equiv 2 \bmod 4$. Then $H^m(\sqrt{2})$ is the free product of the infinite cyclic group $\mathbb{Z}$ and m finite cyclic groups of order two.*

Proof. It is easy to show that the quotient group is isomorphic to the dihedral group D_m of order $2m$. The permutation representations of R, S, and T are

$$\begin{aligned} R &\longrightarrow (1\ 2)(3\ 4)\cdots(2m-1\ 2m), \\ S &\longrightarrow (2\ 3)(4\ 5)\cdots(2m\ 1), \\ T &\longrightarrow (1\ 3\ 5\cdots 2m-1)(2m\ 2m-2\cdots 4\ 2). \end{aligned} \tag{2.7}$$

Then $H^m(\sqrt{2})$ has signature $(0;2^{(m)},\infty,\infty)$, that is, $H^m(\sqrt{2})$ is the free product given in the statement of the theorem. If we denote the normal subgroup by $W_m(\sqrt{2})$, we have $W_m(\sqrt{2}) \cong \mathbb{Z} * \underbrace{C_2 * \cdots * C_2}_{m \text{ times}}$. □

We have already proved that

$$H^m(\sqrt{2}) = \begin{cases} H(\sqrt{2}) & \text{if } m \text{ is odd,} \\ W_m(\sqrt{2}) & \text{if } m \equiv 2 \mod 4. \end{cases} \tag{2.8}$$

Because of this we are only left to consider the case where m is a multiple of four. Now let $m = 4k$, $k \in \mathbb{N}$. Then in $H(\sqrt{2})/H^m(\sqrt{2})$ we have the relations $r^2 = s^4 = 1$, where r and s are the images of R and S, respectively, under the homomorphism of $H(\sqrt{2})$ to $H(\sqrt{2})/H^m(\sqrt{2})$. These relations imply that $H^m(\sqrt{2})$ is a free group.

Theorem 2.4. *The normal subgroup $H^3(\sqrt{3})$ is the free product of four cyclic groups of order 2. Also*

$$\begin{gathered} H(\sqrt{3})/H^3(\sqrt{3}) \cong C_3, \\ H(\sqrt{3}) = H^3(\sqrt{3}) \cup SH^3(\sqrt{3}) \cup S^2H^3(\sqrt{3}), \\ H^3(\sqrt{3}) = \langle R\rangle * \langle S^3\rangle * \langle SRS^5\rangle * \langle S^2RS^4\rangle. \end{gathered} \tag{2.9}$$

Proof. The proof is similar to that of Theorem 2.1. □

The following results are easy to see.

Theorem 2.5. *Let $m \equiv \pm 1 \mod 6$. Then $H^m(\sqrt{3}) = H(\sqrt{3})$.*

Theorem 2.6. *Let $m \equiv \pm 2 \mod 6$. Then $H^m(\sqrt{3}) = W_m(\sqrt{3})$.*

Theorem 2.7. *Let $m \equiv 3 \mod 6$. Then $H^m(\sqrt{3}) = H^3(\sqrt{3})$.*

Therefore the only case left is that when m is divisible by 6. A similar discussion will show that $H^m(\sqrt{3})$ is free in this case.

Theorem 2.8. *The normal subgroup $H^3(\sqrt{n})$, $n > 3$ square-free integer, is the free product of three cyclic groups of order 2 and an infinite cyclic group. Also*

$$\begin{gathered} H(\sqrt{n})/H^3(\sqrt{n}) \cong C_3, \\ H(\sqrt{n}) = H^3(\sqrt{n}) \cup SH^3(\sqrt{n}) \cup S^2H^3(\sqrt{n}), \\ H^3(\sqrt{n}) = \langle R\rangle * \langle S^3\rangle * \langle SRS^{-1}\rangle * \langle S^2RS^{-2}\rangle. \end{gathered} \tag{2.10}$$

