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VARIATION OF CONSTANTS FORMULAE FOR DIFFERENCE EQUATIONS WITH ADVANCED ARGUMENTS

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ABSTRACT. The construction of a variation of constants formula for the difference equation with advanced arguments $y(n+1) = A(n)y(n) + B(n)y(g(n)) + f(n)$, $g(n) \geq n+1$, is given for specific sequential spaces.

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1. Introduction. Differential equations with advanced arguments is an active field of research attracting the attention of the analysts [3, 4, 7]. The slow development of the theory of difference equations with advanced arguments could be explained, among other reasons, by the absence of a concise theory of equation with advance for equations with continuous time variable.

In this paper, we attack a concrete problem arising in the theory of linear equations with advance.

It is a well-known fact that the solution $y_f(n)$ of the initial value problem

$$y(n+1) = A(n)y(n) + f(n), \quad y(0) = 0, \quad n \in \mathbb{N} = \{0, 1, 2, \dots\}, \tag{1.1}$$

where we assume that all matrices $A(n)$ are invertible, is given by the following variation of constants formula [1]:

$$y_f(n) = \sum_{m=0}^{n-1} \Phi(n)\Phi^{-1}(m+1)f(m), \tag{1.2}$$

where $\Phi(n)$ denotes the fundamental matrix of equation

$$x(n+1) = A(n)x(n), \tag{1.3}$$

that is,

$$\Phi(n) = \prod_{s=0}^{n-1} A(s), \quad n \geq 0, \qquad \prod_{s=0}^{-1} A(s) = I. \tag{1.4}$$

Such a formula for the initial value problem

$$y(n+1) = A(n)y(n) + B(n)y(g(n)) + f(n), \qquad y(n) = 0, \tag{1.5}$$

where $g(n)$ is a sequence of advance satisfying $g(n) \geq n+1$, up to the knowledge of the authors of the present paper, has not been found.

A variation of constants formula for the nonhomogeneous equation

$$y(n+1) = A(n)y(n) + B(n)y(g(n)) + f(n), \tag{1.6}$$

cannot be obtained from a straightforward application of the ideas of the method of Lagrange for ordinary differential equations or linear difference equations without advance, since a formal application of this method leads to serious algebraic obstacles.

It is pertinent to emphasize that the analysis of equation (1.6) presents many features distinguishing this equation to the ordinary difference equation (1.1). Some examples, where these properties can be appreciated, are given in [2]. In the case of nonhomogeneous equation (1.6) for a general matrices $A(n)$, $B(n)$ and a general forcing term $f(n)$, the equation (1.6) may have no solutions. This situation makes relevant the problem of existence and uniqueness of the solutions of this equation. Moreover, even in the case when all matrices $B(n)$ are invertible, a hypothesis not considered in this paper, equation (1.6) cannot be treated as a difference equation of higher order since the sequence of advances $\{g(n)\}$ is not constant. From the algebraic point of view, the analysis of equation (1.6) seems to be a challenging problem.

Although simpler, a similar situation we encounter in the study of the linear equation

$$y(n+1) = A(n)y(n) + B(n)y(g(n)), \quad n \in \mathbb{N} = \{0,1,2,\dots\} \tag{1.7}$$

for a prescribed initial value $y(0) = \xi$, studied in [2], where some results of existence and uniqueness of solutions of equation (1.7) are obtained if we restrict these problems to specific sequential spaces. Those results rely on the following set of hypotheses:

(C1) $\sum_{m=0}^{n-1} |\Phi(n)\Phi^{-1}(m)| \leq M$, $\forall n \in \mathbb{N}$, $M =$ constant,

(C2) $|\Phi(n)\Phi^{-1}(m)| \leq K\alpha(n)\beta(m)^{-1}$, $n \geq m$,

(C3) $\sum_{m=0}^{\infty} |\Phi^{-1}(m+1)B(m)\Phi(g(m))| < 1$,

where $\{\alpha(n)\}$ and $\{\beta(n)\}$ in C2 are sequences of positive numbers.

Each of conditions (C1), (C2), and (C3) allow to define a sequential space $\mathcal{S}$, where it is possible to define a generating matrix Ψ of equation (1.7), such that the initial value problem

$$y(n+1) = A(n)y(n) + B(n)y(g(n)), \qquad y(0) = \xi, \tag{1.8}$$

has a solution in $\mathcal{S}$ given by

$$y(n,\xi) = \Psi(n)\xi. \tag{1.9}$$

The aim of this paper is to give a variation of constants formula for the equation (1.6) under each condition (C1), (C2), and (C3). In general, this formula have the form

$$y(n) = \Phi(n) \sum_{k=0}^{\infty} \mathbb{C}_k(f)(n), \tag{1.10}$$

where $\{\mathbb{C}_k\}$ is a sequence of bounded functionals. We prove the convergence of this series depending on which condition (C1), (C2), and (C3) is the problem (1.5) considered.

