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INTUITIONISTIC FUZZY IDEALS OF BCK-ALGEBRAS

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ABSTRACT. We consider the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigate some related properties.

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1. Introduction. After the introduction of the concept of fuzzy sets by Zadeh [9] several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. The first author (together with Hong, Kim, Kim, Meng, Roh, and Song) considered the fuzzification of ideals and subalgebras in BCK-algebras (cf. [3, 4, 5, 6, 7, 8]). In this paper, using the Atanassov's idea, we establish the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigate some of their properties. We introduce the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigate some related properties.

2. Preliminaries. First we present the fundamental definitions. By a *BCK-algebra* we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$,

(II) $(x*(x*y))*y=0$,

(III) $x*x=0$,

(IV) $0*x=0$,

(V) $x*y=0$ and $y*x=0$ imply that $x=y$

for all $x,y,z\in X$.

A partial ordering "$\leq$" on X can be defined by $x\leq y$ if and only if $x*y=0$. A nonempty subset S of a BCK-algebra X is called a *subalgebra* of X if $x*y\in S$ whenever $x,y\in S$. A nonempty subset I of a BCK-algebra X is called an *ideal* of X if

(i) $0\in I$,

(ii) $x*y\in I$ and $y\in I$ imply that $x\in I$ for all $x,y\in X$.

By a *fuzzy set* μ in a nonempty set X we mean a function $\mu: X\to[0,1]$, and the complement of μ, denoted by $\bar{\mu}$, is the fuzzy set in X given by $\bar{\mu}(x)=1-\mu(x)$ for all $x\in X$. A fuzzy set μ in a BCK-algebra X is called a *fuzzy subalgebra* of X if $\mu(x*y)\geq$

$\min\{\mu(x),\mu(y)\}$ for all $x,y \in X$. A fuzzy set μ in a BCK-algebra X is called a *fuzzy ideal* of X if

(i) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(ii) $\mu(x) \geq \min\{\mu(x*y),\mu(y)\}$ for all $x,y \in X$.

An intuitionistic fuzzy set (briefly, IFS) A in a nonempty set X is an object having the form

$$A = \{(x,\alpha_A(x),\beta_A(x)) \mid x \in X\}, \tag{2.1}$$

where the functions $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \alpha_A(x) + \beta_A(x) \leq 1 \quad \forall x \in X. \tag{2.2}$$

An intuitionistic fuzzy set $A = \{(x,\alpha_A(x),\beta_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (α_A,β_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\alpha_A,\beta_A)$ for the IFS $A = \{(x,\alpha_A(x),\beta_A(x)) \mid x \in X\}$.

3. Intuitionistic fuzzy ideals.

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. An IFS $A = (\alpha_A,\beta_A)$ in X is called an *intuitionistic fuzzy subalgebra* of X if it satisfies:

(IS1) $\alpha_A(x*y) \geq \min\{\alpha_A(x),\alpha_A(y)\}$,

(IS2) $\beta_A(x*y) \leq \max\{\beta_A(x),\beta_A(y)\}$,

for all $x,y \in X$.

Example 3.2. Consider a BCK-algebra $X=\{0,a,b,c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $A = (\alpha_A,\beta_A)$ be an IFS in X defined by

$$\begin{aligned} \alpha_A(0) &= \alpha_A(a) = \alpha_A(c) = 0.7 > 0.3 = \alpha_A(b), \\ \beta_A(0) &= \beta_A(a) = \beta_A(c) = 0.2 < 0.5 = \beta_A(b). \end{aligned} \tag{3.1}$$

Then $A = (\alpha_A,\beta_A)$ is an intuitionistic fuzzy subalgebra of X.

