

ON THE DISTRIBUTION OF RAMIFICATION POINTS IN TRIGONAL CURVES

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ABSTRACT. We study the distribution of the total and ordinary ramification points of a trigonal curve over the intersection of this curve with rational curves on a rational normal scroll. We show, among other results, that these intersections may contain all the ramification points of the trigonal curve.

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1. Introduction. Trigonal curves are canonically immersed in rational normal scrolls, which may be viewed as the join of two rational normal curves lying in hyperplanes of complementary dimension. The purpose of this note is to find relations and bounds for the number of total and ordinary ramification points of a trigonal curve that lie on the intersection with a rational curve on the scroll. The motivation comes from results obtained by Coppens [2] relating the number of total ramification points having one or other Weierstrass sequence in a trigonal curve. In a more geometrical approach to that question, Stöhr and Viana [4] showed that a ramification point have one or other Weierstrass sequence depending on whether or not it lies on a certain rational curve (the directrix of the scroll). We use this same approach in what follows. This simplifies the treatment of the question and adds a geometrical meaning to the results (cf. [3]).

2. Preliminaries. Let C be a nonsingular trigonal curve of genus $g \geq 5$ defined over an algebraically closed field k of a characteristic zero and canonically embedded in $\mathbb{P}^{g-1}(k)$. We know from [1] that such a curve lies in a rational normal scroll S_{mn} that may be described as the join of the rational normal curves

$$D := \{(a^n : a^{n-1}b : \dots : b^n : 0 : \dots : 0) \in \mathbb{P}^{m+n+1}(k) \mid (a : b) \in \mathbb{P}^1(k)\} \quad (1)$$

and

$$E := \{(0 : \dots : 0 : a^m : a^{m-1}b : \dots : b^m) \in \mathbb{P}^{m+n+1}(k) \mid (a : b) \in \mathbb{P}^1(k)\}, \quad (2)$$

where m and n are positive integers satisfying $m \leq n$, $(n-2)/2 \leq m \leq 2n+2$ and $m+n+2 = g$.

We denote the lines of the ruling by

$$L_{b/a} := \overline{(a^n : a^{n-1}b : \dots : b^n : 0 : \dots : 0), (0 : \dots : 0 : a^m : a^{m-1}b : \dots : b^m)} \quad (3)$$

with $b/a \in \mathbb{P}^1(k) = k \cup \{\infty\}$ and we observe that the following sets form an open

covering for S_{mn} .

$$U_0 := S_{mn} \setminus (L_\infty \cup E) = \{(a^0 : \dots : a^n : a^0 b : \dots : a^m b) \mid (a, b) \in k^2\}, \tag{4}$$

$$U_n := S_{mn} \setminus (L_0 \cup E) = \{(a^n : \dots : a^0 : a^m b : \dots : a^0 b) \mid (a, b) \in k^2\}, \tag{5}$$

$$U_{n+1} := S_{mn} \setminus (L_\infty \cup D) = \{(a^0 b : \dots : a^n b : a^0 : \dots : a^m) \mid (a, b) \in k^2\}, \tag{6}$$

$$U_{g-1} := S_{mn} \setminus (L_0 \cup D) = \{(a^n b : \dots : a^0 b : \dots : a^m : \dots : a^0) \mid (a, b) \in k^2\}. \tag{7}$$

Note that each of these sets is isomorphic to the affine plane k^2 and, in [4], the following theorem was proved.

THEOREM [1, Thm. 1.1]. *Associating to each plane curve the Zariski closure of its image under the local parametrization $k^2 \hookrightarrow S_{mn}$, defined by*

$$(a, b) \mapsto (a^0 : \dots : a^n : a^0 b : \dots : a^m b), \tag{8}$$

one obtains a bijective correspondence between the canonical curves on the scroll $S_{mn} \subset \mathbb{P}^{m+n+1}(k)$ and the affine plane curves given by the irreducible equation

$$c_3(X)Y^3 + c_2(X)Y^2 + c_1(X)Y + c_0(X) = 0 \tag{9}$$

satisfying $c_3(X) \neq 0$, $\deg c_i(X) \leq d_i$ for each $i = 0, 1, 2, 3$ and $\deg c_i(X) = d_i$ for at least one i , where $d_0 = 2n - m + 2$, $d_1 = n + 2$, $d_2 = m + 2$, and $d_3 = 2m - n + 2$.