2. Notations and preliminaries. In what follows, $\mathbb{V}$ denote the vector space $\mathbb{R}^r$ or $\mathbb{C}^r$. $|x|$ denote some norm in $\mathbb{V}$; for A, an $r\times r$ matrix, $|A|$ denote the corresponding matrix norm. The following sequential spaces will be used:

$$\begin{aligned}
\ell^\infty &= \{f:\mathbb{N}\longrightarrow\mathbb{V}:|f|^\infty<\infty\}, \quad |f|^\infty=\sup\{|f(n)|:n\in\mathbb{N}\},\\
\ell^1 &= \{f:\mathbb{N}\longrightarrow\mathbb{V},|f|^1<\infty\}, \quad |f|^1=\sum_{n=0}^{\infty}|f(n)|.
\end{aligned} \tag{2.1}$$

For a sequence $\{F(n)\}$ of invertible matrices we will define

$$\begin{aligned}
\ell_F^\infty &= \{f:\mathbb{N}\longrightarrow\mathbb{V}:F^{-1}f\in\ell^\infty\}, \qquad |f|_F^\infty=|F^{-1}f|^\infty,\\
\ell_F^1 &= \{f:\mathbb{N}\longrightarrow\mathbb{V}:F^{-1}f\in\ell^1\}, \qquad |f|_F^1=|F^{-1}f|^1.
\end{aligned} \tag{2.2}$$

Also, we will use the notation

$$\langle f\rangle_k(n)=f(n+k). \tag{2.3}$$

3. Variation of constants formulas. In this section, we obtain different formulas for the solution of the initial value problem (1.5). We seek this solution in the form (1.10). Plugging series (1.10) into (1.6), we obtain

$$\sum_{k=0}^{\infty}\Delta\mathbb{C}_k(f)(n)=\Phi^{-1}(n+1)f(n)+\sum_{k=0}^{\infty}D(n)\mathbb{C}_k(f)(g(n)), \tag{3.1}$$

where we have denoted

$$D(n)=\Phi^{-1}(n+1)B(n)\Phi(g(n)). \tag{3.2}$$

We define $\mathbb{C}_0(f)$ as the solution of the initial value problem

$$\Delta\mathbb{C}_0(f)(n)=\Phi^{-1}(n+1)f(n), \qquad y(0)=0. \tag{3.3}$$

Consequently, from the general theory of difference systems [1] we have

$$\mathbb{C}_0(f)(n)=\sum_{j=0}^{n-1}\Phi^{-1}(j+1)f(j). \tag{3.4}$$

For $k\geq 1$, let $\mathbb{C}_k(f)$ be the solution of

$$\Delta\mathbb{C}_k(f)(n)=D(n)\mathbb{C}_{k-1}(f)(g(n)), \qquad y(0)=0, \tag{3.5}$$

that is,

$$\mathbb{C}_k(f)(n)=\sum_{j=0}^{n-1}D(j)\mathbb{C}_{k-1}(f)(g(j)), \quad k>1. \tag{3.6}$$

Definition 3.1. We call the series (1.10), with the functionals $\mathbb{C}_k(f)$ defined by (3.4) and (3.6), the formal variation of constants formula of equation (1.6).

The series (1.10) give a solution of the problem (1.5) if it converges. In the sequel, we show this convergence under each one of the conditions (C1), (C2), and (C3).

Formally, let us denote the right member of (1.10) by

$$\mathcal{K}(f)(n)=\Phi(n)\sum_{k=0}^{\infty}\mathbb{C}_k(f)(n), \tag{3.7}$$

which we call resolvent operator of equation (1.6).

3.1. Variation of constants in the space ℓ^∞

Lemma 3.1. *Under conditions (C1) and $f\in\ell^\infty$, the coefficients of the formal variation of constants formula satisfy the estimate*

$$|\Phi(n)\mathbb{C}_k(f)(n)|\le M\left(M|\{B(n)\}|^\infty\right)^k|f|^\infty,\quad n,k\in\mathbb{N}. \tag{3.8}$$

Proof. For $k=0$, formula (3.8) follows from (3.4). Let us assume valid the estimate (3.8) for $k-1$:

$$|\Phi(n)\mathbb{C}_{k-1}(f)(n)|\le M\left(M|\{B(n)\}|^\infty\right)^{k-1}|f|^\infty. \tag{3.9}$$

For k we obtain

$$\begin{aligned}|\Phi(n)\mathbb{C}_k(f)(n)|&=\left|\sum_{j=0}^{n-1}\Phi(n)\Phi^{-1}(j+1)B(j)\Phi(g(j))\mathbb{C}_{k-1}(f)(g(j))\right|\\&\le M\left(M|\{B(n)\}|^\infty\right)^{k-1}|\{B(n)\}|^\infty\sum_{j=0}^{n-1}|\Phi(n)\Phi^{-1}(j+1)||f|^\infty\end{aligned} \tag{3.10}$$

from where (3.8) follows. □

From Lemma 3.1, we have the following theorem.