Proposition 3.3. *Every intuitionistic fuzzy subalgebra $A = (\alpha_A,\beta_A)$ of X satisfies the inequalities $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} \alpha_A(0) &= \alpha_A(x*x) \geq \min\{\alpha_A(x),\alpha_A(x)\} = \alpha_A(x), \\ \beta_A(0) &= \beta_A(x*x) \leq \max\{\beta_A(x),\beta_A(x)\} = \beta_A(x). \end{aligned} \tag{3.2}$$

This completes the proof. □

Definition 3.4. An IFS $A = (\alpha_A, \beta_A)$ in X is called an *intuitionistic fuzzy ideal* of X if it satisfies the following inequalities:

(IF1) $\alpha_A(0) \ge \alpha_A(x)$ and $\beta_A(0) \le \beta_A(x)$,

(IF2) $\alpha_A(x) \ge \min\{\alpha_A(x * y), \alpha_A(y)\}$,

(IF3) $\beta_A(x) \le \max\{\beta_A(x * y), \beta_A(y)\}$,

for all $x, y \in X$.

Example 3.5. Let $X = \{0,1,2,3,4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Define an IFS $A = (\alpha_A, \beta_A)$ in X as follows:

$$\begin{aligned} \alpha_A(0) = \alpha_A(2) = 1, \qquad \alpha_A(1) = \alpha_A(3) = \alpha_A(4) = t, \\ \beta_A(0) = \beta_A(2) = 0, \qquad \beta_A(1) = \beta_A(3) = \beta_A(4) = s, \end{aligned} \tag{3.3}$$

where $t \in [0,1]$, $s \in [0,1]$, and $t + s \le 1$. By routine calculation we know that $A = (\alpha_A, \beta_A)$ is an *intuitionistic fuzzy ideal* of X.

Lemma 3.6. *Let an* IFS $A = (\alpha_A, \beta_A)$ *in* X *be an intuitionistic fuzzy ideal of* X. *If the inequality* $x * y \le z$ *holds in* X, *then*

$$\alpha_A(x) \ge \min\{\alpha_A(y), \alpha_A(z)\}, \qquad \beta_A(x) \le \max\{\beta_A(y), \beta_A(z)\}. \tag{3.4}$$

Proof. Let $x, y, z \in X$ be such that $x * y \le z$. Then $(x * y) * z = 0$, and thus

$$\begin{aligned} \alpha_A(x) &\ge \min\{\alpha_A(x * y), \alpha_A(y)\} \\ &\ge \min\{\min\{\alpha_A((x * y) * z), \alpha_A(z)\}, \alpha_A(y)\} \\ &= \min\{\min\{\alpha_A(0), \alpha_A(z)\}, \alpha_A(y)\} \\ &= \min\{\alpha_A(y), \alpha_A(z)\}, \\ \beta_A(x) &\le \max\{\beta_A(x * y), \beta_A(y)\} \\ &\le \max\{\max\{\beta_A((x * y) * z), \beta_A(z)\}, \beta_A(y)\} \\ &= \max\{\max\{\beta_A(0), \beta_A(z)\}, \beta_A(y)\} \\ &= \max\{\beta_A(y), \beta_A(z)\}, \end{aligned} \tag{3.5}$$

this completes the proof. □

Lemma 3.7. *Let* $A = (\alpha_A, \beta_A)$ *be an intuitionistic fuzzy ideal of* X. *If* $x \le y$ *in* X, *then*

$$\alpha_A(x) \ge \alpha_A(y), \qquad \beta_A(x) \le \beta_A(y), \tag{3.6}$$

that is, α_A *is order-reserving and* β_A *is order-preserving.*

Proof. Let $x, y \in X$ be such that $x \le y$. Then $x * y = 0$ and so

$$\alpha_A(x) \ge \min\{\alpha_A(x*y), \alpha_A(y)\} = \min\{\alpha_A(0), \alpha_A(y)\} = \alpha_A(y),$$
$$\beta_A(x) \le \max\{\beta_A(x*y), \beta_A(y)\} = \max\{\beta_A(0), \beta_A(y)\} = \beta_A(y). \tag{3.7}$$

This completes the proof. □

Theorem 3.8. *If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X, then for any $x, a_1, a_2, \dots, a_n \in X$, $(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$ implies*

$$\alpha_A(x) \ge \min\{\alpha_A(a_1), \alpha_A(a_2), \dots, \alpha_A(a_n)\},$$
$$\beta_A(x) \le \max\{\beta_A(a_1), \beta_A(a_2), \dots, \beta_A(a_n)\}. \tag{3.8}$$

