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REGULAR L-FUZZY TOPOLOGICAL SPACES AND THEIR TOPOLOGICAL MODIFICATIONS

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ABSTRACT. For L a continuous lattice with its Scott topology, the functor ι_L makes every regular L-topological space into a regular space and so does the functor ω_L the other way around. This has previously been known to hold in the restrictive class of the so-called weakly induced spaces. The concepts of H-Lindelöfness (á la Hutton compactness) is introduced and characterized in terms of certain filters. Regular H-Lindelöf spaces are shown to be normal.

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1. Introduction. The two functors that provide a working link between the category **TOP**(L) of L-(fuzzy)-topological spaces and **TOP** are the Lowen functors ι_L and ω_L. For a wide class of lattices L's, ι_L is a right adjoint and left inverse of ω_L. Therefore, it is of interest to know how various L-topological invariants behave with respect to these functors.

In this paper, we show that when L is a continuous lattice with its Scott topology then ι_L maps the category **Reg**(L) of L-regular spaces onto the category **Reg** of regular spaces. This improves upon and extends a result of Liu and Luo [6] which showed (with different but equivalent terminology) that ι_L maps weakly induced L-regular spaces to regular spaces (with L a completely distributive lattice with its upper topology). As a consequence, we have that ω_L (**Reg**) consists precisely of L-regular spaces of ω_L(**TOP**). Some generalities about L-regular spaces are included and stated in a slightly more general situation, viz. for L-topologies that admit a certain type of approximating relation. This captures complete L-regularity and zero-dimensionality.

We also introduce the concept of H-Lindelöfness (compatible with compactness in the sense of Hutton [2]) and characterize it in terms of closed filters. Finally, we prove that H-Lindelöf and L-regular spaces are L-normal.

2. Notation and some terminology. All the fuzzy topological concepts that concern us are standard. We nevertheless recall some of them.

Let $L = (L, ')$ be a complete lattice (bottom denoted 0) endowed with an order-reversing involution $'$. Thus L satisfies the de Morgan laws. For X a set, L^X is the set of all maps from X to L (called L-sets). Then $(L^X, ')$ is a complete lattice under pointwisely defined ordering and the order-reversing involution. The de Morgan laws are also inherited by L^X. An L-topology on X is a family of elements of L^X (called open L-sets) such that any supremum and any finite infimum of open L-sets are open. The L-topology of an L-topological space (L-ts) X is denoted $o(X)$. Members of $\kappa(X) = \{k \in L^X : k' \in o(X)\}$ are called closed. For each $a \in L^X$, we let $\operatorname{Int} a = \bigvee\{u \in o(X) : u \le a\}$ and $\overline{a} = (\operatorname{Int}(a'))'$. If X and Y are two L-ts's, then $f : X \to Y$ is continuous if uf (the composition of f and u) is in $o(X)$ whenever $u \in o(Y)$. The weakest L-topology on X making f continuous is denoted by $f^{\leftarrow}(o(Y))$. We say that $S \subset L^X$ generates $o(X)$ if $o(X) = \bigcap\{T : S \subset T, \text{an } L\text{-topology on } X\}$. If $\mathcal{T}$ is a family of L-topologies on X, then the supremum L-topology $\bigvee \mathcal{T}$ is generated by $\bigcup \mathcal{T}$. In particular, $\bigvee_{j\in J} \pi_j^{\leftarrow}(o(X_j))$ is the product L-topology on $\prod_{j\in J} X_j$ (π_j being the jth projection). The set of all restrictions $\{u \mid A : u \in o(X)\}$ is the subspace L-topology on $A \subset X$.

Given $\alpha, \beta \in L$ we let $\alpha \ll \beta$ whenever for any $B \subset L$ with $\beta \le \bigvee B$ there is a finite $B_0 \subset B$ such that $\alpha \le \bigvee B_0$. Then L is called continuous if $\alpha = \bigvee\{\beta \in L : \beta \ll \alpha\}$ for every $\alpha \in L$. We write $\Downarrow\alpha = \{\beta \in L : \beta \ll \alpha\}$ and dually for $\Uparrow\alpha$. Each continuous L has the interpolation property: $\alpha \ll \beta$ implies $\alpha \ll \gamma \ll \beta$ for some $\gamma \in L$. The Scott topology $\sigma(L)$ on a continuous L is one which has $\{\Uparrow\alpha : \alpha \in L\}$ as a base. We write ΣL for $(L, \sigma(L))$ (see [1] for details).