In what follows, we always denote an equation as the above by $f_0(X, Y) = 0$ and we call C_0 the plane curve defined by it. Observe that if C is the correspondent trigonal curve in S_{mn} , then C_0 is isomorphic to the affine curve $C \cap U_0$. Furthermore, there is a plane curve which we call C_n that is isomorphic to $C \cap U_n$ under the local parametrization of S_{mn} defined by $(a : b) \mapsto (a^n : \dots : a^0 : a^m b : \dots : a^0 b)$. Denoting by $f_n(X, Y) = 0$ the equation of C_n , we must have $f_n(X, Y) = \tilde{c}_3(X)Y^3 + \tilde{c}_2(X)Y^2 + \tilde{c}_1(X)Y + \tilde{c}_0(X)$, where $\tilde{c}_i(X) := c_i(1/X)X^{(2n-m+2-i(n-m))}$, $i = 0, 1, 2, 3$ (to see that, observe that if $(a^0 : \dots : a^n : a^0 b : \dots : a^m b) = (\tilde{a}^n : \dots : \tilde{a}^0 : \tilde{a}^m \tilde{b} : \dots : \tilde{a}^0 \tilde{b})$, then $\tilde{a} = 1/a$ and $\tilde{b} = b/a^{n-m}$). Similarly, we denote by $f_{n+1}(X, Y) = 0$ the equation of the plane curve C_{n+1} which is isomorphic to the affine curve $C \cap U_{n+1}$ through the local parametrization of S_{mn} , defined by $(a, b) \mapsto (a^0 b : \dots : a^n b : a^0 : \dots : a^m)$, and as above we may conclude that $f_{n+1}(X, Y) = c_0(X)Y^3 + c_1(X)Y^2 + c_2(X)Y + c_3(X)$. Finally, we denote by $f_{g-1}(X, Y) = 0$ the equation of the plane curve C_{g-1} which is isomorphic to the affine curve $C \cap U_{g-1}$ through the local parametrization of S_{mn} , defined by $(a, b) \mapsto (a^n b : \dots : a^0 b : a^0 : \dots : a^m)$, and one may easily verify that $f_{g-1}(X, Y) = \tilde{c}_0(X)Y^3 + \tilde{c}_1(X)Y^2 + \tilde{c}_2(X)Y + \tilde{c}_3(X)$.

A point $P \in C$ is a *ramification point* with respect to the trigonal morphism of C if the line L of the scroll passing through P is the tangent line at P , and the *ramification index* e_P is the intersection multiplicity of C and L at P . We say that P is a *total ramification point* (respectively, an *ordinary ramification point*) when $e_P = 3$ (respectively, when $e_P = 2$). Observe that maps $k^2 \hookrightarrow U_0$ and $k^2 \hookrightarrow U_{n+1}$ (respectively, $k^2 \hookrightarrow U_n$ and $k^2 \hookrightarrow U_{g-1}$) take the vertical line $X = a$ of k^2 into L_a (respectively, into $L_{1/a}$), $a \in k$. Thus, we call a point of C_i ($i = 0, n, n + 1, g - 1$) a *ramification point* if it has a vertical tangent line and we classify it as a *total ramification point* or an *ordinary ramification*

point according to its intersection multiplicity with the tangent line as above. Equivalently, a point $P = (a, b) \in C_i (i = 0, n, n + 1, g - 1)$ is a total (respectively, an ordinary) ramification point if and only if $\text{ord}_P(x - a) = 3$ (respectively, $\text{ord}_P(x - a) = 2$), where x is the rational function of $k(C_i) = k(C)$, defined by $(a, b) \rightarrow a$, and $\text{ord}_P(h)$ is the order at P of the rational function $h \in k(C_i)$.

We end this section with the following observations.