Theorem 3.1. *Let us assume condition (C1). If $M|\{B(n)\}|^\infty<1$, then the series (1.10) converges uniformly on all $\mathbb{N}$ and gives the unique solution of the initial value problem (1.5) in the space ℓ^∞. Moreover, under these conditions, the series (3.7) defines a bounded operator*

$$\mathcal{K}_\infty:\ell^\infty\longrightarrow\ell^\infty, \tag{3.11}$$

which norm satisfies

$$|\mathcal{K}_\infty(f)|^\infty\le\frac{M}{1-M|\{B(n)\}|^\infty}|f|^\infty. \tag{3.12}$$

The conditions of Theorem 3.1 and the estimation given by Lemma 3.1 yield the following formula:

$$\mathcal{K}_\infty(f)(n)=\Phi(n)\mathbb{C}_0(f)(n)+\cdots+\Phi(n)\mathbb{C}_k(f)(n)+O_\infty(f,k)(n), \tag{3.13}$$

where the remainder $O_\infty(f,k)(n)$ has the estimate

$$|O_\infty(f,k)(n)| \le \frac{M(M|\{B(n)\}|^\infty)^{k+1}}{1-M|\{B(n)\}|^\infty}|f|^\infty. \tag{3.14}$$

Formula (3.13) says that the solution $\mathcal{K}_\infty(f)(n)$ is approximated by

$$\Phi(n)\mathbb{C}_0(f)(n)+\cdots+\Phi(n)\mathbb{C}_k(f)(n), \tag{3.15}$$

with an error given by $O_\infty(f,k)(n)$. This approximation is valid on all $\mathbb{N}$.

3.2. Variation of constants in the space ℓ_α^∞

Lemma 3.2. *Under condition C2, let us suppose that the sequence $\{B(n)\}$ is summable in the following sense:*

$$\delta = \sum_{j=0}^{\infty}\alpha(g(j))\beta(j+1)^{-1}|B(j)| < \infty. \tag{3.16}$$

Then the functionals $\mathbb{C}_k(f)$ defined by (3.4) and (3.6) have the following estimates:

$$|\Phi(n)\mathbb{C}_k(f)(n)| \le K\alpha(n)(K\delta)^k|f|_{\langle\beta\rangle_1}^1, \quad \forall n,k\in\mathbb{N}. \tag{3.17}$$

Proof. For $k=0$ we have

$$|\Phi(n)\mathbb{C}_0(f)(n)| \le K\sum_{j=0}^{n-1}\alpha(n)\beta(j+1)^{-1}|f(j)| \le K\alpha(n)|f|_{\langle\beta\rangle_1}^1. \tag{3.18}$$

Thus, (3.17) is valid for $k=0$. Let us assume the validity of the assertion for $k-1$:

$$|\Phi(n)\mathbb{C}_{k-1}(f)(n)| \le K\alpha(n)(K\delta)^{K-1}|f|_{\langle\beta\rangle_1}^1. \tag{3.19}$$

From (3.6) we obtain

$$\begin{aligned}|\Phi(n)\mathbb{C}_k(f)(n)| &= \left|\sum_{j=o}^{n-1}\Phi(n)\Phi^{-1}(j+1)B(j)\Phi(g(j))\mathbb{C}_{k-1}(f)(g(j))\right| \\ &\le \alpha(n)K^2(K\delta)^{k-1}\sum_{j=0}^{n-1}\alpha(g(j))\beta(j+1)^{-1}|B(j)||f|_{\langle\beta\rangle_1}^1\end{aligned} \tag{3.20}$$

implying (3.17). □

From the above lemma we obtain the following.

Theorem 3.2. *Under condition C2, if $f\in\ell_{\langle\beta\rangle_1}^1$ and $K\delta<1$, then series (1.10) converges uniformly in the norm of the space ℓ_α^∞. Moreover, the series (3.7) defines the operator*

$$\mathcal{K}_{\beta,\alpha}:\ell_{\langle\beta\rangle_1}^1 \longrightarrow \ell_\alpha^\infty, \tag{3.21}$$

satisfying the estimate

$$|\mathcal{K}_{\beta,\alpha}(f)|_\alpha^\infty \le \frac{K}{1-K\delta}|f|_{\langle\beta\rangle_1}^1. \tag{3.22}$$

