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ON CERTAIN SUFFICIENT CONDITIONS FOR STARLIKENESS

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ABSTRACT. We consider certain properties of $f(z)f''(z)/f'^2(z)$ as a sufficient condition for starlikeness.

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1. Introduction and preliminaries. Let A denote the class of functions $f(z)$ which are analytic in the unit disc $U = \{z : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$.

For a function $f(z) \in A$ we say that it is *starlike* in the unit disc U if and only if

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0 \tag{1.1}$$

for all $z \in U$. We denote by S^* the class of all such functions. We denote by K the class of *convex* functions in the unit disc U, i.e., the class of univalent functions $f(z) \in A$ for which

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0, \tag{1.2}$$

for all $z \in U$.

Both of the above mentioned classes are subclasses of univalent functions in U and more $K \subset S^*$ ([1, 2]).

Let $f(z)$ and $g(z)$ be analytic in the unit disc. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if $g(z)$ is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In this paper, we use the method of differential subordinations. The general theory of differential subordinations introduced by Miler and Mocanu is given in [5]. Namely, if $\phi : C^2 \to C$ (where C is the complex plane) is analytic in domain D, if $h(z)$ is univalent in U, and if $p(z)$ is analytic in U with $(p(z), zp'(z)) \in D$ when $z \in U$, then we say that $p(z)$ satisfies a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1.3}$$

We say that the univalent function $q(z)$ is *dominant* of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). If $\tilde{q}(z)$ is a dominant of (1.3) and $\tilde{q}(z) \prec q(z)$ for all dominants of (1.3), then we say that $\tilde{q}(z)$ is the *best dominant* of the differential subordination (1.3).

In the following section, we need the following lemma of Miller and Mocanu [6].

Lemma 1.1 [6]. *Let $q(z)$ be univalent in the unit disc U, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(U)$, with $\phi(\omega) \neq 0$ when $\omega \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

(i) *$Q(z)$ is starlike in the unit disc U,*

(ii) $\operatorname{Re}\{z(h'(z)/Q(z))\} = \operatorname{Re}\{\theta'(q(z))/\phi(q(z)) + z(Q'(z)/Q(z))\} > 0$, $z \in U$.

If $p(z)$ is analytic in U, with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{1.4}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.4).

Even more we need the following lemma, which in more general form is due to Hallenbeck and Ruscheweyh [3].

Lemma 1.2 [3]. *Let $G(z)$ be a convex univalent in U, $G(0) = 1$. Let $F(z)$ be analytic in U, $F(0) = 1$ and let $F(z) \prec G(z)$ in U. Then for all $n \in \mathbf{N}_0$*

$$(n+1)z^{-n-1}\int_0^z t^n F(t)\,dt \prec (n+1)z^{-n-1}\int_0^z t^n G(t)\,dt. \tag{1.5}$$

2. Main results and consequences.

In this part, we use Lemmas 1.1 and 1.2 to obtain some conditions for $f(z)f''(z)/f'^2(z)$ which lead to starlikeness.

Theorem 2.1. *If $f \in A$ and*

$$\frac{f(z)f''(z)}{f'^2(z)} \prec 2 - \frac{2}{(1-z)^2} = h(z) \tag{2.1}$$

then $f \in S^*$.

Proof. We choose $p(z) = z(f'(z)/f(z))$; $q(z) = (1-z)/(1+z)$; $\phi(\omega) = 1/\omega^2$; $\theta(\omega) = 1 - (1/\omega)$. Then $q(z)$ is univalent in U; $\theta(\omega)$ and $\phi(\omega)$ are analytic with domain $D = \mathbf{C}\backslash\{0\}$ which contains $q(U) = \{z : \operatorname{Re}(z) > 0\}$ and $\phi(\omega) \neq 0$ when $\omega \in q(U)$. Further

$$Q(z) = zq'(z)\phi(q(z)) = -\frac{2z}{(1-z)^2} \tag{2.2}$$

is starlike in U, and for the function

$$h(z) = \theta(q(z)) + Q(z) = \frac{2z(z-2)}{(1-z)^2} = 2 - \frac{2}{(1-z)^2} \tag{2.3}$$

we have

$$\operatorname{Re}\left\{z\frac{h'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{2}{1-z}\right\} > 0, \quad z \in U. \tag{2.4}$$

Also, p is analytic in U, $p(0) = q(0) = 1$ and $p(U) \subset D$ because $0 \notin p(U)$. Therefore the conditions of Lemma 1.1 are satisfied and we obtain that if

$$\theta(p(z)) + zp'(z)\phi(p(z)) = \frac{f(z)f''(z)}{f'^2(z)} \prec 2 - \frac{2}{(1-z)^2} = h(z) \tag{2.5}$$

then

$$\frac{zf'(z)}{f(z)} = p(z) \prec q(z) = \frac{1-z}{1+z}, \tag{2.6}$$

i.e., $f \in S^*$. □

Example 2.2. The function $f(z) = z - z^2/2$ belongs to the class A and $f(z)f''(z)/f'^2(z) = 1/2 - (1-z)^2/2$ is subordinated to $2 - 2/(1-z)^2$. So, from Theorem 2.1 $f \in S^*$. Obtaining starlikeness from $zf'(z)/f(z) = (2-2z)/(2-z)$ needs one step more.