We also recall that L is a frame provided $\alpha \wedge \bigvee B = \bigvee\{\alpha \wedge \beta : \beta \in B\}$ for every $\alpha \in L$ and $B \subset L$.

Given $a \in L^X$ and $\alpha \in L$, we let $[a \gg \alpha] = \{x \in X : a(x) \gg \alpha\}$, $[a \not\le \alpha] = \{x \in X : a(x) \not\le \alpha\}$, etc. The constant member of L^X with value α is denoted α as well, and $\alpha 1_A = \alpha \wedge 1_A$, where 1_A is the characteristic function of $A \subset X$. If $\mathcal{A} \subset L^X$, we let $\mathcal{A}' = \{a' : a \in \mathcal{A}\}$, $\overline{\mathcal{A}} = \{\overline{a} : a \in \mathcal{A}\}$, and similarly for $\operatorname{Int}\mathcal{A}$. We include for record.

Remark 2.1. Let L be a complete lattice and X a nonempty set. The following statements are equivalent:

(1) L is continuous;

(2) $a = \bigvee_{\alpha\in L} \alpha 1_{[a\gg\alpha]}$ for every $a \in L^X$;

(3) $[a \not\le \alpha] = \bigcup_{\beta\not\le\alpha} [a \gg \beta]$ for every $a \in L^X$ and $\alpha \in L$.

3. L-topologies with approximating relation. Let $L = (L, ')$ be a complete lattice. An L-ts X is called *L-regular* [3] if for every $u \in o(X)$ there exists $\mathcal{V} \subset o(X)$ such that $u = \bigvee \mathcal{V}$ and $\overline{v} \le u$ for all $v \in \mathcal{V}$. This is the case if and only if $u = \bigvee \mathcal{V} = \bigvee \overline{\mathcal{V}}$.

It is clear that X is L-regular if and only if for every basic open u one has $u = \bigvee\{v \in o(X) : \overline{v} \le u\}$.

To avoid repetitions of some argument used in [5], we introduced an auxiliary relation $\prec$ on the L-topology $o(X)$ of an L-ts X.

Definition 3.1. Let $\prec$ be a binary relation on $o(X)$ satisfying the following conditions for all $u, v, w_1, w_2 \in o(X)$:

(1) $0 \prec u$;

(2) $v \prec u$ implies $v \le u$;

(3) $w_1 \le v \prec u \le w_2$ implies $w_1 \prec w_2$;

(4) $w_1 \prec u$ and $w_2 \prec u$ imply $w_1 \vee w_2 \prec u$;

(5) $u \prec w_1$ and $u \prec w_2$ imply $u \prec w_1 \wedge w_2$.

We say X is $\prec$*-regular* if for each open u there exists $\mathcal{V} \subset o(X)$ such that $u = \bigvee \mathcal{V}$ and $v \prec u$ for all $v \in \mathcal{V}$.

Examples. (1) X is L-regular if and only if it is $\prec$-regular with $v \prec u$ defined by $\overline{v} \le u$.

(2) X is completely L-regular [3] if and only if it is $\prec$-regular, where $v \prec u$ if and only if $v \le L_1' f \le R_0 f \le u$ for some $f \in C(X, I(L))$; see [5] for details and notice that (4) and (5) of Definition 3.1 require L to be meet-continuous (cf. Section 5).

(3) X is zero-dimensional if and only if it is $\prec$-regular and $v \prec u$, whenever $v \le w \le u$ for some closed and open w (cf. [9]).

