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RANDOM TRILINEAR FORMS AND THE SCHUR MULTIPLICATION OF TENSORS

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ABSTRACT. We obtain estimates for the distribution of the norm of the random trilinear form $A:\ell_r^M\times\ell_p^N\times\ell_q^K\to\mathbb{C}$, defined by $A(e_i,e_j,e_k)=a_{ijk}$, where the a_{ijk}'s are uniformly bounded, independent, mean zero random variables. As an application, we make progress on the problem when $\ell_r\check{\otimes}\ell_p\check{\otimes}\ell_q$ is a Banach algebra under the Schur multiplication.

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1. Introduction and notation. We adopt the standard notation $\ell_p^N(1\le p<\infty)$ for the complex vector space $\mathbb{C}^N$ equipped with the norm

$$\|x\|_p:=\left(\sum_{n=1}^N|x_n|^p\right)^{1/p}. \tag{1.1}$$

The usual modifications are made to define ℓ_∞^N and the infinite dimensional sequence spaces $\ell_p(1\le p\le\infty)$. All of these are Banach spaces.

Let $A:\ell_p^N\to\ell_q^M(1\le p,q\le\infty)$ be a linear map and define the operator norm by

$$\|A\|_{p\to q}:=\sup\{\|Ax\|_q:\|x\|_p=1\}. \tag{1.2}$$

The map A can be represented as an $M\times N$ matrix (a_{ij}) with respect to the standard bases. Motivated by problems on absolutely summing operators, Bennett [1] and Bennett, Goodman and Newman [2] obtained estimates for the probability distribution of $\|A\|_{p\to q}$ when the a_{ij}'s are independent, mean zero random variables bounded by (2.3). They showed that, for all $1\le p,q\le\infty$, the expectation $\mathscr{E}(\|A\|_{p\to q})$ is of the same order as the smallest possible value of $\|A\|_{p\to q}$ when all the matrix entries a_{ij} are ±1.

Notice that these results can also be interpreted as estimates for the norms of random bilinear forms. Problems involving the von Neumann inequality led Varopoulos [9] to work with norms of random trilinear forms on ℓ_2^N. His results were extended and refined by Mantero and Tonge [7]. Let $A:\ell_{p_1}^N\times\cdots\times\ell_{p_n}^N\to\mathbb{C}$ be an n-linear form with $A(e_{k_1},\dots,e_{k_n})=a_{k_1,\dots,k_n}$, where the e_k's are the standard unit basis vectors. There is a natural norm

$$\|A\|_{p_1,\dots,p_n}:=\sup\Big\{|A(x_1,\dots,x_n)|:\|x_i\|_{p_i}\le1(1\le i\le n)\Big\}. \tag{1.3}$$

In [7], it was shown that when the $a_{k_1,\dots,k_n}$'s are independent random variables taking the values ±1 with equal probability, the expectation $\mathscr{E}(\|A\|_{p_1,\dots,p_n})$ is of the same order as the smallest conceivable value: the least possible value of $\|A\|_{p_1,\dots,p_n}$ when each $a_{k_1\cdots k_n}$ is ± 1. These results turned out to be useful in the study of Banach algebra structures on the tensor products $\ell_{p_1}\otimes\cdots\otimes\ell_{p_n}$. However, open problems were left, even in the case $n=3$.

We address these problems. They involve the Schur product of tensors $A, B\in\ell_r\otimes\ell_p\otimes\ell_q$ when $A=(a_{ijk})$ and $B=(b_{ijk})$ this is given by

$$A*B := (a_{ijk}b_{ijk}). \tag{1.4}$$

The injective tensor product $\ell_r\check{\otimes}\ell_p\check{\otimes}\ell_q$ is the completion of $\ell_r\otimes\ell_p\otimes\ell_q$ under the norm

$$\|(a_{ijk})\|_{rpq} := \sup\left\{\left|\sum_{ijk}a_{ijk}x_iy_jz_k\right| : \|x\|_{r'}\le 1,\ \|y\|_{p'}\le 1,\ \|z\|_{q'}\le 1\right\}. \tag{1.5}$$

Here, the index p' is the one conjugate to p, that is $1/p+1/p'=1$. Notice that $\|A\|_{rpq}=\|A\|_{r',p',q'}$.