Corollary 2.3. *Let $f \in A$.*

(i) *Let $D = \{z : \operatorname{Re} z < 1.5\} \cup \{z : \operatorname{Re} z \geq 1.5,\ |\operatorname{Im} z| > \sqrt{-3+2\operatorname{Re} z}\}$. If $f(z)f''(z)/f'^2(z) \in D$, $z \in U$, then $f \in S^*$;*

(ii) *if $\operatorname{Re}\{f(z)f''(z)/f'^2(z)\} < 3/2$, $z \in U$, then $f \in S^*$;*

(iii) *if $|f(z)f''(z)/f'^2(z)| < 3/2$, $z \in U$, then $f \in S^*$.*

Proof. (i) We have that $f(z)f''(z)/f'^2(z)$ and $h(z)$ defined by (2.1) are analytic in U; $f(0)f''(0)/f'^2(0) = h(0) = 0$ and $h(z)$ is univalent in U (it is one to one mapping because only one of the points $1+\sqrt{2/(2-\omega)}$ is in U). So, we get that (2.1) is equivalent with

$$\frac{f(z)f''(z)}{f'^2(z)} \in h(U), \quad z \in U, \tag{2.7}$$

and it is enough to prove that $h(U) = D$. After some transformations we obtain

$$|h(e^{i\theta}) - 2| = \frac{1}{2\sin^2\theta/2}, \qquad \arg\{h(e^{i\theta}) - 2\} = -\theta, \tag{2.8}$$

i.e.,

$$\operatorname{Re}\{h(e^{i\theta})\} - 2 = \frac{1}{2}\left(ctg^2\frac{\theta}{2} - 1\right), \qquad \operatorname{Im}\{h(e^{i\theta})\} = -\,ctg\,\frac{\theta}{2}. \tag{2.9}$$

So

$$\operatorname{Im}\{h(e^{i\theta})\} = \pm\sqrt{-3+2\operatorname{Re} h(e^{i\theta})} \tag{2.10}$$

and because of $h(0) = 0 < 3/2$ we can say that $h(U) = D$. Parts (ii) and (iii) follow directly from (i). □

Example 2.4. The function $f(z) = 1 - e^{-z}$ is in A and the real part of $f(z)f''(z)/f'^2(z) = 1 - e^z$ is smaller than $3/2$ for all $z \in U$. So $f(z)$ is starlike according to Corollary 2.3(ii). It have been more complicated to realize it from $zf'(z)/f(z) = z/(e^z - 1)$.

Now, using Lemma 1.2 we prove a theorem which we used to improve the results from Corollary 2.3(ii) and (iii) and to obtain some other results.

Theorem 2.5. *Let $f \in A$. If $f(z)f''(z)/f'^2(z) \prec h(z)$, $h(0) = 0$ and $h(z)$ is a convex univalent in U then*

$$\frac{f(z)}{zf'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)\,dt. \tag{2.11}$$

Proof. Let $F(z) = (f(z)/f'(z))' = 1 - f(z)f''(z)/f'^2(z)$ and $G(z) = 1 - h(z)$, $z \in U$. Then $G(z)$ is a convex univalent in U, $G(0) = 1$; $F(z)$ is analytic in U, $F(0) = 1$. Further we have that

$$1 - \frac{f(z)f''(z)}{f'^2(Z)} = F(z) \prec G(z) = 1 - h(z). \tag{2.12}$$

Therefore the conditions of Lemma 1.2 are satisfied and for $n = 0$ we obtain

$$\frac{1}{z}\int_0^z F(t)\,dt \prec \frac{1}{z}\int_0^z G(t)\,dt. \tag{2.13}$$

If we apply the definitions of $F(z)$ and $G(z)$ in the result above and use the following fact which is true because $F(z)$ is analytic

$$\int_0^z \left(\frac{f(t)}{f'(t)}\right)' dt = \frac{f(z)}{f'(z)} - \frac{f(0)}{f'(0)} = \frac{f(z)}{f'(z)}, \tag{2.14}$$

we obtain that

$$\frac{f(z)}{zf'(z)} \prec \frac{1}{z}\int_0^z (1 - h(t))\,dt = 1 - \frac{1}{z}\int_0^z h(t)\,dt. \tag{2.15}$$

□

Remark 2.6. If $h(z)$ is convex, from [4], $1 - (1/z)\int_0^z h(t)\,dt$ is also convex.