Proof. Using induction on n and Lemmas 3.6 and 3.7, the proof is straightforward. □

Theorem 3.9. *Every intuitionistic fuzzy ideal of X is an intuitionistic fuzzy subalgebra of X.*

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of X. Since $x * y \le x$ for all $x, y \in X$, it follows from Lemma 3.7 that

$$\alpha_A(x*y) \ge \alpha_A(x), \qquad \beta_A(x*y) \le \beta_A(x), \tag{3.9}$$

so by (IF2) and (IF3),

$$\alpha_A(x*y) \ge \alpha_A(x) \ge \min\{\alpha_A(x*y), \alpha_A(y)\} \ge \min\{\alpha_A(x), \alpha_A(y)\},$$
$$\beta_A(x*y) \le \beta_A(x) \le \max\{\beta_A(x*y), \beta_A(y)\} \le \max\{\beta_A(x), \beta_A(y)\}. \tag{3.10}$$

This shows that $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy subalgebra of X. □

The converse of Theorem 3.9 may not be true. For example, the intuitionistic fuzzy subalgebra $A = (\alpha_A, \beta_A)$ in Example 3.2 is not an intuitionistic fuzzy ideal of X since

$$\beta_A(b) = 0.5 > 0.2 = \min\{\beta_A(b*a), \beta_A(a)\}. \tag{3.11}$$

We now give a condition for an intuitionistic fuzzy subalgebra to be an intuitionistic fuzzy ideal.

Theorem 3.10. *Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy subalgebra of X such that*

$$\alpha_A(x) \ge \min\{\alpha_A(y), \alpha_A(z)\}, \qquad \beta_A(x) \le \max\{\beta_A(y), \beta_A(z)\} \tag{3.12}$$

for all $x, y, z \in X$ satisfying the inequality $x * y \le z$. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X.

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy subalgebra of X. Recall that $\alpha_A(0) \ge \alpha_A(x)$ and $\beta_A(0) \le \beta_A(x)$ for all X. Since $x * (x * y) \le y$, it follows from the hypothesis that

$$\alpha_A(x) \ge \min\{\alpha_A(x*y), \alpha_A(y)\}, \qquad \beta_A(x) \le \max\{\beta_A(x*y), \beta_A(y)\}. \tag{3.13}$$

Hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X. □

Lemma 3.11. *An* IFS $A = (\alpha_A, \beta_A)$ *is an intuitionistic fuzzy ideal of* X *if and only if the fuzzy sets* α_A *and* $\bar{\beta}_A$ *are fuzzy ideals of* X.

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of X. Clearly, α_A is a fuzzy ideal of X. For every $x, y \in X$, we have

$$\begin{aligned}\bar{\beta}_A(0) &= 1-\beta_A(0) \geq 1-\beta_A(x) = \bar{\beta}_A(x),\\ \bar{\beta}_A(x) &= 1-\beta_A(x) \geq 1-\max\{\beta_A(x*y),\beta_A(y)\}\\ &= \min\{1-\beta_A(x*y), 1-\beta_A(y)\}\\ &= \min\{\bar{\beta}_A(x*y),\bar{\beta}_A(y)\}.\end{aligned} \tag{3.14}$$

Hence $\bar{\beta}_A$ is a fuzzy ideal of X.

Conversely, assume that α_A and $\bar{\beta}_A$ are fuzzy ideals of X. For every $x, y \in X$, we get

$$\alpha_A(0) \geq \alpha_A(x), \qquad 1-\beta_A(0) = \bar{\beta}_A(0) \geq \bar{\beta}_A(x) = 1-\beta_A(x), \tag{3.15}$$

that is, $\beta_A(0) \leq \beta_A(x)$; $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\}$ and

$$\begin{aligned}1-\beta_A(x) = \bar{\beta}_A(x) &\geq \min\{\bar{\beta}_A(x*y),\bar{\beta}_A(y)\}\\ &= \min\{1-\beta_A(x*y), 1-\beta_A(y)\}\\ &= 1-\max\{\beta_A(x*y),\beta_A(y)\},\end{aligned} \tag{3.16}$$

that is, $\beta_A(x) \leq \max\{\beta_A(x*y), \beta_A(y)\}$. Hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X. □

Theorem 3.12. *Let* $A = (\alpha_A, \beta_A)$ *be an IFS in* X. *Then* $A = (\alpha_A, \beta_A)$ *is an intuitionistic fuzzy ideal of* X *if and only if* $\Box A = (\alpha_A, \bar{\alpha}_A)$ *and* $\Diamond A = (\bar{\beta}_A, \beta_A)$ *are intuitionistic fuzzy ideals of* X.