Proposition 3.2. *Let L be a complete lattice and let X be any of $\prec$-regular spaces of Example 3. The following hold*

(1) *If $f : Y \to X$ is continuous, then Y is $\prec$-regular with respect to $f^{\leftarrow}(o(X))$.*

(2) *Every subspace of X is $\prec$-regular.*

If L is a frame, then

(3) *$u = \bigvee\{v : v \prec u\}$ for every subbasic open $u \in L^X$.*

(4) *If $\mathcal{T}$ is a family of $\prec$-regular L-topologies on X, then $\bigvee \mathcal{T}$ is $\prec$-regular.*

(5) *$\prec$-regularity is preserved by arbitrary products.*

Proof. The argument given in [5, Remark 2.5 and Lemma 2.3] for the case (2) of Example 3 goes unchanged in the remaining cases. □

Proposition 3.3. *Let L be a continuous lattice. For X an L-topological space, the following are equivalent:*

(1) *X is $\prec$-regular.*

(2) *$u = \bigvee\{v : v \prec u\}$ for every (basic) open u.*

(3) *$[u \gg \alpha] = \bigcup_{v \prec u} [v \gg \alpha]$ for every (basic) open u and $\alpha \in L$.*

(4) *$[u \not\le \alpha] = \bigcup_{v \prec u} [v \not\le \alpha]$ for every (basic) open u and $\alpha \in L$.*

Proof. (1)$\Longrightarrow$(2). Obvious.

(2)$\Longrightarrow$(3). Let $\alpha \ll u(x) = \bigvee\{v(x) : v \prec u\}$. Select $\beta \in L$ such that $\alpha \ll \beta \ll u(x)$. There is a finite family $\mathcal{V} \subset o(X)$ such that $\beta \le (\bigvee \mathcal{V})(x)$ and $w \prec u$ for every $w \in \mathcal{V}$. Put $v = \bigvee \mathcal{V}$. Then $v \prec u$ and $\alpha \ll \beta \le v(x)$. Thus $\alpha \ll v(x)$ with $v \prec u$. This proves the nontrivial inclusion of (3).

(3)$\Longrightarrow$(4). If $u(x) \not\le \alpha$, there is a β such that $\beta \ll u(x)$ and $\beta \not\le \alpha$. By (3), $\beta \ll v(x)$ for some $v \prec u$. Then $v(x) \not\le \alpha$, i.e., $[u \not\le \alpha] \subset \bigcup_{v \prec u} [v \not\le \alpha]$. The reverse inclusion is obvious.

(4)$\Longrightarrow$(1). Let $u \ne 0$. Then $\Delta = \{(x, \beta) \in X \times L : u(x) \not\le \beta\} \ne \emptyset$. For every pair $(x, \beta) \in \Delta$ select $v_{x\beta} \prec u$ such that $v_{x\beta}(x) \not\le \beta$. Clearly, $\bigvee\{v_{x\beta} : (x, \beta) \in \Delta\} \le u$. To show the converse, assume there exists $y \in X$ such that

$$\gamma = \bigvee \{v_{x\beta}(y) : (x, \beta) \in \Delta\} \not\ge u(y). \tag{3.1}$$

Then $(y, \gamma) \in \Delta$, hence $v_{y\gamma}(y) \not\le \gamma$. But from (3.1) we have $v_{x\beta}(y) \le \gamma$ for all $(x, \beta) \in \Delta$, in particular $v_{y\gamma}(y) \le \gamma$, a contradiction. □

Remark 3.4. (1) The proof of (4)$\Longrightarrow$(1) is a complete lattice proof. Since there is a direct and obvious complete lattice argument for (2)$\Longrightarrow$(4), therefore (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(4) hold true for any complete lattice L.

(2) With L a complete chain without elements isolated from below (e.g., with $L = [0,1]$), conditions (3) and (4) coincide. When expressed in terms of fuzzy points (these are L-sets of the form $\alpha 1_{\{x\}}$) and with $v \prec u$ if and only if $\overline{v} \le u$, these conditions become the definitions of fuzzy regularity given by numerous authors, e.g., [10], thereby showing that all those definitions are equivalent to the one of Hutton-Reilly [3].

(3) For L a frame, the open L-set u in conditions (3) and (4) of Proposition 3.3 can be assumed to be in any family that generates the L-topology (on account of Proposition 3.2(3)); cf. [8, Lemma 3(iii)].

Now we show that the regularity axiom of Liu and Luo [6] is equivalent to the L-regularity for any complete L in which primes are order generating. We recall that $p \in L$ is called prime whenever $\alpha \wedge \beta \le p$ implies $\alpha \le p$ or $\beta \le p$. The set of all primes is order generating if $\alpha = \bigwedge\{p \ge \alpha : p \text{ is prime}\}$ for every $\alpha \in L$. The dual concept is that of a coprime element. In our case, i.e., in $(L, ')$, an element $q \in L$ is coprime if and only if q' is prime. We have the following.