In [7], it was shown that the Schur product extends continuously to $\ell_r\check{\otimes}\ell_p\check{\otimes}\ell_q$ when

(i) the sum of the reciprocals of any two of p,q, and r is at least $3/2$,

(ii) $1\le p,\ q,r\le 2$ and the sum of the reciprocals of all three is at least 2, or

(iii) at least one of p, q, and r is 1 or ∞.

Cases were also identified where the Schur product did not extend continuously to $\ell_r\check{\otimes}\ell_p\check{\otimes}\ell_q$. We extend knowledge of such cases by providing an estimate for the distribution of $\|A\|_{rpq}$ when the a_{ijk}'s are uniformly bounded, independent, mean zero random variables. Our methods build on the techniques of [1, 2, 7].

Although the problem we consider is relevant to many issues in the geometry of Banach spaces or Banach algebras (see, for example, Diestel, Jarchow, and Tonge [4]), we are not aware of any progress in the last few years. Recent work on the Schur product (see, for example, Horn and Johnson [6]) or on random matrices (see, for example, Girko [5]) mostly focuses on other issues. There is one notable exception, namely, the body of work on completely bounded operators. The basic theory can be found in Paulsen [8], and interesting results closely related to the operator algebra theory, developed by Varopoulos [9] and his group appear in Blecher and Le Merdy [3] and references therein. None of this, however, appears to be directly applicable to the problem we treat in this paper.

2. The probabilistic estimate. We consistently use $\mathcal{P}$ to denote probability and $\mathcal{E}$ to denote mathematical expectation.

Proposition 2.1. *Let $1\le p\le 2$ and $2\le q,r<\infty$. Let $A=(a_{ijk})\in\ell_r^M\check{\otimes}\ell_p^N\check{\otimes}\ell_q^K$ and suppose that the a_{ijk}'s are independent, mean zero random variables, and that each $|a_{ijk}|\le 1$. Then there are positive constants C_1 and C_2, independent of M, N, and K, such that*

$$\mathcal{P}\Big(\|A\|_{rpq}^r \ge C_1MN^{(r/p)-(r/2)}+C_2(N+K)N^{(r/p)-1}K^{(r/q)-(2/q)}\Big)<1. \tag{2.1}$$

PROOF. Our argument is an adaptation of the work in [1, 2]. Note that

$$\|A\|_{rpq}^r = \sup\left\{\sum_{i=1}^M \left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r : \|y\|_{p'} \le 1, \|z\|_{q'} \le 1\right\}. \tag{2.2}$$

As in [1] or [2], for any positive λ and any nonzero $(y_j)_{j=1}^N$ and $(z_k)_{k=1}^K$, we have

$$\mathcal{P}\left(\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right| \ge \lambda\right) \le 2\exp\left(-\frac{\lambda^2}{4}\sum_{j,k=1}^{N,K} y_j^2 z_k^2\right). \tag{2.3}$$

If $\mu > 0$, then

$$\begin{aligned}\mathcal{E}\left(\exp\left(\mu\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r\right)\right) &= \int_0^\infty e^{\mu\lambda^r} d\mathcal{P}\left(\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right| \le \lambda\right)\\ &= 1 + \int_0^\infty \mu r\lambda^{r-1}e^{\mu\lambda^r}\mathcal{P}\left(\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right| > \lambda\right)d\lambda,\end{aligned} \tag{2.4}$$

and an application of (2.3) gives

$$\mathcal{E}\left(\exp\left(\mu\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r\right)\right) \le 1 + \int_0^{N^{1/p}K^{1/q}} \mu r\lambda^{r-1}e^{\mu\lambda^r}\cdot 2e^{-\lambda^2/4\sum_{j,k=1}^{N,K} y_j^2 z_k^2}d\lambda. \tag{2.5}$$

Since $2 \le r < \infty$, if $0 \le \mu \le (N^{1/p}K^{1/q})^{2-r}/8\sum_{j,k=1}^{N,K} y_j^2 z_k^2$, we can find a constant C_1, independent of M, N, or K, such that

$$\begin{aligned}\mathcal{E}\left(\exp\left(\mu\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r\right)\right) &\le 1 + 2\mu r\int_0^\infty \lambda^{r-1}e^{-\lambda^2/8\sum_{j,k=1}^{N,K} y_j^2 z_k^2}d\lambda\\ &= 1 + C_1\mu\left(\sum_{j,k=1}^{N,K} y_j^2 z_k^2\right)^{r/2}.\end{aligned} \tag{2.6}$$