Proof. If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X, then $\alpha_A = \bar{\bar{\alpha}}_A$ and β_A are fuzzy ideals of X from Lemma 3.11, hence $\Box A = (\alpha_A, \bar{\alpha}_A)$ and $\Diamond A = (\bar{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of X. Conversely, if $\Box A = (\alpha_A, \bar{\alpha}_A)$ and $\Diamond A = (\bar{\beta}_A, \beta_A)$ are intuitionistic fuzzy ideals of X, then the fuzzy sets α_A and $\bar{\beta}_A$ are fuzzy ideals of X, hence $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X. □

For any $t \in [0,1]$ and a fuzzy set μ in a nonempty set X, the set

$$U(\mu;t) = \{x \in X \mid \mu(x) \geq t\} \tag{3.17}$$

is called an *upper* t*-level cut* of μ and the set

$$L(\mu;t) = \{x \in X \mid \mu(x) \leq t\} \tag{3.18}$$

is called a *lower* t*-level cut* of μ.

Theorem 3.13. *An* IFS $A = (\alpha_A, \beta_A)$ *is an intuitionistic fuzzy ideal of* X *if and only if for all* $s, t \in [0,1]$, *the sets* $U(\alpha_A;t)$ *and* $L(\beta_A;s)$ *are either empty or ideals of* X.

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of X and $U(\alpha_A;t) \neq \emptyset \neq L(\beta_A;s)$ for any $s,t \in [0,1]$. It is clear that $0 \in U(\alpha_A;t) \cap L(\beta_A;s)$ since $\alpha_A(0) \geq t$ and $\beta_A(0) \leq s$. Let $x,y \in X$ be such that $x * y \in U(\alpha_A;t)$ and $y \in U(\alpha_A;t)$. Then $\alpha_A(x*y) \geq t$ and $\alpha_A(y) \geq t$. It follows that

$$\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\} \geq t \tag{3.19}$$

so that $x \in U(\alpha_A;t)$. Hence $U(\alpha_A;t)$ is an ideal of X. Now let $x,y \in X$ be such that $x*y \in L(\beta_A;s)$ and $y \in L(\beta_A;s)$. Then $\beta_A(x*y) \leq s$ and $\beta_A(y) \leq s$, which imply that

$$\beta_A(x) \leq \max\{\beta_A(x*y), \beta_A(y)\} \leq s. \tag{3.20}$$

Thus $x \in L(\beta_A;s)$, and therefore $L(\beta_A;s)$ is an ideal of X. Conversely, assume that for each $t,s \in [0,1]$, the sets $U(\alpha_A;t)$ and $L(\beta_A;s)$ are either empty or ideals of X. For any $x \in X$, let $\alpha_A(x) = t$ and $\beta_A(x) = s$. Then $x \in U(\alpha_A;t) \cap L(\beta_A;s)$, and so $U(\alpha_A;t) \neq \emptyset \neq L(\beta_A;s)$. Since $U(\alpha_A;t)$ and $L(\beta_A;s)$ are ideals of X, therefore $0 \in U(\alpha_A;t) \cap L(\beta_A;s)$. Hence $\alpha_A(0) \geq t = \alpha_A(x)$ and $\beta_A(0) \leq s = \beta_A(x)$ for all $x \in X$. If there exist $x', y' \in X$ such that $\alpha_A(x') < \min\{\alpha_A(x'*y'), \alpha_A(y')\}$, then by taking

$$t_0 = \frac{1}{2}(\alpha_A(x') + \min\{\alpha_A(x'*y'), \alpha_A(y')\}), \tag{3.21}$$