If condition C2 is accomplished with $\alpha = \beta$, then we denote the operator (3.21) by $\mathcal{K}_\alpha = \mathcal{K}_{\alpha,\alpha}$. Therefore

$$\mathcal{K}_\alpha : \ell^1_{\langle\alpha\rangle_1} \longrightarrow \ell^\infty_\alpha, \tag{3.23}$$

$$|\mathcal{K}_\alpha(f)|^\infty_\alpha \le \frac{K}{1-K_\delta}|f|^1_{\langle\alpha\rangle_1}. \tag{3.24}$$

Lemma 3.2 and the estimate given by Theorem 3.2 yield to the following asymptotic formula:

$$\mathcal{K}_\alpha(f) = \Phi(n)\mathbb{C}_0(f) + \cdots + \Phi(n)\mathbb{C}_k(f) + O_\alpha(f,k), \tag{3.25}$$

where

$$|O_\alpha(f,k)| \le \frac{K\alpha(n)(K\delta)^{k+1}}{1-K\delta}|f|^1_{\langle\alpha\rangle_1}, \tag{3.26}$$

with

$$K\delta = K\sum_{j=0}^\infty \alpha(g(j))\alpha(j+1)^{-1}|B(j)| < 1. \tag{3.27}$$

If the sequence α is monotone in the following sense:

$$\alpha(n) \le \mathbb{C}\alpha(n+1), \quad \forall n \in \mathbb{N}, \tag{3.28}$$

where $\mathbb{C}$ is a positive constant, then we obtain the following.

Theorem 3.3. *If the conditions C2 with $\alpha = \beta$, $K\delta < 1$ and (3.28) are fulfilled, then the operator (3.23) satisfies*

$$\mathcal{K}_\alpha : \ell^1_\alpha \longrightarrow \ell^\infty_\alpha, \tag{3.29}$$

$$|\mathcal{K}_\alpha(f)|^\infty_\alpha \le \frac{KC}{1-K\delta}|f|^1_\alpha. \tag{3.30}$$

In the form (3.29) the operator $\mathcal{K}_\alpha$ is more useful in applications to nonlinear equations with advanced arguments [6].

3.3. Variation of constants in the space ℓ^∞_Φ

Lemma 3.3. *Under condition C3, if $f \in \ell^1_{\langle\Phi\rangle_1}$, then the coefficients of the series (1.10) satisfy*

$$|\mathbb{C}_k(f)(n)| \le \mu^k |f|^1_{\langle\Phi\rangle_1}, \quad \forall n,k \in \mathbb{N}, \tag{3.31}$$

where

$$\mu = \sum_{m=0}^\infty |\Phi^{-1}(m+1)B(m)\Phi(g(m))|. \tag{3.32}$$

Proof. For $k = 0$ the assertion is clear. If

$$|\mathbb{C}_{k-1}(f)(n)| \le \mu^{k-1}|f|^{1}_{\langle\Phi\rangle_1}, \tag{3.33}$$

then

$$\begin{aligned} |\mathbb{C}_k(f)(n)| &= \left|\sum_{j=0}^{n-1} D(j)\mathbb{C}_{k-1}(f)(g(j))\right| \le \sum_{j=0}^{n-1}|D(j)||\mathbb{C}_{k-1}(f)(g(j))| \\ &\le \mu^{k-1}\sum_{j=0}^{n-1}|D(j)||f|_{\langle\Phi\rangle_1} = \mu^k|f|^{1}_{\langle\Phi\rangle_1}. \end{aligned} \tag{3.34}$$

This estimate completes the proof. □

From the above lemma we deduce the following.

Theorem 3.4. *Under condition C3, if $f \in \ell^1_{\langle\Phi\rangle_1}$ and $\mu < 1$, then the series*

$$\sum_{k=0}^{\infty}\mathbb{C}_k(f)(n) \tag{3.35}$$

converges in the metric of the space ℓ^∞_Φ. Moreover, the series (3.7) defines the operator

$$\mathcal{K}_\Phi : \ell^1_{\langle\Phi\rangle_1} \longrightarrow \ell^\infty_\Phi, \tag{3.36}$$

satisfying the estimate

$$|\mathcal{K}_\Phi(f)|^{\infty}_{\Phi} \le (1-\mu)^{-1}|f|^{1}_{\langle\Phi\rangle_1}. \tag{3.37}$$

The estimate given by Lemma 3.3 and the conditions of Theorem 3.4 yield the following asymptotic formulas:

$$\mathcal{K}_\infty(f)(n) = \Phi(n)\mathbb{C}_0(f)(n) + \cdots + \Phi(n)\mathbb{C}_k(f)(n) + O_\Phi(f,k)(n), \tag{3.38}$$

where the remainder $O_\Phi(f,k)(n)$ satisfies

$$|O_\Phi(f,k)(n)| \le (1-\mu)^{-1}\mu^k|f|^{\infty}_{\langle\Phi\rangle_1}. \tag{3.39}$$