Next, applying independence, we obtain

$$\begin{aligned}\mathcal{E}\left(\exp\left(\mu\sum_{i=1}^M\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r\right)\right) &= \prod_{i=1}^M \mathcal{E}\left(\exp\left(\mu\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r\right)\right)\\ &\le \prod_{i=1}^M\left(1 + C_1\mu\left(\sum_{j,k=1}^{N,K} y_j^2 z_k^2\right)^{r/2}\right)\\ &\le \exp\left(C_1M\mu\left(\sum_{j,k=1}^{N,K} y_j^2 z_k^2\right)^{r/2}\right).\end{aligned} \tag{2.7}$$

Consequently, for any $\nu > 0$, we have

$$\mathcal{P}\left(\mu\sum_{i=1}^M\left|\sum_{j,k=1}^{N,K} a_{ijk}y_j z_k\right|^r \ge C_1M\mu\left(\sum_{j,k=1}^{N,K} y_j^2 z_k^2\right)^{r/2} + \nu\right) \le e^{-\nu}. \tag{2.8}$$

Now, if $\|y\|_{p'} \le 1$ and $\|z\|_{q'} \le 1$, then since $1 \le p \le 2 \le q < \infty$, we have $\sum_{j,k=1}^{N,K} y_j^2 z_k^2 \le N^{(2/p)-1}$. Using the result of the entropy argument in the proof [7, Thm. 1], we get

$$\mathcal{P}\left(\|a\|_{rpq}^r \ge C_1 M N^{(r/p)-(r/2)} + \frac{\nu}{\mu}\right) \le e^{D(N+K)} e^{-\nu/2^{(3r+1)}}, \tag{2.9}$$

where D is some positive constant independent of M, N, and K.

Take $\mu = 1/8(N^{1/p}K^{1/q})^{2-r}N^{-(2/p)+1}$ and $\nu = 2^{3r+2}D(N+K)$, and set $C_2 = 2^{3r+5}D$ in (2.9) to get

$$\mathcal{P}\left(\|A\|_{rpq}^r \ge C_1 M N^{(r/p)-(r/2)} + C_2(N+K)(N^{1/p}K^{1/q})^{r-2}N^{(2/p)-1}\right) \le e^{-D(N+K)}. \tag{2.10}$$

Since $e^{-D(N+K)} < 1$ for large N and K, the result follows. □

What we need later is an immediate corollary.

Corollary 2.2. *Let $1 \le p \le 2$ and $2 \le q, r < \infty$. Then there is an $A = (a_{ijk}) \in \ell_r^M \breve{\otimes} \ell_p^N \breve{\otimes} \ell_q^K$, with each $a_{ijk} = \pm 1$, such that*

$$\|A\|_{rpq}^2 < C \max\left(M^{2/r}N^{(2/p)-1}, N^{2/p}K^{(2/q)(1-2/r)}, N^{(2/p)-(2/r)}K^{(2/r)+(2/q)(1-2/r)}\right), \tag{2.11}$$

where C is a positive constant independent of M, N, and K.

The next proposition and its corollary are obtained by making minor adjustments to the arguments above. We present them without proof.

Proposition 2.3. *Let $2 \le p, q, r < \infty$. Let $A = (a_{ijk}) \in \ell_r^M \breve{\otimes} \ell_p^N \breve{\otimes} \ell_q^K$ and suppose that the a_{ijk}'s are independent, mean zero random variables, and that each $|a_{ijk}| \le 1$. Then there are positive constants C_1 and C_2, independent of M, N and K, such that*

$$\mathcal{P}\left(\|A\|_{rpq}^r \ge C_1 M + C_2(N+K)N^{(r/p)-(2/p)}K^{(r/q)-(2/q)}\right) < 1. \tag{2.12}$$

Corollary 2.4. *Let $2 \le p, q, r < \infty$. Then there is an $A = (a_{ijk}) \in \ell_r^M \breve{\otimes} \ell_p^N \breve{\otimes} \ell_q^K$, with each $a_{ijk} = \pm 1$, such that*

$$\|A\|_{rpq}^2 < C\max\left(M^{2/r}, N^{(2/r)+(2/p)(1-2/r)}, K^{(2/q)(1-2/r)}, N^{(2/p)(1-2/r)}K^{(2/r)+(2/q)(1-2/r)}\right), \tag{2.13}$$

where C is a positive constant independent of M, N, and K.

