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SUBSEQUENCES AND CATEGORY

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ABSTRACT. If a sequence of functions diverges almost everywhere, then the set of subsequences which diverge almost everywhere is a residual set of subsequences.

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1. Introduction. In [1], Bilyeu, Lewis, and Kallman proved a general theorem about rearrangements of a series of Banach space valued functions. This theorem settled a question on rearrangements of Fourier series posed by Kac and Zygmund. Kallman [3] proved an analog of this theorem for subseries of a series of Banach space valued functions. The purpose of this paper is to complete the cycle of these ideas by proving an analogous theorem (Theorem 1.1) for subsequences of a sequence of Banach space valued functions. Theorem 1.1 does not seem to follow directly from results of [1] or [3]. Other than [1, 3], the only precedent for Theorem 1.1 seems to be a paper [7] on subsequences of a sequence of complex numbers.

Let S be the set of all sequences $s = (s_1, s_2, \dots)$, where $1 \le s_1 < s_2 < \cdots$ is a strictly increasing sequence of positive integers. S is a closed subset of the countable product of the positive integers, and so S is a complete separable metric space. Given any sequence of objects $a_1, a_2, \dots$, one can identify the set of its subsequences both as a set and as a topological space with S. In this context, it is natural to identify a collection of subsequences with a subset of S and ask if it is first category, second category, or residual ([5] or [6]). Define an equivalence relation $\sim$ on S as follows: if $s, t \in S$, then $s \sim t$ if and only if $s_n = t_n$ for all sufficiently large n. Intuitively this states that $s \sim t$ if and only if s and t agree from some point on. It is simple to check that any nonempty subset of S which is saturated with respect to $\sim$ is dense.

The main result of this paper is the following theorem, which is proved in Section 2.

THEOREM 1.1. *Let (X, μ) be a regular locally compact σ-finite measure space, Z a separable Banach space, and $f_n : X \mapsto Z$ a sequence of Borel measurable functions. Suppose that the sequence $f_n(x)$ diverges for μ-a.e., $x \in X$. Then $[s \in S \mid f_{s_n}(x)$ diverges for μ-a.e. $x \in X]$ is a residual set in S.*

Just as in [1, 3], this measure-category result has a category-category analog which is discussed in Section 3.

2. Proof of Theorem 1.1. The following special case of Theorem 1.1 will be proved first.

Lemma 2.1. *Let K be a compact Hausdorff space, Z a Banach space, and $f_n : K \mapsto Z$ a sequence of continuous functions, and $\delta > 0$. Suppose that for every $x \in K$ and positive integer N, there exists a pair of integers $n = n(x,N)$ and $m = m(x,N)$ so that $N \le n \le m$ and $\|f_m(x) - f_n(x)\| > \delta$. Then $[s \in S \mid f_{s_n}(x)$ diverges for every $x \in K]$ is a residual set in S.*

Proof. If m,n is a pair of integers such that $1 \le n \le m$ and $s \in S$, let $g_{s,m,n} : K \mapsto [0,+\infty)$ be defined by $g_{s,m,n}(x) = \|f_{s_m}(x) - f_{s_n}(x)\|$. $g_{s,m,n}$ is continuous. Consider

$$A = \bigcap_{N\ge 1} \bigcup_{N\le n_1\le m_1,\dots,N\le n_p\le m_p} \Big[s \in S \mid \cup_{1\le i\le p}\, g^{-1}_{s,m_i,n_i}((\delta,+\infty)) = K\Big]. \tag{2.1}$$

Fix $1 \le n \le m$ and $s \in S$. Then $V = [t \in S \mid t_m = s_m$ and $t_n = s_n]$ is an open neighborhood of s in S. Hence, if $t \in V$, then $g_{t,m,n} = g_{s,m,n}$. This in turn implies that A is a G_δ subset of S. Furthermore, A is saturated with respect to the equivalence relation $\sim$ and therefore is a dense G_δ if it is nonempty.

A is nonempty since $t = (1,2,3,\dots)$ is in A. To see this, fix $N \ge 1$. For $N \le n \le m$, let $U(m,n) = g^{-1}_{t,m,n}((\delta,+\infty))$. Note that the collection $\{U(m,n)\}_{N\le n\le m}$ is an open covering of K by hypothesis and so has a finite subcover, say $U(m_1,n_1),\dots,U(m_p,n_p)$. One easily concludes from this that $t \in A$.