Let us assume the condition

$$L = \sum_{m=0}^{\infty}|\Phi^{-1}(m)A^{-1}(m)\Phi(m)| = \sum_{m=0}^{\infty}|\Phi^{-1}(m+1)\Phi(m)| < \infty. \tag{3.40}$$

Theorem 3.5. *If C3, $\mu < 1$ and (3.40) are assumed, then the series (3.7) defines an operator*

$$\mathcal{K}_\Phi : \ell^\infty_\Phi \longrightarrow \ell^\infty_\Phi, \tag{3.41}$$

satisfying the estimate

$$|\mathcal{K}_\Phi(f)(n)|^{\infty}_{\Phi} \le \frac{L}{1-\mu}|f|^{\infty}_{\Phi}. \tag{3.42}$$

Proof. The proof follows from (3.37). □

4. An integral equation for the operator $\mathcal{K}$. We have deduced the existence of a variation of constants formula under different condition (C1), (C2), and (C3). The above methodology gives the erroneous impression of a certain chaotic set of these formulas depending on the sequential space we have posed the initial value problem (1.5). We demonstrate that all obtained operators $\mathcal{K}_\infty, \mathcal{K}_\alpha$, and $\mathcal{K}_\Phi$ satisfy a common equation, from where they can be deduced.

Formally, let us write (3.7) in the following form:

$$\mathcal{K}(f)(n) = \Phi(n)\mathbb{C}_0(f)(n) + \Phi(n)\sum_{k=1}^{\infty}\mathbb{C}_k(f)(n). \tag{4.1}$$

Using the definition of the functionals $\mathbb{C}_k$, we may write

$$\mathcal{K}(f)(n) = \Phi(n)\sum_{i=0}^{n-1}\Phi^{-1}(i+1)f(i) + \Phi(n)\sum_{k=1}^{\infty}\sum_{i=0}^{n-1}D(i)\mathbb{C}_{k-1}(f)(g(i)). \tag{4.2}$$

Let us interchange the sums in this formula. We obtain

$$\mathcal{K}(f)(n) = \Phi(n)\sum_{i=0}^{n-1}\left(\Phi^{-1}(i+1)f(i) + \Phi^{-1}(i+1)B(i)\mathcal{K}(f)(g(i))\right). \tag{4.3}$$

In the next theorem, $\mathcal{S}$ and $\tilde{\mathcal{S}}$ denote sequential spaces defined on $\mathbb{N}$.

Theorem 4.1. *Let $\mathcal{K} : \mathcal{S} \to \tilde{\mathcal{S}}$ be a bounded operator such that for each $f \in \mathcal{S}$ the sequence $\mathcal{K}(f)$ satisfies the integral equation (4.3), then the sequence $\mathcal{K}(f)(n)$ is a solution of the initial value problem (1.5) belongs to $\tilde{\mathcal{S}}$.*

Proof. The proof is obtained by a straightforward calculation. □

Theorem 4.1 allows the possibility of giving other variation of constants formula, different from $\mathcal{K}_\infty, \mathcal{K}_\alpha$, and $\mathcal{K}_\Phi$.

Let us consider problem (1.5) under the following conditions:

(D1) $\sum_{n=0}^{\infty}|\Phi(n)| < \infty$,

(D2) $|\{\Phi^{-1}(n+1)B(n)\}|^\infty < \infty$,

(D3) $\mathcal{S} = \ell^1_{\langle\Phi\rangle_1}$, $\tilde{\mathcal{S}} = \ell^1$.

Let us consider the integral equation

$$y(n) = \Phi(n)\sum_{i=0}^{n-1}\Phi^{-1}(i+1)f(i) + \Phi(n)\sum_{i=0}^{n-1}\Phi^{-1}(i+1)B(i)y(g(i)). \tag{4.4}$$

Under conditions (D1), (D2), and (D3) and

$$|\{\Phi^{-1}(n+1)B(n)\}|^\infty\sum_{n=0}^{\infty}|\Phi(n)| < 1, \tag{4.5}$$

the right-hand side of the integral equation (4.4), for each fixed $f \in \ell^1_{\langle\Phi\rangle_1}$, defines an operator $\mathcal{U} : \ell^1 \to \ell^1$, that acts like a contraction on ℓ^1 into itself. Let us denote the unique fixed point of this contraction by $\mathcal{T}(f)$. Thus $\mathcal{T}:\ell^1_{\langle\Phi\rangle_1}\to\ell^1$. From

$$\begin{aligned}\mathcal{T}(f)(n)-\mathcal{T}(h)(n)=\Phi(n)\sum_{i=0}^{n-1}\Phi^{-1}(i+1)(f-h)(i)\\+\Phi(n)\sum_{i=0}^{n-1}\Phi^{-1}(i+1)B(i)(\mathcal{T}(f)-\mathcal{T}(h))(g(i)),\end{aligned}\tag{4.6}$$

we obtain

$$|\mathcal{T}(f)-\mathcal{T}(h)|^1\le\rho|f-h|^1_{\langle\Phi\rangle_1},\tag{4.7}$$

where

$$\rho=\frac{\sum_{n=0}^{\infty}|\Phi(n)|}{1-|\{\Phi^{-1}(n+1)B(n)\}|^{\infty}\sum_{n=0}^{\infty}|\Phi(n)|}.\tag{4.8}$$