3. Application to the question of the continuity of Schur multiplication. Now, we turn to the problem left unsolved in Mantero and Tonge [7]: under what circumstances is $\ell_r \breve{\otimes} \ell_p \breve{\otimes} \ell_q$ a Banach algebra under Schur multiplication? We give further instances when this is not a Banach algebra. To do this, we use the previous results to show that the following is true for appropriate values of p, q, and r:

For each positive B, it is possible to find integers M, N, and K and an $A = (a_{ijk}) \in \ell_r^M \breve{\otimes} \ell_p^N \breve{\otimes} \ell_q^K$, with each $a_{ijk} = \pm 1$ for which $\|A * A\|_{rpq} > B\|A\|_{rpq}^2$.

For this, it is important to note that, trivially, if $A = (a_{ijk}) \in \ell_r^M \breve{\otimes} \ell_p^N \breve{\otimes} \ell_q^K$ has each $a_{ijk} = \pm 1$, then

$$\|A * A\|_{rpq} = M^{1/r}N^{1/p}K^{1/q}. \tag{3.1}$$

Proposition 3.1. *Let* $1 < \min(p,q) \le 2 \le \max(p,q) < \infty$ *and* $2 \le r < \infty$. *Then* $\ell_r \breve{\otimes} \ell_p \breve{\otimes} \ell_q$ *is not a Banach algebra under Schur multiplication when*

$$\frac{1}{\min(p,q)} < \frac{2}{r} \cdot \frac{1}{\max(p,q)} + \frac{1}{2}. \tag{3.2}$$

Proof. We consider the case where $1 < p \le 2 \le q < \infty$. The other case is similar.

Fix $B > 0$. By (3.1) and Corollary 2.2, it is enough to show that we can find positive integers M, N, and K with

$$\begin{aligned} &M^{1/r} N^{1/p} K^{1/q} \\ &\quad > BC \max\left(M^{2/r} N^{(2/p)-1}, N^{2/p} K^{(2/q)(1-2/r)}, N^{(2/p)-(2/r)} K^{(2/r)+(2/q)(1-2/r)}\right), \end{aligned} \tag{3.3}$$

where C is a fixed positive number, independent of B, M, N, or K. We can achieve this with $N = K = M^t$, where $t > 0$ provided that

$$\begin{aligned} M^{t/q} &> BC M^{(1/r)+t((1/p)-1)}, \\ M^{1/r} &> BC M^{t((1/p)+(1/q)-(4/qr))}. \end{aligned} \tag{3.4}$$

Inequalities (3.4) hold simultaneously for large M if there is a $t > 0$ that satisfies

$$t\left(\frac{1}{p} + \frac{1}{q} - \frac{4}{qr}\right) < \frac{1}{r} < t\left(\frac{1}{q} - \frac{1}{p} + 1\right). \tag{3.5}$$

Such a positive t exists if and only if

$$\frac{1}{p} < \frac{2}{r} \cdot \frac{1}{q} + \frac{1}{2}. \tag{3.6}$$

□

This result is illustrated in Figure 1. Mantero and Tonge [7] showed that, for $2 \le r < \infty$, $\ell_r \breve{\otimes} \ell_p \breve{\otimes} \ell_q$ is a Banach algebra under Schur multiplication in the diagonally shaded region, but is not in the horizontally shaded region. Our results assert that $\ell_r \breve{\otimes} \ell_p \breve{\otimes} \ell_q$ is not a Banach algebra under Schur multiplication in the heavily shaded region. We use the same shading conventions in all subsequent figures.

If we change the role of the indices p, q, and r, we obtain the following result which is illustrated and compared to existing knowledge in Figure 2.

Proposition 3.2. *Let* $1 < r \le 2 \le p, q < \infty$. *Then* $\ell_r \breve{\otimes} \ell_p \breve{\otimes} \ell_q$ *is not a Banach algebra under Schur multiplication when*

$$\frac{1}{p} \cdot \frac{1}{q} > \frac{1}{2r} - \frac{1}{4}. \tag{3.7}$$

Next, we make use of Corollary 2.4.