Remark 3.5. Let L be a complete lattice in which primes are order generating. For X an L-ts, the following are equivalent:

(1) X is L-regular.

(2) (Liu and Luo [6]) for every $x \in X$, coprime q, and $k \in \kappa(X)$, whenever $k(x) \not\ge q$, there exists $h \in \kappa(X)$ such that $h(x) \not\ge q$ and $k \le \operatorname{Int} h$.

Proof of Remark 3.5(2). Observe that condition (4) of Proposition 3.3 (cf. also Remark 3.4(1)) can be written as follows (with $v \prec u$ if and only if $\overline{v} \le u$) : $[u \not\le p] = \bigcup_{\overline{v} \le u} [v \not\le p]$ for every open u and each prime p. And this is just the dual form of (2). $\square$

4. The topological modifications of L-regular spaces.

The main topic of this paper requires the lattice L to carry a topology such that $C(Y,L)$ is an L-topology for every topological space Y. Among examples of such lattices are the continuous lattices with their Scott topologies.

If L is a continuous lattice, then ΣL is a topological lattice (see [1, Chapter II, Corollary 4.16, Proposition 4.17]). The family $[Y,\Sigma L]$ of all continuous functions from a topological space Y to ΣL is, therefore, closed under finite suprema and finite infima (both formed in L^Y). However, by using the interpolation property of the relation $\ll$, for every $\alpha \in L$ and $\mathcal{U} \subset [Y,\Sigma L]$ one has $[\bigvee \mathcal{U} \gg \alpha] = \bigcup\{[\bigvee \mathcal{V} \gg \alpha] : \mathcal{V} \subset \mathcal{U} \text{ is finite}\}$, an open subset of Y. Thus $[Y,\Sigma L]$ is an L-topology on the set Y. For every topological space Y, $\omega_{\Sigma L} Y$ denotes the set Y provided with the L-topology $[Y,\Sigma L]$. One then says that $\omega_{\Sigma L} Y$ is *topologically generated* from Y.

Now, for X an L-topological space, let $\iota_{\Sigma L} X$ be the topological space with X as the underlying set and with the weak topology generated by $o(X)$ and ΣL, i.e., $\iota_{\Sigma L} X$ has $\bigvee\{u^{\leftarrow}(\sigma(L)) : u \in o(X)\}$ as a topology. It is called the *topological modification* of X.

Then $\omega_{\Sigma L} : \mathbf{TOP} \to \mathbf{TOP}(L)$ and $\iota_{\Sigma L} : \mathbf{TOP}(L) \to \mathbf{TOP}$ (with preservation of mappings) are the Lowen functors (cf. [4, 5]).

We have $o(X) \subset o(\omega_{\Sigma L}\iota_{\Sigma L}X)$ and $\iota_{\Sigma L}\omega_{\Sigma L} = \mathrm{id}_{\mathbf{TOP}}$. Hence $\omega_{\Sigma L}$ is an injection. We also recall that if Y is a topological space, then χY denotes the set Y endowed with the L-topology $\{1_U : U \text{ open in } X\}$. Clearly, $\iota_{\Sigma L}\chi Y = Y$.

Sometimes it may be more convenient to write $(X, \omega_{\Sigma L}(T))$ for the space topologically generated from (X,T), and similarly for $\iota_{\Sigma L}$.

Lemma 4.1. *Let L be a continuous lattice. For every L-regular space X, $\iota_{\Sigma L}X$ is a regular topological space.*

Proof. It suffices to show that every point of an arbitrary subbasic open set of $\iota_{\Sigma L}Y$ has an open neighborhood whose closure is in the set (this is Proposition 3.2(3) with $L = \{0,1\}$). So, let u be open in X, $\alpha \in L$, and let $x \in [u \gg \alpha]$. By Proposition 3.3(3) there is an open v in X such that $\overline{v} \leq u$ and $x \in [v \gg \alpha]$. Select $\gamma \in L$ such that $\alpha \ll \gamma \ll v(x)$. Then

$$x \in [v \gg \gamma] \subset [\overline{v} \geq \gamma] \subset [u \gg \alpha]. \tag{4.1}$$

Now it suffices to note that, by Remark 2.1,

$$[\overline{v} \geq \gamma] = X \backslash [\mathrm{Int}(v') \not\leq \gamma'] = X \backslash \bigcup_{\beta \not\leq \gamma'} [\mathrm{Int}(v') \gg \beta]. \tag{4.2}$$

Thus $[\overline{v} \geq \gamma]$ is closed, hence $\iota_{\Sigma L}X$ is regular.