Finally, note that the Cauchy criterion for convergence implies that if $s \in A$, then $f_{s_n}(x)$ diverges for every $x \in K$. Hence, $A \subseteq [s \in S \mid f_{s_n}(x)$ diverges for every $x \in K]$. This proves Lemma 2.1. □

Proof of Theorem 1.1. We may assume that μ is a probability measure since μ is σ-finite. If $q \ge 1$, let

$$D_q = \bigcap_{N\ge 1} \bigcup_{N\le n\le m} \left[x \in X \mid \|f_m(x) - f_n(x)\| > \frac{1}{q}\right]. \tag{2.2}$$

Each D_q is a Borel subset of X, $D_q \subseteq D_{q+1}$, and the Cauchy criterion for convergence implies that $\cup_{q\ge1} D_q = [x \in X \mid f_n(x)$ diverges$]$. $\mu(\cup_{q\ge1} D_q) = 1$ by assumption. Use a vector-valued version of Lusin's Theorem [2] to choose, for each q, a compact subset K_q of D_q so that each $f_n \mid K_q$ is continuous and $\mu(D_q - K_q) < 1/q$. $R_q = [s \in S \mid f_{s_n}(x)$ diverges for every $x \in K_q]$ is a residual subset of S by Lemma 2.1. Hence, $R = \cap_{q\ge1} R_q$ is a residual set in S and is contained in $[s \in S \mid f_{s_n}(x)$ diverges for μ-a.e., $x \in X]$ since $\mu(\cup_{q\ge1} K_q) = 1$. This proves Theorem 1.1. □

3. Sequences of functions with the Baire property.

Theorem 1.1 may be regarded as a measure-category result. The purpose of this section is to prove a category-category analog of Theorem 1.1 (cf. [1, Thm. 1.2] and [3, Thm. 3.1]).

Let X be a Polish space. A subset of X is said to have the Baire property if there exists an open set U in X so that $A \triangle U$ is first category. The collection of all subsets of X with the Baire property is a σ-algebra which includes the analytic sets in X. Let Z be any other Polish space. A function $f : X \mapsto Z$ is said to have the Baire property if U open in Z implies that $f^{-1}(U)$ has the Baire property in X. Any Borel function $f : X \mapsto Z$ is a function with the Baire property. See [4, 5] or [6] for a thorough discussion of this circle of ideas. The following theorem is then a category-category analog of Theorem 1.1.

Theorem 3.1. *Let X be a Polish space, Z a separable Banach space, and $f_n : X \mapsto Z$ a sequence of functions with the Baire property. Suppose that $[x \in X \mid f_n(x)$ diverges$]$ is a residual subset of X. Then $[s \in S \mid f_{s_n}(x)$ diverges on a residual subset of $X]$ is a residual subset of S.*

The following proposition, of independent interest, is needed to prove Theorem 3.1.

Proposition 3.2. *Let Z be a Banach space and let $\{z_n\}_{n\geq 1}$ be a sequence in Z. Let $A = [s \in S \mid z_{s_n}$ converges$]$. Then either $A = S$ or A is of first category in S.*

Proof. For $k \geq 1$ define

$$B_k = \bigcap_{N\geq 1} \bigcup_{N\leq n\leq m} \left[s \in S \mid \|z_{s_m} - z_{s_n}\| > \frac{1}{k} \right]. \tag{3.1}$$

Note that $B_k \subseteq B_{k+1}$. Each set in square brackets is open in S. Hence, this formula shows that B_k is a G_δ. B_k is dense if it is nonempty since it is saturated with respect to the equivalence relation $\sim$. Therefore, B_k is a residual set in S if it is nonempty since any dense G_δ is residual.

The Cauchy criterion for convergence implies that $A^c = \cup_{k\geq 1} B_k$. Hence, either $A = S$ or A^c is residual in S; or either $A = S$ or A is of first category in S. This proves Proposition 3.2. □

Proof of Theorem 3.1. Check that the mapping $(x,s) \mapsto f_{s_n}(x)$, $X \times S \mapsto Z$, is a function with the Baire property for every $n \geq 1$. Hence,

$$B = [(x,s) \mid f_{s_n}(x) \text{ diverges}] = \bigcup_{k\geq 1} \bigcap_{N\geq 1} \bigcup_{N\leq n\leq m} \left[(x,s) \mid \|f_{s_m}(x) - f_{s_n}(x)\| > \frac{1}{k} \right] \tag{3.2}$$

is a subset of $X \times S$ with the Baire property. For each $x \in X$, let B_x^c be the projection of $B^c \cap ((x) \times S)$ onto S. The hypotheses of Theorem 3.1 plus Proposition 3.2 imply that each B_x^c is a first category subset of S, except for a first category set of x's. But then B^c is itself a first category subset of $X \times S$ [6, Thm. 15.4] and so B_s^c, the projection of $B^c \cap (X \times (s))$ onto X, is a first category subset of X, except for a first category set of s's (Theorem of Kuratowski-Ulam, [6, Thm. 15.1]). Hence, B_s, the projection of $B \cap (X \times (s))$ onto X, is a residual subset of X for all except a first category set of s's. This proves Theorem 3.1. □

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