Proposition 3.3. *Let* $2 \le p, q, r < \infty$. *Then* $\ell_r \breve{\otimes} \ell_p \breve{\otimes} \ell_q$ *is not a Banach algebra under Schur multiplication when*

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}, \qquad \frac{1}{q} + \frac{1}{r} > \frac{1}{2}, \qquad \text{or} \qquad \frac{1}{r} + \frac{1}{p} > \frac{1}{2}. \tag{3.8}$$

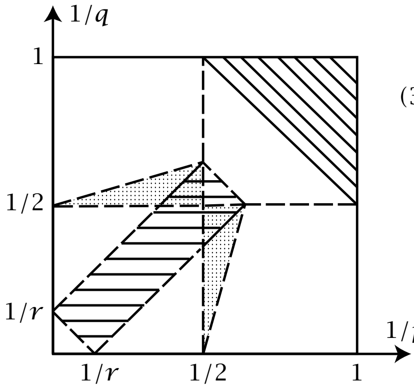

Figure 1

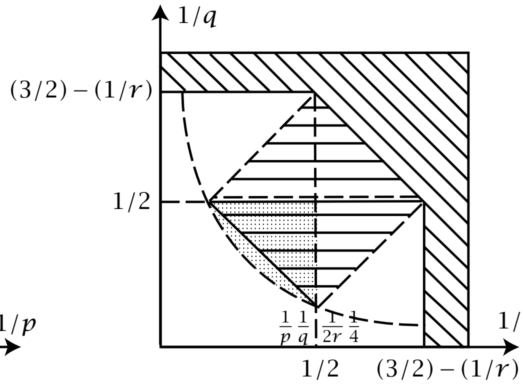

Figure 2

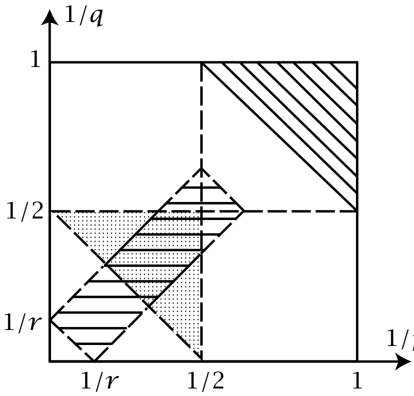

Figure 3

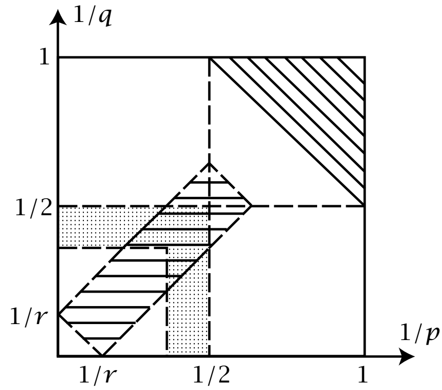

Figure 4

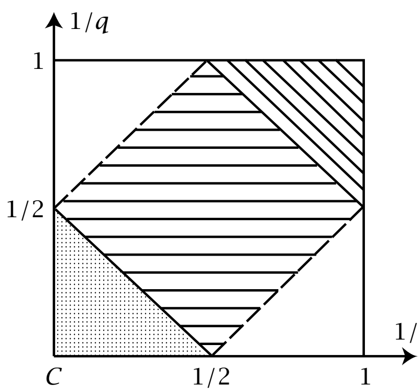

Figure 5

Proof. Fix $B > 0$. By (3.1) and Corollary 2.4, it is enough to show that we can find positive integers M, N, and K, with

$$M^{1/r}N^{1/p}K^{1/q} > BC\max\left(M^{2/r}, N^{(2/r)+(2/p)(1-2/r)}K^{(2/q)(1-2/r)}, N^{(2/p)(1-2/r)}K^{(2/r)+(2/q)(1-2/r)}\right), \tag{3.9}$$

where C is a fixed positive constant, independent of B, M, N, or K. We can achieve this with $N = K = M^t$, where $t > 0$ provided that

$$\begin{aligned} M^{t(1/p+1/q)} &> BCM^{1/r}, \\ M^{1/r} &> BCM^{(2t/r)+t(1/p+1/q)(1-4/r)}. \end{aligned} \tag{3.10}$$

Inequalities (3.10) hold simultaneously for large M if there is a $t > 0$ that satisfies

$$t\left(\frac{2}{r}+\left(\frac{1}{p}+\frac{1}{q}\right)\left(1-\frac{4}{r}\right)\right) < \frac{1}{r} < t\left(\frac{1}{p}+\frac{1}{q}\right). \tag{3.11}$$

Such a positive t exists if and only if

$$\frac{1}{p}+\frac{1}{q} > \frac{1}{2}. \tag{3.12}$$

The other results follow in a similar manner when the roles of p, q, and r are permuted. $\square$

The results in Proposition 3.3 are illustrated and compared to previous knowledge in Figures 3 and 4. The special case when $r = 2$ is worth recording separately in Figure 5.

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