Now it is more convenient to write (X,T) for an L-ts X with the L-topology T. In [6], (X,T) is said to be *weakly induced* if $1_{[u \not\leq \alpha]} \in T$ for every $u \in T$ and $\alpha \in L$. Let $[T] = \{U \subset X : 1_U \in T\}$. In what follows, we write "L-regular" on account of Remark 3.5. □

Corollary 4.2 [6]. *Let L be completely distributive. If (X,T) is a weakly induced L-regular space, then $(X,[T])$ is regular.*

Proof. First, recall that a completely distributive L is continuous and the sets $\{\beta \in L : \beta \not\leq \alpha\}$ $(\alpha \in L)$ form a subbase for its Scott topology (see [1, e.g., Chapter IV, Exercise 2.31 and Chapter III, Exercise 3.23]). Thus (X,T) is weakly induced if and only if $\iota_{\Sigma L}(T) \subset [T]$. Finally, notice that $[T] \subset \iota_{\Sigma L}(T)$ always since $[1_U \gg \alpha] \in \{\emptyset, U, X\}$ for every $\alpha \in L$. □

Theorem 4.3. *Let L be a continuous lattice. Then the following hold:*

$$\iota_{\Sigma L}\big(\boldsymbol{Reg}(L)\big) = \boldsymbol{Reg}. \tag{4.3}$$

$$\omega_{\Sigma L}(\boldsymbol{Reg}) = \boldsymbol{Reg}(L) \cap w_{\Sigma L}(\boldsymbol{TOP}). \tag{4.4}$$

Proof. (1) That $\iota_{\Sigma L}$ maps $\mathbf{Reg}(L)$ into $\mathbf{Reg}$ is stated in Lemma 4.1. The mapping is onto since for any topological regular $X, \chi X$ is L-regular and $\iota_{\Sigma L}\chi X = X$.

(2) If X is a regular topological space and u is open in $\omega_{\Sigma L}X$, then for every $\alpha \in L$ there is a family $\mathcal{W}_\alpha$ of open subsets of X such that

$$[u \gg \alpha] = \bigcup \mathcal{W}_\alpha = \bigcup \overline{\mathcal{W}}_\alpha. \tag{4.5}$$

By Remark 2.1 and the first equality of (4.5), we obtain

$$\begin{aligned} u &= \bigvee_{\alpha\in L} \alpha 1_{[u\gg\alpha]} = \bigvee_{\alpha\in L}\Big(\alpha\wedge \bigvee_{W\in\mathcal{W}_\alpha} 1_W\Big) \\ &= \bigvee_{\alpha\in L}\bigvee_{W\in\mathcal{W}_\alpha} \alpha 1_W \le \bigvee_{\alpha\in L}\bigvee_{W\in\mathcal{W}_\alpha} \overline{\alpha 1_W}. \end{aligned} \tag{4.6}$$

(Note that there is no distributivity used in arriving at the third equality: always $\alpha\wedge\bigvee B = \bigvee\{\alpha\wedge\beta : \beta\in B\}$ provided $B\subset\{0,1\}$ as is the case above).

Since $\overline{\alpha 1_W}\le \alpha 1_{\overline{W}}$, the same argument shows, by using the second equality of (4.5), that we actually have

$$u = \bigvee_{\alpha\in L}\bigvee_{W\in\mathcal{W}_\alpha} \alpha 1_W = \bigvee_{\alpha\in L}\bigvee_{W\in\mathcal{W}_\alpha} \overline{\alpha 1_W}. \tag{4.7}$$

This shows that $\omega_{\Sigma L}X$ is L-regular.

