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RESEARCH NOTES

ON A DENSITY PROBLEM OF ERDÖS

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ABSTRACT. For a positive integer n, let $P(n)$ denotes the largest prime divisor of n and define the set: $\mathcal{S}(x) = \mathcal{S} = \{n \le x : n \text{ does not divide } P(n)!\}$. Paul Erdös has proposed that $|S| = o(x)$ as $x \to \infty$, where $|S|$ is the number of $n \in S$. This was proved by Ilias Kastanas. In this paper we will show the stronger result that $|S| = O(xe^{-1/4}\sqrt{\log x})$.

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Introduction. For a positive integer n, let $P(n)$ denote the largest prime divisor of n and define the set

$$\mathcal{S}(x) = \mathcal{S} = \{n \le x : n \text{ does not divide } P(n)!\}. \tag{1}$$

Paul Erdös [1] proposed that $|S| = o(x)$ as $x \to \infty$, where $|S|$ is the number of $n \in S$. A solution [3] was provided by Ilias Kastanas. There was also [3] a claim of proving that $|S(x)| = O(x/\log x)$. In this paper, we show the stronger result.

THEOREM. *For some constant $a > 0$, we have*

$$|S| = O\left(xe^{-a\sqrt{\log x}}\right). \tag{2}$$

In fact, $a = 1/4$ suffices.

LEMMA 1. *Let $\nu(n)$ be the number of distinct prime divisors of n. Define*

$$S_1 = \{n \le x : \nu(n) > 4k\log\log x,\ k \ge 1\}. \tag{3}$$

Then

$$|S_1| = O\left(\frac{x}{(\log x)^k}\right) \tag{4}$$

uniformly in k.

PROOF. It is well known [2] that if $d(m)$ is the number of divisors of m, then $\sum_{m\le x} d(m) = O(x\log x)$. Since $d(m) \ge 2^{\nu(m)}$,

$$O(x\log x) \ge \sum_{m\le x} 2^{\nu(m)} \ge \sum_{m\in S_1} 2^{\nu(m)} \ge \sum_{m\in S_1} (\log x)^{(4\log 2)k} \ge |S_1|(\log x)^{k+1}, \tag{5}$$

and the lemma follows. □

Lemma 2. *Let $C(x) = C = (\log x)^k$, where $k = k(x)$ will be chosen later. Define*

$$S_2 = \{n \le x : p^2 \mid n \text{ for some prime } p > C\}. \tag{6}$$

Then

$$|S_2| = O\left(\frac{x}{(\log x)^k}\right). \tag{7}$$

Proof. Since $n \in S_2$ if and only if $n = tp^2$ for some $C < p \le \sqrt{x}$ and some $t \le x/p^2$,

$$|S_2| = \sum_{C<p\le\sqrt{x}} \left[\frac{x}{p^2}\right] \le \sum_{C<p\le\sqrt{x}} \frac{x}{p^2} = O\left(\frac{x}{C}\right) = O\left(\frac{x}{(\log x)^k}\right). \tag{8}$$

The first big O in (8) follows since $\sum_{p>C} 1/p^2 \le \int_{[C]}^{\infty} du/u^2 = 1/[C] = O(1/C)$, for $C \ge 1$. □

Lemma 3. *Let*

$$S_3 = \{n \le x : p^\alpha \mid n \text{ for some } \alpha \ge T \text{ and some prime } P \le C\}, \tag{9}$$

where $T = 2\log C$. Then

$$|S_3| = O\left(\frac{x}{(\log x)^k}\right). \tag{10}$$

Proof.

$$\begin{aligned} |S_3| &= \sum_{\substack{p\le C\\ \alpha\ge T}} \left[\frac{x}{p^\alpha}\right] \le 2\sum_{p\le C} \frac{x}{p^T} \le 2\left(\frac{x}{2^T}\right)\sum_{p} \frac{4}{p^2} = O\left(\frac{x}{2^T}\right) \\ &= O\left(\frac{x}{C^{2\log 2}}\right) \le O\left(\frac{x}{C}\right) = O\left(\frac{x}{(\log x)^k}\right). \end{aligned} \tag{11}$$