Thus the operator $\mathcal{T}$ is continuous and satisfies equation (4.3) for any $f\in\ell^1_{\langle\Phi\rangle_1}$. From Theorem 4.1, $\mathcal{T}(f)$ is a solution of the problem (1.5) if $f\in\ell^1_{\langle\Phi\rangle_1}$. We emphasize that the operator $\mathcal{T}$ was not obtained by means of Theorems 3.1, 3.2, and 3.3.

5. Applications.

In this section, we exhibit some examples of the obtained results.

5.1. Perturbed systems.

Let us recall, the notion of the generating matrix of a linear system with advance which was introduced in [2]. Let $\mathcal{S}$ denote a space of sequences.

Definition 5.1. A sequence of $r\times r$ matrices $\{\Psi(n)\}$, $\Psi(0)=1$, is called a generating matrix of system (1.7) in the space $\mathcal{S}$, if and only if the following hypotheses hold:

(i) the sequence $\{\Psi(n)\}$ is a solution of the equation (1.7) on $\mathbb{N}$,
(ii) $\Psi(n)$ is invertible for all values of $n\ge 0$,
(iii) $\{\Psi(n)\}\in\mathcal{S}$.

Let $\tilde{\Psi}$ be the generating matrix of

$$y(n+1)=A(n)y(n)+\big(B(n)+\mathbb{C}(n)\big)y\big(g(n)\big)\tag{5.1}$$

in the space $\mathcal{S}$. The question for an estimate of $\Psi(n)-\tilde{\Psi}(n)$ is an important and difficult question in the theory of equations with advance. Such estimates are not obtained by discrete inequalities, because such a theory has not been developed for equations with advance. In our paper we give a partial solution to this problem.

Theorem 5.1. *Let us assume condition (C1). If*

$$M|\{B(n)\}|^{\infty}<1,\qquad M|\{B(n)+\mathbb{C}(n)\}|^{\infty}<1,\tag{5.2}$$

then there exist the generating matrices Ψ and $\tilde{\Psi}$ of systems (1.7) and (5.1), respectively, in the space ℓ^{∞}, satisfying the following estimate:

$$|\Psi-\tilde{\Psi}|^{\infty}\le\frac{M}{1-M|\{B(n)+\mathbb{C}(n)\}|^{\infty}}|\{\mathbb{C}(n)\Psi(n)\}|^{\infty}.\tag{5.3}$$

Proof. The existence of generating matrices Ψ and $\tilde{\Psi}$ follows from [2, Theorem 2]. From (1.7) and (5.1), we obtain that the sequence $\Psi - \tilde{\Psi}$ satisfies

$$y(n+1) = A(n)y(n) + (B(n) + \mathbb{C}(n))y(g(n)) - \mathbb{C}(n)\Psi(g(n)). \tag{5.4}$$

Applying the estimate (3.12) to this equation we obtain (5.3). □

If we formally replace $B(n) = 0$ and $\mathbb{C}(n) = B(n)$ in (5.2), then from Theorem 5.1 it follows that

$$|\Psi - \Phi|^{\infty} \leq K|\{B(n)\Phi(n)\}|^{\infty}, \tag{5.5}$$

where

$$K = \frac{M}{1 - M|\{B(n)\}|^{\infty}}. \tag{5.6}$$

In applications, a more general estimate than (5.5) is needed

$$|\Psi(n)\psi^{-1}(m) - \Phi(n)\Phi^{-1}(m)|^{\infty} \leq K \sup_{n \geq m} |B(n)\Phi(n)\Phi^{-1}(m)|, \quad n \geq m. \tag{5.7}$$

The estimate (5.7) can be obtained by a slight modification of Lemma 3.1 and Theorem 3.1, where it is necessary to fix an initial time $n_0 = m$ instead of $n_0 = 0$.

A similar estimate to (5.7) can be obtained by using the condition C2.