Conversely, if $\omega_{\Sigma L}X$ is L-regular, then $X = \iota_{\Sigma L}\chi X$ is regular by Lemma 4.1. □

Remark 4.4. (1) Let L be a continuous frame (then it becomes completely distributive on account of the order reversing involution; cf. [1, Chapter I, Theorem 3.15]). Then the inclusion $\omega_{\Sigma L}(\mathbf{Reg})\subset\mathbf{Reg}(L)$ obviously follows from Proposition 3.2(4). Indeed, for X a regular space, the L-topology of $\omega_{\Sigma L}X$ is the supremum of two L-regular L-topologies: the one of χX and the one consisting of all constant L-sets (cf. [5, Proposition 1.5.1(7)]).

(2) The equality (4.4) of Theorem 4.3 is available in [12] with $L=[0,1]$ and in [6] with L completely distributive. Theorem 4.3 is also a supplement to the discussion about regularity in fuzzy topology given in [7].

(3) We recall that an L-ts X is an L-T_3 space if and only if it is L-regular and points of X can be separated by open L-sets. By [5, Remark 8.4], we obtain: $\iota_{\Sigma L}(L\text{-}T_3) = T_3$ and $\omega_{\Sigma L}(T_3) = L\text{-}T_3\cap\omega_{\Sigma L}(\mathbf{TOP})$.

We close this section with some remarks about maximal L-regular spaces. Following [11], we say that X is *maximal* L-regular if the only L-regular L-topology on the set X which is stronger than the original one is L^X (the discrete L-topology).

Proposition 4.5. *Let L be a continuous lattice. Every maximal L-regular space with a nondiscrete topological modification is topologically generated (from a maximal regular space).*

Proof. Let (X,T) be maximal L-regular and let $\iota_{\Sigma L}(T)$ be nondiscrete. We have $T\subset\omega_{\Sigma L}(\iota_{\Sigma L}(T))$ and the latter L-topology is L-regular by Theorem 4.3. Assume $\omega_{\Sigma L}(\iota_{\Sigma L}(T)) = L^X$. Then, by acting with $\iota_{\Sigma L}$, we have $\iota_{\Sigma L}(T) = \iota_{\Sigma L}(L^X)$, a discrete topology. This contradiction shows that $T = \omega_{\Sigma L}(\iota_{\Sigma L}(T))$. Thus (X,T) is topologically generated from $(X,\iota_{\Sigma L}(T))$. The latter space is maximal regular. For, if $\iota_{\Sigma L}(T)\subsetneq S\subsetneq\mathcal{P}(X)$ with S regular, then $T=\omega_{\Sigma L}(\iota_{\Sigma L}(T))\subsetneq\omega_{\Sigma L}(S)\subsetneq\omega_{\Sigma L}(\mathcal{P}(X)) = L^X$. Since $\omega_{\Sigma L}(S)$ is L-regular, this contradicts the maximality of T (recall that $\omega_{\Sigma L}$ is injective). □

Remark 4.6. From the above proof it is clear that Proposition 4.5 can be stated for any topological property $\mathbf{P}$ and any L-topological property L-$\mathbf{P}$ for which there holds a counterpart of Theorem 4.3. This is, for instance, the case of complete L-regularity by [5, Theorem 8.5]. See also Remark 4.4(3).

5. H-Lindelöfness. An L-ts X is called H-Lindelöf if for every $k \in \kappa(X)$, whenever $k \le \bigvee \mathcal{U}$ with $\mathcal{U} \subset o(X)$, there exists a countable subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $k \le \bigvee \mathcal{U}_0$. If $\mathcal{U}_0$ is finite, then X is called H-compact [2]. It is clear that H-Lindelöfness is preserved under continuous surjections. Also, the characterizations of H-compactness in terms of certain filters have their counterparts for H-Lindelöf spaces.

Definition 5.1 (cf. [2])**.** Let $\mathcal{F} \subset L^X$ be nonempty and let $a \in L^X$. We say that:

(1) $\mathcal{F}$ has the countable intersection property relative to a if $\bigwedge \mathcal{F}_0 \not\le a$ for every countable $\mathcal{F}_0 \subset \mathcal{F}$,

(2) $\mathcal{F}$ is a filter if it is closed under finite infima and such that if $f \in \mathcal{F}$ and $f \le a$, then $a \in \mathcal{F}$. (A filter $\mathcal{F}$ is called closed if $\mathcal{F} \subset \kappa(X)$.)