In the following corollaries, we deliver some interesting results using Theorem 2.5.

Corollary 2.7. *Let $f \in A$. If $|f(z)f''(z)/f'^2(z)| < 2$ then $f \in S^*$.*

Proof. From $|f(z)f''(z)/f'^2(z)| < 2$, $z \in U$, because $h(z) = 2z$ is univalent and $f(0)f''(0)/f'^2(0) = h(0) = 0$ we get that

$$\frac{f(z)f''(z)}{f'^2(z)} \prec 2z = h(z). \tag{2.16}$$

Further, $h(z)$ is convex, so the conditions from Theorem 2.5 are satisfied, and we obtain

$$\frac{f(z)}{zf'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)\,dt = 1 - z, \tag{2.17}$$

i.e.,

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > 0. \tag{2.18}$$

Because of that $\operatorname{Re}\{zf'(z)/f(z)\} > 0$, i.e., $f \in S^*$. □

Remark 2.8. The result from Corollary 2.7 is the same as in [7] (Theorem 1, for $a = 0$ and $b = -1$) and it is better than the result from Corollary 2.3(iii).

Example 2.9. The same function as in Example 2.4, $f(z) = 1 - e^{-z}$, can be used to illustrate Corollary 2.7:

$$\left|\frac{f(z)f''(z)}{f'^2(z)}\right| = |1 - e^z| < |1 - e| < 2, \quad z \in U, \tag{2.19}$$

and $f(z)$ is starlike.

Corollary 2.10. *Let $f \in A$.*

(i) *If $f(z)f''(z)/f'^2(z) \prec 2\alpha z/(1+z) = h(z)$, $0 < \alpha \le 1/2(1-\ln 2)$, then $f \in S^*$.*

(ii) *If $\operatorname{Re}\{f(z)f''(z)/f'^2(z)\} < 1/2(1-\ln 2) = 1.629445\ldots$, $z \in U$, then $f \in S^*$.*

Proof. (i) From $h(0) = 0$ and $h(z)$ is a convex function in the unit disc U, by Theorem 2.5 we get that

$$\frac{f(z)}{zf'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)\,dt = 1 - 2\alpha + 2\alpha\frac{\ln(1+z)}{z} = g(z). \tag{2.20}$$

Now, from

$$\begin{aligned}\operatorname{Re}\{g(z)\} &= 1 - 2\alpha + \frac{2\alpha}{|z|^2}[x\ln|1+z| + y\arg(1+z)],\\ \operatorname{Im}\{g(z)\} &= \frac{2\alpha}{|z|^2}[x\arg(1+z) - y\ln|1+z|],\end{aligned} \tag{2.21}$$

where $z = x + iy$, it follows that $g(U)$ is symmetric with respect to the x-axis. It is also convex (Remark 2.6) and so

$$\operatorname{Re}\{g(z)\} > \min\{g(1), g(-1)\} = g(1) = 1 - 2\alpha + 2\alpha\ln 2 > 0, \quad z \in U. \tag{2.22}$$

Thus, from $f(z)/zf'(z) \prec g(z)$ we get that $\operatorname{Re}\{f(z)/zf'(z)\} > 0$, $z \in U$ and $\operatorname{Re}\{z(f'(z)/f(z))\} > 0$, $z \in U$, i.e., $f \in S^*$.

(ii) $f(z)f''(z)/f'^2(z)$ is analytic in the unit disc U, $h(z)$ is univalent in U and $f(0)f''(0)/f'^2(0) = h(0) = 0$. Therefore the condition from (i)

$$\frac{f(z)f''(z)}{f'^2(z)} \prec \frac{2\alpha z}{1+z} = h(z) \tag{2.23}$$

is equivalent with

$$\frac{f(z)f''(z)}{f'^2(z)} \in h(U), \quad z \in U. \tag{2.24}$$

Now, from $\operatorname{Re}\{h(e^{i\theta})\} = \alpha$ and $h(0) = 0 < \alpha$ we get that $h(z)$ maps the unit disc U into the half plane with real part less than α. So the condition from (i) is equivalent with

$$\operatorname{Re}\left\{\frac{f(z)f''(z)}{f'^2(z)}\right\} < \alpha, \quad z \in U. \tag{2.25}$$

If we put $\alpha = 1/2(1-\ln 2)$ here, using (i) we obtain the statement of (ii). □

Remark 2.11. Because $1/2(1-\ln 2) = 1.629445\cdots > 1.5$, the result from Corollary 2.10(ii) is better than the result from the Corollary 2.3(ii).