Theorem 5.2. *Let us assume condition C2. If*

$$\sum_{m=0}^{\infty} \frac{\alpha(g(m))}{\beta(m+1)}|B(m)| < 1, \qquad \sum_{m=0}^{\infty} \frac{\alpha(g(m))}{\beta(m+1)}|B(m) + \mathbb{C}(m)| < 1, \tag{5.8}$$

then there exist the generating matrices Ψ and $\tilde{\Psi}$ of systems (1.7) and (5.1) in the space ℓ_{α}^{∞} satisfying

$$|\Psi - \tilde{\Psi}|_{\alpha}^{\infty} \leq \frac{K}{1 - K_{\tilde{\delta}}}|\{\mathbb{C}(n)\Psi(n)\}|_{\langle\beta\rangle_1}^{1}, \tag{5.9}$$

where

$$\tilde{\delta} = \sum_{m=0}^{\infty} \alpha(g(m))\beta(m+1)^{-1}|B(m) + \mathbb{C}(m)|. \tag{5.10}$$

Proof. The existence of matrices Ψ and $\tilde{\Psi}$ follows from [2, Theorem 4]. From (1.7) and (5.1) we obtain that the sequence $\Psi - \tilde{\Psi}$ satisfies equation (5.3). Applying the estimate (3.22) to this equation we obtain (5.9). □

From Theorem 5.2, it follows

$$|\Psi - \Phi|_{\alpha}^{\infty} \leq \frac{K}{1 - K\tilde{\delta}}|\{B(n)\Phi(n)\}|_{\langle\beta\rangle_1}^{1}, \tag{5.11}$$

where $\tilde{\delta}$ is defined by (5.10) with $B(n) = 0$.

It is useful to point out the forthcoming estimate, following from Lemma 3.2 and Theorem 3.2,

$$|\{\Psi(n)\Psi^{-1}(m)-\Phi(n)\Phi^{-1}(m)\}|_{\alpha}^{\infty} \le \frac{K}{1-K\tilde{\delta}}|\{B(n)\Phi(n)\Phi^{-1}(m)\}|_{\alpha}^{1}, \quad n \ge m. \tag{5.12}$$

In the sequential space ℓ_{Φ}^{∞} we may enunciate the following.

Theorem 5.3. *Let us assume condition C3. If*

$$\sum_{m=0}^{\infty} |\Phi^{-1}(m+1)(B(m)+\mathbb{C}(m))\Phi(g(m))| < 1, \tag{5.13}$$

then there exist the generating matrices Ψ *and* $\tilde{\Psi}$ *of systems (1.7) and (5.1) in the space* ℓ_{Φ}^{∞} *satisfying*

$$|\Psi-\tilde{\Psi}|_{\Phi}^{\infty} \le (1-\tilde{\mu})^{-1}|\{\mathbb{C}(n)\Psi(n)\}|_{\langle\Phi\rangle_1}^{1}, \tag{5.14}$$

where

$$\tilde{\mu} = \sum_{m=0}^{\infty} |\Phi^{-1}(m+1)(B(m)+\mathbb{C}(n))\Phi(g(m))|. \tag{5.15}$$

Proof. The existence of the generating matrices Ψ and $\tilde{\Psi}$ follows from [2, Theorem 6]. The sequence $\Psi-\tilde{\Psi}$ satisfies equation (5.3). Applying the estimate (3.37) to this equation we obtain (5.14). □

Theorem 5.3 implies

$$|\Psi-\Phi|_{\Phi}^{\infty} \le (1-\tilde{\mu})^{-1}|\{B(n)\Phi(n)\}|_{\langle\Phi\rangle_1}^{1}. \tag{5.16}$$

From Lemma 3.3 and Theorem 3.4 we obtain

$$|\Psi(n)\Psi^{-1}(m)-\Phi(n)\Phi^{-1}(m)|_{\Phi}^{\infty} \le (1-\tilde{\mu})^{-1}|\{B(n)\Phi(n)\}|_{\langle\Phi\rangle_1}^{1}, \quad n \ge m. \tag{5.17}$$

5.2. Equations with periodic solutions. Let us consider equation (1.6), where the sequences $\{A(n)\},\{B(n)\}$, and $\{g(n)\}$ are periodic with a same period $T \in \mathbb{N}$, $T > 0$:

$$A(n+T) = A(n), \quad B(n+T) = B(n), \quad g(n+T) = g(n), \quad \forall n \in \mathbb{N}. \tag{5.18}$$

Using the Floquet formula [1], the fundamental matrix of equation (1.3) can be written in the form

$$\Phi(n) = P(n)\mathbb{C}^n, \tag{5.19}$$

where $P(n)$ is a T-periodic sequence and $\mathbb{C} = \sqrt[T]{\Phi(T)}$. Let us assume that all the eigenvalues of the invertible matrix $\mathbb{C}$ satisfy $|\mu| < 1$. In this case, it is easy to prove the existence of a positive constant K such that

$$|\Phi(n)\Phi^{-1}(m)| \le Kr^{n-m}, \quad n \ge m,\ 0 < r < 1. \tag{5.20}$$

Thus, condition (C1) is satisfied with $M = (1-r)^{-1}K$. If the T-periodic sequence $\{B(n)\}$ satisfies $K|\{B(n)\}|^\infty < 1-r$, then for any T-periodic sequence $\{f(n)\}$, the initial value problem (1.5) has a unique bounded solution. This bounded solution must be T-periodic and is the unique T-periodic solution of equation (1.6). This T-periodic solution can be approximated by the asymptotic formula (3.13), where the sum

$$\Phi(n)\mathbb{C}_0(f)(n)+\cdots+\Phi(n)\mathbb{C}_k(f)(n) \tag{5.21}$$

is T-periodic.