Theorem 5.2. *Let L be a complete lattice and let X be an L-ts. The following are equivalent:*

(1) *X is H-Lindelöf.*

(2) *Every family $\mathcal{K} \subset \kappa(X)$ with the countable intersection property relative to an open u satisfies $\bigwedge \mathcal{K} \not\le u$.*

(3) *Every closed filter $\mathcal{K}$ with the countable intersection property relative to an open u satisfies $\bigwedge \mathcal{K} \not\le u$.*

Proof. (1)$\Rightarrow$(2). Assume $\bigwedge \mathcal{K} \le u$. Then $u' \le \bigvee \mathcal{K}'$ and there is a countable $\mathcal{C} \subset \mathcal{K}'$ such that $u' \le \bigvee \mathcal{C}$, a contradiction with the countable intersection property of $\mathcal{K}$.

(2)$\Rightarrow$(3). Obvious.

(3)$\Rightarrow$(1). Let $k \le \bigvee \mathcal{U}$. Assume that $\mathcal{U}$ does not have a countable subfamily which covers k. Let $\langle \mathcal{U}' \rangle$ be the closed filter generated by $\mathcal{U}'$, i.e., let

$$\langle \mathcal{U}' \rangle = \{f \in \kappa(X) : \exists \text{ finite } \mathcal{C}_f \subset \mathcal{U}' \text{ s.t. } \bigwedge \mathcal{C}_f \le f\}. \tag{5.1}$$

We claim that $\langle \mathcal{U}' \rangle$ has the countable intersection property relative to k'. Suppose that this is not the case. Then for some countable $\mathcal{F} \subset \langle \mathcal{U}' \rangle$ one has $\bigwedge \mathcal{F} \le k'$. Thus

$$k \le \bigvee \mathcal{F}' \le \bigvee_{f \in \mathcal{F}} \Big(\bigwedge \mathcal{C}_f\Big)' = \bigvee \Big(\bigcup_{f \in \mathcal{F}} \mathcal{C}'_f\Big) \tag{5.2}$$

and $\bigcup_{f \in \mathcal{F}} \mathcal{C}'_f$ is a countable subfamily of $\mathcal{U}$, a contradiction with our assumption about $\mathcal{U}$. Therefore $\langle \mathcal{U}' \rangle$ has the countable intersection property relative to k', i.e., $\bigwedge \langle \mathcal{U}' \rangle \not\le k'$. Hence $k \not\le \bigvee \langle \mathcal{U}' \rangle'$ and since $\bigvee \mathcal{U} \le \bigvee \langle \mathcal{U}' \rangle'$, we conclude that $k \not\le \bigvee \mathcal{U}$. This contradiction completes the proof. □

Remark 5.3. There is no counterpart of Theorem 4.3 for H-Lindelöfness and Lindelöfness:

(1) The set $X = L = [0,1]$ (with $\alpha' = 1 - \alpha$) equipped with the L-topology $[0,1/4]^X \cup \{1_X\}$ is H-Lindelöf (as each open cover of a nonzero closed L-set must contain 1_X), while $\iota_{\Sigma L} X$ is an uncountable discrete space.

(2) An L-ts topologically generated from a Lindelöf space need not be H-Lindelöf. Indeed, let X be an uncountable Lindelöf topological space. Put $L = \mathcal{P}(X)$ with usual complement as its order-reversing involution (note that $\mathcal{P}(X)$ is a continuous lattice). Then the cover of 1_X consisting of all constant L-sets having values $\{x\}$ with $x \in X$

(these are all open in $\omega_{\Sigma L}X$) does not have a countable subcover. Therefore $\omega_{\Sigma L}X$ fails to be H-Lindelöf.

(3) However, if $\omega_{\Sigma L}X$ is H-Lindelöf, then X is Lindelöf. Indeed, χX carries a weaker L-topology than $\omega_{\Sigma L}X$, so that χX is H-Lindelöf, and the latter is equivalent to the statement that X is a Lindelöf space.

(4) All the above discussion applies unchanged to the case of H-compactness and compactness.

It is clear that for any complete L, every H-compact and L-regular space X is L-normal, i.e., whenever $k \le u$ (k is closed and u is open), there exists an open v with $k \le v \le \overline{v} \le u$ [3]. In what follows we show that H-compactness can be replaced by H-Lindelöfness provided L is meet-continuous, i.e., for every $\alpha \in L$ and every directed subset $\mathcal{D} \subset L$ there holds: $\alpha \wedge \bigvee \mathcal{D} = \bigvee \{\alpha \wedge \delta : \delta \in \mathcal{D}\}$. We recall that every continuous L is meet-continuous [1]. Also, on account of the order-reversing involution, the dual law is valid too.