The first inequality in (11) is valid because $\sum_{\alpha\ge T} 1/p^\alpha \le 1/p^T + 1/p^{T+1} + \cdots \le 2/p^T$ and the second is valid because $p^T = 2^T(p/2)^T \ge 2^T(p/2)^2$ for $T \ge 2$. □

Lemma 4. *Let $S'(x) = S - (S_1 \cup S_2 \cup S_3)$, then, for any $n \in S'$, we have*

$$P(n) \le 2CT. \tag{12}$$

Proof. Let $n \in S'$. Then $n \in S$ and so n does not divide $P(n)!$. There exists a prime p_0 dividing n such that

$$\nu_{p_0}(n) > \nu_{p_0}(P(n)!), \tag{13}$$

where $\nu_p(m)$ denotes the largest integer t such that p^t divides m. Since $\nu_{p_0}(P(n)!) \ge 1$, (13) implies that $\nu_{p_0}(n) \ge 2$. Since $n \notin S_2$, it follows that $p_0 \le C$. Also $n \notin S_3$, so that $T \ge \nu_{p_0}(n)$. Hence,

$$T \ge \nu_{p_0}(n) > \nu_{p_0}(P(n)!) \ge \frac{P(n)}{2p_0} \ge \frac{P(n)}{2C} \tag{14}$$

which implies (12). Note that the third inequality in (14) is true because $\nu_p(m!) \geq m/2p$ for any m and any $p \mid m$. This is because

$$\nu_p(m!) \geq \left[\frac{m}{p}\right] > \frac{m}{p} - 1 \geq \frac{m}{2p}, \quad p \neq m, \tag{15}$$

and $[m/p] = 1 \geq m/2p$ if $p = m$. □

Proof of the Theorem. For $n \in S'$, we have $n \notin S_1 \cup S_2 \cup S_3$. Thus, $\nu_p(n) \leq 4k \log\log x$. Also, if p^α is any prime power dividing n, then one of the following two possibilities must occur:

(a) $p \leq C$ and $\alpha \leq T$,

(b) $p > C$ and $\alpha = 0$ or 1.

Case (a) generates at most $C(T+1) \leq 2CT$ prime powers. For Case (b), the number of prime powers p^α with $p > C$ and $\alpha \leq 1$ is at most $P(n)$. By Lemma 4, this is at most $2CT$. Hence, the number of possible prime powers p^α that divide an $n \in S'$ is at most $4CT$. But such an n can be the product of at most $4k \log\log x$ distinct prime powers. Therefore,

$$\begin{aligned} |S'| &\leq (4CT)^{4k \log\log x} = (8C \log C)^{4k \log\log x} \\ &= e^{4k \log\log x (\log 8 + k \log\log x + \log k + \log\log\log x)} \leq e^{8k^2 (\log\log x)^2} \end{aligned} \tag{16}$$

since $\log 8 + \log(k \log\log x) \leq k \log\log x$, for $k \log\log x \geq 4$.

Choosing

$$k = \frac{1}{4} \cdot \frac{\sqrt{\log x}}{\log\log x}, \tag{17}$$

(16) gives $|S'| \leq e^{(1/2)\log x} = x^{1/2}$. Hence,

$$S' = O\left(xe^{-(1/4)\sqrt{\log x}}\right). \tag{18}$$

From (17), we have $x/(\log x)^k = xe^{-1/4\sqrt{\log x}}$. Lemmas 1, 2, and 3 imply that

$$|S_i| = O\left(xe^{-1/4\sqrt{\log x}}\right), \quad i = 1, 2, 3. \tag{19}$$

Finally, $S = S' \cup [S \cap (S_1 \cup S_2 \cup S_3)]$. Hence, (18) and (19) yield

$$|S| \leq |S'| + |S_1| + |S_2| + |S_3| = O\left(xe^{-(1/4)\sqrt{\log x}}\right), \tag{20}$$

and (2) follows with $a = 1/4$. □

Remark. If $\pi(x)$ is the number of prime integers that are less than or equal to x, an early version of the prime numbers theorem asserts that

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\left(xe^{-a\sqrt{\log x}}\right), \tag{21}$$

for some constant a. Although the big O terms in (19) and (2) are similar, there is no apparent relationship between the PNT and (2).

References

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