Proof. If we add the relation $X^3 = 1$ to the presentation of $H(\sqrt{n})$ we have

$$H(\sqrt{n})/H^3(\sqrt{n}) = \langle R,S;R^2 = 1, X^3 = 1\rangle = \langle S;S^3 = 1\rangle \cong C_3. \tag{2.11}$$

Thus $|H(\sqrt{n}) : H^3(\sqrt{n})| = 3$. Let $\{I,S,S^2\}$ be a Schreier transversal for $H^3(\sqrt{n})$. Then all the possible products are

$$\begin{array}{lll} S_{IR} = IRI = R, & S_{IS} = ISS^{-1} = I, & S_{SR} = SRS^{-1}, \\ S_{S^2} = SSS^{-2} = I, & S_{S^2R} = S^2RS^{-2}, & S_{S^3} = S^3I = S^3. \end{array} \tag{2.12}$$

Therefore, $H^3(\sqrt{n})$ is generated by $x_1 = R$, $x_2 = S^3$, $x_3 = SRS^{-1}$, and $x_4 = S^2RS^{-2}$. Using the Reidemeister rewriting process, we get the relations

$$\begin{aligned}\tau(IRRI) &= \tau(RR) = S_{IR}\cdot S_{R^2} = R^2 = I,\\ \tau(SRRS^{-1}) &= S_{IS}\cdot S_{SR}\cdot S_{SR}\cdot S_{IS}^{-1} = ISRS^{-1}SRS^{-1}I = I,\\ \tau(SSRRS^{-1}S^{-1}) &= S_{IS}\cdot S_{S^2}\cdot S_{S^2R}\cdot S_{S^2R}\cdot S_{S^2}^{-1}\cdot S_{IS}^{-1} = IIS^2RS^{-2}S^2RS^{-2}II = I.\end{aligned} \tag{2.13}$$

The permutation representations of R, S, and T are

$$R \longrightarrow (1)(2)(3), \qquad S \longrightarrow (1\ 2\ 3), \qquad T \longrightarrow (1\ 2\ 3). \tag{2.14}$$

Then $H^3(\sqrt{n})$ has the signature $(0;2^{(3)},\infty;1)$, that is, $H^3(\sqrt{n})$ is the free product given in the statement of the theorem. □

Theorem 2.9. *Let m be a positive odd integer and $n > 3$ is a square-free integer. Then*

$$H^m(\sqrt{n}) \cong \mathbb{Z} * \underbrace{C_2 * \cdots * C_2}_{m\ \text{times}}. \tag{2.15}$$

Proof. Since $H(\sqrt{n})/H^m(\sqrt{n}) = \langle S; S^m = I\rangle \cong C_m$, the permutation representations of R, S, and T are

$$R \longrightarrow (1)(2)\cdots(m), \qquad S \longrightarrow (1\ 2\cdots m), \qquad T \longrightarrow (1\ 2\cdots m). \tag{2.16}$$

By the permutation method, we find the signature of $H^m(\sqrt{n})$ as $(0;2^{(m)},\infty;1)$. Therefore, $H^m(\sqrt{n})$ is isomorphic to the free product of m cyclic groups of order 2 and an infinite cyclic group. □

Let m be a positive even integer and $n > 3$ is a square-free integer. Then we have

$$H(\sqrt{n})/H^m(\sqrt{n}) = \langle R, S; R^2 = S^m = (RS)^m = I\rangle, \tag{2.17}$$

that is, the factor group is the group whose signature $(2,m,m)$. If $m = 2$, we have already seen that $H^2(\sqrt{n}) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ which is a normal subgroup of genus 0, then $H(\sqrt{n})/H^2(\sqrt{n})$ is a group of automorphisms of a sphere with two boundary components and two punctures. If $m = 4$, we have a normal subgroup acting on the Euclidean plane. Because, in this case the factor group $(2,4,4)$ is a group of infinite order and $1/4+1/4 = 1/2$. If $m \geq 6$ and even, the factor group $(2,m,m)$ is a group of infinite order and $1/m+1/m = 2/m < 1/2$. Therefore, in this case we have a normal subgroup acting on the hyperbolic 2-space (i.e., upper half plane).

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