we have

$$\alpha_A(x') < t_0 < \min\{\alpha_A(x'*y'), \alpha_A(y')\}. \tag{3.22}$$

Hence $x' \notin U(\alpha_A;t_0)$, $x'*y' \in U(\alpha_A;t_0)$ and $y' \in (\alpha_A;t_0)$, that is, $U(\alpha_A;t_0)$ is not an ideal of X, which is a contradiction. Finally, assume that there exist $a,b \in X$ such that

$$\beta_A(a) > \max\{\beta_A(a*b), \beta_A(b)\}. \tag{3.23}$$

Taking $s_0 := (1/2)(\beta_A(a) + \max\{\beta_A(a*b), \beta_A(b)\})$, then

$$\max\{\beta_A(a*b), \beta_A(b)\} < s_0 < \beta_A(a). \tag{3.24}$$

Therefore $a*b \in L(\beta_A;s_0)$ and $b \in L(\beta_A;s_0)$, but $a \notin L(\beta_A;s_0)$, which is a contradiction, this completes the proof. □

Let Λ be a nonempty subset of $[0,1]$.

Theorem 3.14. *Let $\{I_t \mid t \in \Lambda\}$ be a collection of ideals of X such that*
(i) $X = \cup_{t\in\Lambda} I_t$,
(ii) $s > t$ *if and only if* $I_s \subset I_t$ *for all* $s,t \in \Lambda$.
Then an IFS $A = (\alpha_A, \beta_A)$ *in X defined by*

$$\alpha_A(x) := \sup\{t \in \Lambda \mid x \in I_t\}, \qquad \beta_A(x) := \inf\{t \in \Lambda \mid x \in I_t\} \tag{3.25}$$

for all $x \in X$ is an intuitionistic fuzzy ideal of X.

Proof. According to Theorem 3.13, it is sufficient to show that $U(\alpha_A;t)$ and $L(\beta_A;s)$ are ideals of X for every $t \in [0, \alpha_A(0)]$ and $s \in [\beta_A(0), 1]$. In order to prove that $U(\alpha_A;t)$ is an ideal of X, we divide the proof into the following two cases:

(i) $t = \sup\{q \in \Lambda \mid q < t\}$,

(ii) $t \neq \sup\{q \in \Lambda \mid q < t\}$.

Case (i) implies that

$$x \in U(\alpha_A;t) \Longleftrightarrow x \in I_q \quad \forall q < t \Longleftrightarrow x \in \cap_{q<t} I_q, \tag{3.26}$$

so that $U(\alpha_A;t) = \cap_{q<t} I_q$, which is an ideal of X. For the case (ii), we claim that $U(\alpha_A;t) = \cup_{q\geq t} I_q$. If $x \in \cup_{q\geq t} I_q$, then $x \in I_q$ for some $q \geq t$. It follows that $\alpha_A(x) \geq q \geq t$, so that $x \in U(\alpha_A;t)$. This shows that $\cup_{q\geq t} I_q \subseteq U(\alpha_A;t)$. Now assume that $x \notin \cup_{q\geq t} I_q$. Then $x \notin I_q$ for all $q \geq t$. Since $t \neq \sup\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin I_q$ for all $q > t - \varepsilon$, which means that if $x \in I_q$, then $q \leq t - \varepsilon$. Thus $\alpha_A(x) \leq t - \varepsilon < t$, and so $x \notin U(\alpha_A;t)$. Therefore $U(\alpha_A;t) \subseteq \cup_{q\geq t} I_q$, and thus $U(\alpha_A;t) = \cup_{q\geq t} I_q$ which is an ideal of X. Next we prove that $L(\beta_A;s)$ is an ideal of X. We consider the following two cases:

(iii) $s = \inf\{r \in \Lambda \mid s < r\}$,

(iv) $s \neq \inf\{r \in \Lambda \mid s < r\}$.