5.3. Scalar equations. Let us consider the scalar equation

$$y(n+1) = a(n)y(n)+b(n)y(g(n))+f(n), \tag{5.22}$$

where the complex value sequence $\{a(n)\}$ satisfies $a(n) \neq 0$ for all n. The hypothesis C2 is accomplished with

$$\alpha(n) = \beta(n) = \prod_{m=0}^{n-1} |a(m)|. \tag{5.23}$$

According to Theorem 3.2, if

$$\delta = \sum_{m=0}^{\infty} \prod_{k=m+1}^{g(m)-1} |a(k)b(m)| < 1, \tag{5.24}$$

then the solution of the initial value problem (1.5) is given by the approximate formula (3.25):

$$\begin{aligned} \mathcal{K}_\alpha(f)(n) = & \sum_{m=0}^{n-1} \prod_{k=m+1}^{n-1} a(k)f(m) \\ & + \sum_{m=0}^{n-1} b(m) \prod_{k=m+1}^{n-1} a(k) \sum_{j=0}^{g(m)-1} \prod_{i=j+1}^{g(m)-1} a(i)f(j)+O_\alpha(f,1)(n), \end{aligned} \tag{5.25}$$

where the remainder $O_\alpha(f,1)(n)$ satisfies the estimate

$$|O_\alpha(f,1)(n)| \leq \frac{\alpha\delta^2}{1-\delta}|f|^1_{\langle\alpha\rangle_1}. \tag{5.26}$$

For $g(n) = n+2$, we obtain the formula

$$\begin{aligned} \mathcal{K}_\alpha(f)(n) = & \sum_{m=0}^{n-1} \prod_{k=m+1}^{n-1} a(k)f(m) \\ & + \sum_{m=0}^{n-1} b(m) \prod_{k=m+1}^{n-1} a(k) \sum_{j=0}^{m+1} \prod_{i=j+1}^{m+1} a(i)f(j)+O_\alpha(f,1)(n). \end{aligned} \tag{5.27}$$

5.4. Equations with constant coefficients. Let us consider the case $A(n) = A =$ constant and $B(n) = B =$ constant. If A is an invertible matrix with all eigenvalues μ satisfying $|\mu| < 1$, then from the results in [1, 5] it follows, for some positive K and $0 < r < 1$, the estimate

$$\left|\Phi(n)\Phi^{-1}(m)\right| \le Kr^{n-m}, \quad n \ge m, \tag{5.28}$$

where $\Phi(n) = A^n$. Thus (C1) follows with $M = K/(1-r)$. If $M|B| < 1$, then Theorem 3.1 implies the asymptotic formula (3.13) for the solution of the initial value problem

$$y(n+1) = Ay(n) + By(g(n)) + f(n), \quad y(0) = 0, \quad g(n) \ge n+1, \ f \in \ell^\infty. \tag{5.29}$$

In this particular case (3.13) has the form

$$\begin{aligned} \mathcal{K}_\infty(f)(n) = &\sum_{m=0}^{n-1} A^{n-m-1} f(m) \\ &+ \sum_{m=0}^{n-1} \sum_{k=0}^{g(m)-1} A^{n-1-m} B A^{g(m)-k-1} f(k) + O_\infty(f,1)(n), \end{aligned} \tag{5.30}$$

where the remainder $O_\infty(f,1)(n)$ satisfies

$$\left|O_\infty(f,1)(n)\right|^\infty \le \frac{M}{1-M|B|}\left(M|B|\right)^2 |f|^\infty. \tag{5.31}$$

We point out the following particular cases of (5.30): if the matrices A and B commute, then we obtain the approximate formula

$$\mathcal{K}_\infty(f)(n) = \sum_{m=0}^{n-1} A^{n-m-1} f(m) + B \sum_{m=0}^{n-1} \sum_{k=0}^{g(m)-1} A^{n+g(m)-m-k-2} f(k). \tag{5.32}$$

The particular case $AB = BA$ and $g(n) = n+2$ leads to the solution

$$\mathcal{K}_\infty(f)(n) = \sum_{m=0}^{n-1} A^{n-m-1} f(m) + B \sum_{m=0}^{n-1} \sum_{k=0}^{m+1} A^{n-k} f(k) + O_\infty(f,1)(n). \tag{5.33}$$

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