Theorem 5.4. *Let L be a meet-continuous lattice. Then every L-regular and H-Lindelöf space is L-normal.*

Proof. Let k be closed, u be open, and $k \le u$ in an L-regular H-Lindelöf space X. By L-regularity there exist $\mathcal{U} \subset o(X)$ and $\mathcal{K} \subset \kappa(X)$ such that $u = \bigvee \mathcal{U} = \bigvee \overline{\mathcal{U}}$ and $k = \bigwedge \mathcal{K} = \bigwedge \operatorname{Int} \mathcal{K}$ (the latter on account of the de Morgan laws). By H-Lindelöfness, there exist two countable subfamilies $\mathcal{U}_0 \subset \mathcal{U}$ and $\mathcal{K}_0 \subset \mathcal{K}$ such that $k \le \bigvee \mathcal{U}_0$ and (again by the de Morgan laws) $\bigwedge \mathcal{K}_0 \le u$. Thus

$$k \le \bigvee \mathcal{U}_0 \le \bigvee \overline{\mathcal{U}}_0 \qquad \text{and} \qquad k \le \bigwedge \operatorname{Int} \mathcal{K}_0 \le \bigwedge \mathcal{K}_0 \le u. \tag{5.3}$$

The rest of the proof is exactly the same as that of [5, Theorem 9.11] which shows that second countability plus L-regularity implies L-normality. Note that the proof in [5] uses a result holding for L a meet-continuous lattice. □

Remark 5.5. By [5, Lemma 3.7], every second countable L-ts is H-Lindelöf for any complete L. Therefore Theorem 5.4 extends [5, Theorem 9.11].

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References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, 1980. MR 82h:06005. Zbl 452.06001.

[2] B. Hutton, *Products of fuzzy topological spaces*, Topology Appl. **11** (1980), no. 1, 59–67. MR 80m:54009. Zbl 422.54006.

[3] B. Hutton and I. Reilly, *Separation axioms in fuzzy topological spaces*, Fuzzy Sets and Systems **3** (1980), no. 1, 93–104. MR 81k:54008. Zbl 421.54006.

[4] T. Kubiak, *The topological modification of the L-fuzzy unit interval*, Applications of category theory to fuzzy subsets (Linz, 1989), Kluwer Acad. Publ., Dordrecht, 1992, pp. 275–305. MR 93f:54007. Zbl 766.54006.

[5] ______, *On L-Tychonoff spaces*, Fuzzy Sets and Systems **73** (1995), no. 1, 25–53. MR 96g:54013. Zbl 867.54005.

[6] Y. M. Liu and M. K. Luo, *Separations in lattice-valued induced spaces*, Fuzzy Sets and Systems **36** (1990), no. 1, 55–66. MR 91e:54021. Zbl 703.54004.

[7] M. Macho Stadler and M. A. de Prada Vicente, *Strong separation and strong countability in fuzzy topological spaces*, Fuzzy Sets and Systems **43** (1991), no. 1, 95–116. MR 92m:54017. Zbl 760.54004.

[8] Y. W. Peng, *Topological structure of a fuzzy function space—the pointwise convergent topology and compact open topology*, Kexue Tongbao (English Ed.) **29** (1984), no. 3, 289–292. MR 86g:54014. Zbl 551.54006.

[9] L. Pujate and A. Šostak, *On zero-dimensionality in fuzzy topology*, Serdica **16** (1990), no. 3-4, 285–288. MR 91k:54009. Zbl 724.54010.

[10] M. Sarkar, *On fuzzy topological spaces*, J. Math. Anal. Appl. **79** (1981), no. 2, 384–394. MR 82h:54005. Zbl 457.54006.

[11] A. K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc. **122** (1966), 379–398. MR 32#8303. Zbl 139.15905.

[12] G. P. Wang and L. F. Hu, *On induced fuzzy topological spaces*, J. Math. Anal. Appl. **108** (1985), no. 2, 495–506. MR 86j:54014. Zbl 578.54004.

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