For the case (iii), we have

$$x \in L(\beta_A;s) \Longleftrightarrow x \in I_r \quad \forall s < r \Longleftrightarrow x \in \cap_{s<r} I_r, \tag{3.27}$$

and hence $L(\beta_A;s) = \cap_{s<r} I_r$ which is an ideal of X. For the case (iv) there exists $\varepsilon > 0$ such that $(s, s+\varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\beta_A;s) = \cup_{s\geq r} I_r$. If $x \in \cup_{s\geq r} I_r$, then $x \in I_r$ for some $r \leq s$. It follows that $\beta_A(x) \leq r \leq s$ so that $x \in L(\beta_A;s)$. Hence $\cup_{s\geq r} I_r \subseteq L(\beta_A;s)$. Conversely, if $x \notin \cup_{s\geq r} I_r$, then $x \notin I_r$ for all $r \leq s$, which implies that $x \notin I_r$ for all $r < s + \varepsilon$, that is, if $x \in I_r$, then $r \geq s + \varepsilon$. Thus $\beta_A(x) \geq s + \varepsilon > s$, that is, $x \notin L(\beta_A;s)$. Therefore $L(\beta_A;s) \subseteq \cup_{s\geq r} I_r$ and consequently $L(\beta_A;s) = \cup_{s\geq r} I_r$ which is an ideal of X. This completes the proof. □

A mapping $f : X \to Y$ of BCK-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCK-algebras, then $f(0) = 0$. Let $f : X \to Y$ be a homomorphism of BCK-algebras. For any IFS $A = (\alpha_A, \beta_A)$ in Y, we define a new IFS $A^f = (\alpha_A^f, \beta_A^f)$ in X by

$$\alpha_A^f(x) := \alpha_A(f(x)), \quad \beta_A^f(x) := \beta_A(f(x)) \quad \forall x \in X. \tag{3.28}$$

Theorem 3.15. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If an* IFS $A = (\alpha_A, \beta_A)$ *in Y is an intuitionistic fuzzy ideal of Y, then an* IFS $A^f = (\alpha_A^f, \beta_A^f)$ *in X is an intuitionistic fuzzy ideal of X.*

Proof. We first have that

$$\begin{aligned} \alpha_A^f(x) &= \alpha_A(f(x)) \leq \alpha_A(0) = \alpha_A(f(0)) = \alpha_A^f(0), \\ \beta_A^f(x) &= \beta_A(f(x)) \geq \beta_A(0) = \beta_A(f(0)) = \beta_A^f(0) \end{aligned} \tag{3.29}$$

for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned}
\min\{\alpha_A^f(x*y),\alpha_A^f(y)\} &= \min\{\alpha_A(f(x*y)),\alpha_A(f(y))\}\\
&= \min\{\alpha_A(f(x)*f(y)),\alpha_A(f(y))\}\\
&\le \alpha_A(f(x)) = \alpha_A^f(x),\\
\max\{\beta_A^f(x*y),\beta_A^f(y)\} &= \max\{\beta_A(f(x*y)),\beta_A(f(y))\}\\
&= \max\{\beta_A(f(x)*f(y)),\beta_A(f(y))\}\\
&\ge \beta_A(f(x)) = \beta_A^f(x).
\end{aligned} \tag{3.30}$$

Hence $A^f = (\alpha_A^f, \beta_A^f)$ is an intuitionistic fuzzy ideal of X. □

If we strengthen the condition of f, then we can construct the converse of Theorem 3.15 as follows.

Theorem 3.16. *Let $f : X \to Y$ be an epimorphism of BCK-algebras and let $A = (\alpha_A, \beta_A)$ be an IFS in Y. If $A^f = (\alpha_A^f, \beta_A^f)$ is an intuitionistic fuzzy ideal of X, then $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of Y.*

Proof. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then

$$\begin{aligned}
\alpha_A(x) &= \alpha_A(f(a)) = \alpha_A^f(a) \le \alpha_A^f(0) = \alpha_A(f(0)) = \alpha_A(0),\\
\beta_A(x) &= \beta_A(f(a)) = \beta_A^f(a) \ge \beta_A^f(0) = \beta_A(f(0)) = \beta_A(0).
\end{aligned} \tag{3.31}$$

Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follows that

$$\begin{aligned}
\alpha_A(x) &= \alpha_A(f(a)) = \alpha_A^f(a)\\
&\ge \min\{\alpha_A^f(a*b),\alpha_A^f(b)\}\\
&= \min\{\alpha_A(f(a*b)),\alpha_A(f(b))\}\\
&= \min\{\alpha_A(f(a)*f(b)),\alpha_A(f(b))\}\\
&= \min\{\alpha_A(x*y),\alpha_A(y)\},\\
\beta_A(x) &= \beta_A(f(a)) = \beta_A^f(a)\\
&\le \max\{\beta_A^f(a*b),\beta_A^f(b)\}\\
&= \max\{\beta_A(f(a*b)),\beta_A(f(b))\}\\
&= \max\{\beta_A(f(a)*f(b)),\beta_A(f(b))\}\\
&= \max\{\beta_A(x*y),\beta_A(y)\}.
\end{aligned} \tag{3.32}$$

This completes the proof. □

Let $\mathrm{IF}(X)$ be the family of all intuitionistic fuzzy ideals of X and let $t \in [0,1]$. Define binary relations U^t and L^t on $\mathrm{IF}(X)$ as follows:

$$(A,B) \in U^t \Longleftrightarrow U(\alpha_A;t) = U(\alpha_B;t), \qquad (A,B) \in L^t \Longleftrightarrow L(\beta_A;t) = L(\beta_B;t), \tag{3.33}$$

respectively, for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in $\mathrm{IF}(X)$. Then clearly U^t and L^t are equivalence relations on $\mathrm{IF}(X)$. For any $A = (\alpha_A, \beta_A) \in \mathrm{IF}(X)$, let $[A]_{U^t}$ (respectively, $[A]_{L^t}$) denote the equivalence class of A modulo U^t (respectively, L^t), and denote by $\mathrm{IF}(X)/U^t$ (respectively, $\mathrm{IF}(X)/L^t$) the system of all equivalence classes modulo U^t (respectively, L^t); so

$$\mathrm{IF}(X)/U^t := \{[A]_{U^t} \mid A = (\alpha_A, \beta_A) \in \mathrm{IF}(X)\}, \tag{3.34}$$

respectively,

$$\mathrm{IF}(X)/L^t := \{[A]_{L^t} \mid A = (\alpha_A, \beta_A) \in \mathrm{IF}(X)\}. \tag{3.35}$$

Now let $I(X)$ denote the family of all ideals of X and let $t \in [0,1]$. Define maps f_t and g_t from $\mathrm{IF}(X)$ to $I(X) \cup \{\varnothing\}$ by $f_t(A) = U(\alpha_A; t)$ and $g_t(A) = L(\beta_A; t)$, respectively, for all $A = (\alpha_A, \beta_A) \in \mathrm{IF}(X)$. Then f_t and g_t are clearly well defined.

Theorem 3.17. *For any $t \in (0,1)$ the maps f_t and g_t are surjective from* $\mathrm{IF}(X)$ *to* $I(X) \cup \{\varnothing\}$.

Proof. Let $t \in (0,1)$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in $\mathrm{IF}(X)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in X defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in X$. Obviously $f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \varnothing = L(\mathbf{1}; t) = g_t(\mathbf{0}_\sim)$. Let $G(\neq \varnothing) \in I(X)$. For $G_\sim = (\chi_G, \bar{\chi}_G) \in \mathrm{IF}(X)$, we have $f_t(G_\sim) = U(\chi_G; t) = G$ and $g_t(G_\sim) = L(\bar{\chi}_G; t) = G$. Hence f_t and g_t are surjective. □

Theorem 3.18. *The quotient sets* $\mathrm{IF}(X)/U^t$ *and* $\mathrm{IF}(X)/L^t$ *are equipotent to* $I(X) \cup \{\varnothing\}$ *for every* $t \in (0,1)$.