Example 2.12. For $f(z) = (1-e^{-2z})/2$ we have that $f \in A$ and $f(z)f''(z)/f'^2(z) = 1 - e^{2z}$. Further for $z = e^{i\theta}$ we get

$$\operatorname{Re}\{1 - e^{2z}\} = 1 - e^{2\cos\theta}\cos(2\sin\theta) \tag{2.26}$$

with maximum value 1.603838... which it attains for $\theta = 1.246054\ldots$, i.e., for the solution of the equation

$$\theta + 2\sin\theta = \pi. \tag{2.27}$$

So from Corollary 2.10(ii) we obtain that $f(z)$ is starlike. Starlikeness of $f(z)$ could not have been derived using Corollary 2.3. Also, because for $z = 1$

$$\left|\frac{f(z)f''(z)}{f'^2(z)}\right| = |1-e^{2z}| > 2, \tag{2.28}$$

we cannot use Corollary 2.7.

In the following corollary, we see what is happening if $f(z)f''(z)/f'^2(z)$, $z \in U$, is in the half plane right from $1/2(1-\ln 2)$.

Corollary 2.13. *Let $f \in A$.*
(i) *If $f(z)f''(z)/f'^2(z) \prec -\ln(1+\alpha z) = h(z)$, $0 < \alpha \le 1$, then $f \in S^*$;*
(ii) *If $\mathrm{Re}\{f(z)f''(z)/f'^2(z)\} \ge a > -\ln 2 = -0.6931\ldots$ and $|\mathrm{Im}\{f(z)f''(z)/f'^2(z)\}| < \arccos 1/(2e^a)$, $z \in U$, then $f \in S^*$.*

Proof. (i) $h(0) = 0$ and $h(z)$ is a univalent function in the unit disc U because $h(z)$ is analytic in U and it is one to one mapping. From $\alpha \le 1$ we get that

$$\mathrm{Re}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} = \mathrm{Re}\left\{\frac{1}{1+\alpha z}\right\} > 0, \quad z \in U, \tag{2.29}$$

i.e., $h(z)$ is a convex function in the unit disc U. Therefore from Theorem 2.5 we obtain

$$\frac{f(z)}{zf'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)\,dt = \left(1 + \frac{1}{\alpha z}\right)\ln(1+\alpha z) = g(z). \tag{2.30}$$

Now, $g(U)$ is symmetric with respect to the x-axis and $g(z)$ is a convex function (Remark 2.6). So for $z \in U$

$$\begin{aligned}\mathrm{Re}\{g(z)\} &> \min\{g(1), g(-1)\}\\ &= \min\left\{\left(1+\frac{1}{\alpha}\right)\ln(1+\alpha), \left(1-\frac{1}{\alpha}\right)\ln(1-\alpha)\right\} \ge 0\end{aligned} \tag{2.31}$$

and from $f(z)/zf'(z) \prec g(z)$ we get that $\mathrm{Re}\{f(z)/zf'(z)\} > 0$, i.e., $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, $z \in U$, and $f \in S^*$.

(ii) $f(z)f''(z)/f'^2(z)$ is analytic in the unit disc U, $h(z)$ is univalent in U and $f(0)f''(0)/f'^2(0) = h(0) = 0$. Therefore the condition from (i), for $\alpha = 1$

$$\frac{f(z)f''(z)}{f'^2(z)} \prec -\ln(1+z) = h_1(z) \tag{2.32}$$

is equivalent to

$$\frac{f(z)f''(z)}{f'^2(z)} \in h_1(U), \quad z \in U. \tag{2.33}$$

Further, $\mathrm{Re}\{h_1(e^{i\theta})\} = -\ln 2\cos(\theta/2)$ and $\mathrm{Im}\{h_1(e^{i\theta})\} = -\arg(1+e^{i\theta}) = -\theta/2$. So, $\mathrm{Re}\{h_1(e^{i\theta})\} \ge a$ for $|\theta| \ge 2\arccos 1/(2e^a)$, and for such θ we get that $|\mathrm{Im}\{h_1(e^{i\theta})\}| = |\theta/2| \ge \arccos 1/(2e^a)$. From here, because $h_1(0) = 0 > -\ln 2$ is on the same side of the curve $h_1(e^{i\theta})$ with a, it follows that (2.32) is true, i.e., $f \in S^*$. □

Example 2.14. The use of Corollary 2.13 can be illustrated with the function $f(z) = \ln(1+z)$. It belongs to the class A and $f(z)f''(z)/f'^2(z) = -\ln(1+z)$, so from Corollary 2.13(i), for $a = 1$, we get that $f \in S^*$. The starlikeness is not obvious from $zf'(z)/f(z) = z/((1+z)\ln(1+z))$.

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