Proof. For $t \in (0,1)$ let f_t^* (respectively, g_t^*) be a map from $\mathrm{IF}(X)/U^t$ (respectively, $\mathrm{IF}(X)/L^t$) to $I(X) \cup \{\varnothing\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (respectively, $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\alpha_A, \beta_A) \in \mathrm{IF}(X)$. If $U(\alpha_A; t) = U(\alpha_B; t)$ and $L(\beta_A; t) = L(\beta_B; t)$ for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in $\mathrm{IF}(X)$, then $(A, B) \in U^t$ and $(A, B) \in L^t$; hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore the maps f_t^* and g_t^* are injective. Now let $G(\neq \varnothing) \in I(X)$. For $G_\sim = (\chi_G, \bar{\chi}_G) \in \mathrm{IF}(X)$, we have

$$\begin{aligned} f_t^*([G_\sim]_{U^t}) &= f_t(G_\sim) = U(\chi_G; t) = G, \\ g_t^*([G_\sim]_{L^t}) &= g_t(G_\sim) = L(\bar{\chi}_G; t) = G. \end{aligned} \tag{3.36}$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathrm{IF}(X)$ we get

$$\begin{aligned} f_t^*([\mathbf{0}_\sim]_{U^t}) &= f_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) = \varnothing, \\ g_t^*([\mathbf{0}_\sim]_{L^t}) &= g_t(\mathbf{0}_\sim) = L(\mathbf{0}; t) = \varnothing. \end{aligned} \tag{3.37}$$

This shows that f_t^* and g_t^* are surjective. This completes the proof. □

For any $t \in [0,1]$, we define another relation R^t on $\mathrm{IF}(X)$ as follows:

$$(A, B) \in R^t \Longleftrightarrow U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t) \tag{3.38}$$

for any $A = (\alpha_A, \beta_A)$, $B = (\alpha_B, \beta_B) \in \mathrm{IF}(X)$. Then the relation R^t is also an equivalence relation on $\mathrm{IF}(X)$.

Theorem 3.19. *For any* $t \in (0,1)$*, the map* $\phi_t : \mathrm{IF}(X) \to I(X) \cup \{\emptyset\}$ *defined by* $\phi_t(A) = f_t(A) \cap g_t(A)$ *for each* $A = (\alpha_A, \beta_A) \in \mathrm{IF}(X)$ *is surjective.*

Proof. Let $t \in (0,1)$. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathrm{IF}(X)$,

$$\phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) \cap L(\mathbf{1}; t) = \emptyset. \tag{3.39}$$

For any $H \in \mathrm{IF}(X)$, there exists $H_\sim = (\chi_H, \bar{\chi}_H) \in \mathrm{IF}(X)$ such that

$$\phi_t(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H; t) \cap L(\bar{\chi}_H; t) = H. \tag{3.40}$$

This completes the proof. □

Theorem 3.20. *For any* $t \in (0,1)$*, the quotient set* $\mathrm{IF}(X)/R^t$ *is equipotent to* $I(X) \cup \{\emptyset\}$*.*

Proof. Let $t \in (0,1)$ and let $\phi_t^* : \mathrm{IF}(X)/R^t \to I(X) \cup \{\emptyset\}$ be a map defined by $\phi_t^*([A]_{R^t}) = \phi_t(A)$ for all $[A]_{R^t} \in \mathrm{IF}(X)/R^t$. If $\phi_t^*([A]_{R^t}) = \phi_t^*([B]_{R^t})$ for any $[A]_{R^t}$, $[B]_{R^t} \in \mathrm{IF}(X)/R^t$, then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is, $U(\alpha_A; t) \cap L(\beta_A; t) = U(\alpha_B; t) \cap L(\beta_B; t)$, hence $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that ϕ_t^* is injective. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \mathrm{IF}(X)$,

$$\phi_t^*([\mathbf{0}_\sim]_{R^t}) = \phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0}; t) \cap L(\mathbf{1}; t) = \emptyset. \tag{3.41}$$

If $H \in \mathrm{IF}(X)$, then for $H_\sim = (\chi_H, \bar{\chi}_H) \in \mathrm{IF}(X)$, we have

$$\phi_t^*([H_\sim]_{R^t}) = \phi(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H; t) \cap L(\bar{\chi}_H; t) = H. \tag{3.42}$$

Hence ϕ_t^* is surjective, this completes the proof. □

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