REMARK 2.1. It is easy to check the following facts:

- (i) If (a, b) is a total ramification point of C_0 , then $c_3(a) \neq 0$.
- (ii) A point $(a, 0) \in k^2$ is a total (respectively, an ordinary) ramification point of C_0 if and only if $c_0(a) = 0, c_1(a) = 0$ and $c_2(a) = 0$ (respectively, $c_2(a) \neq 0$).
- (iii) A point $(a, 0) \in k^2$ is a total (respectively, an ordinary) ramification point of C_n if and only if $\tilde{c}_0(a) = 0, \tilde{c}_1(a) = 0$ and $\tilde{c}_2(a) = 0$ (respectively, $\tilde{c}_2(a) \neq 0$).
- (iv) A point $(a, 0) \in k^2$ is a total (respectively, an ordinary) ramification point of C_{n+1} if and only if $c_3(a) = 0, c_2(a) = 0$ and $c_1(a) = 0$ (respectively, $c_1(a) \neq 0$).
- (v) The point $(a, 0) \in k^2$ is a total (respectively, an ordinary) ramification point of C_{g-1} if and only if $\tilde{c}_3(a) = 0, \tilde{c}_2(a) = 0$ and $\tilde{c}_1(a) = 0$ (respectively, $\tilde{c}_1(a) \neq 0$).

REMARK 2.2. If $\text{deg } c_3(X) \leq 2m - n$, then $X^2 \mid \tilde{c}_3(X)$ and $(0, 0) \in C_{g-1}$. Since this curve is nonsingular, we must have $X \nmid \tilde{c}_2(X)$, i.e., $\text{deg } c_2(X) = m + 2$. Similarly, if $\text{deg } c_0(X) \leq 2n - m$, then we must have $\text{deg } c_1(X) = n + 2$.

REMARK 2.3. If $(a, 0) \in k^2$ is a ramification point of C_{n+1} , then a is a simple root of $c_3(X)$. In fact, if a is a multiple root of $c_3(X)$ and also a root of $c_2(X)$, then $(a, 0)$ is a singular point of C_{n+1} , which is absurd. Similarly, if $(a, 0) \in k^2$ is a ramification point of C_0 , then a is a simple root of $c_0(X)$.

3. The main results. The only rational curves on S_{mn} are the lines of the ruling and the intersection of the scroll with the hyperplanes of $\mathbb{P}^{m+n-1}(k)$. We want to find bounds for the number of ramified points that lie on the intersection of a trigonal curve on the scroll with rational curves of the latter type. When $m = n$, all such rational curves are linearly equivalent as divisors on S_{mn} and after a change of variables, it is enough to consider the points on $C \cap D$. When $m < n$ though, E is a distinguished curve on the scroll (e.g., is the only curve with negative self-intersection) and then we must also consider the points on $C \cap E$ (in this case, E is called the directrix of the scroll).

To study the occurrence of ramification points on the intersections $C \cap D$ and $C \cap E$, we use the local parametrizations and the curves C_0, C_n, C_{n+1} , and C_{g-1} introduced above. We see that in order to count such points, it suffices to count the number of ramification points of zero ordinate in C_0 and in C_{n+1} , then the number of ramification points with zero abscissa on C_n , and finally to check if the origin of the plane containing C_{g-1} is a ramification point of that curve.

PROPOSITION 3.1. *Let σ (respectively, ρ) denote the number of total (respectively, ordinary) ramification points of C lying over E and let η (respectively, ξ) denote the number of total (respectively, ordinary) ramification points of C lying over D . Then one of the following possibilities occurs:*

- (i) $0 \leq \sigma + \rho \leq 2m - n + 2, 0 \leq \sigma + \rho + \eta \leq m + 2, 0 \leq \eta + \xi + \sigma \leq n + 2.$
- (ii) $\sigma + \rho = 2m - n + 2, \xi = 0$ and $n - m + 1 \leq \eta \leq n + 2 - \sigma.$
- (iii) $\rho = 0, \xi + \eta = 2n - m + 2, 0 \leq \sigma \leq 2m - n + 2$ and $0 \leq \sigma + \eta \leq m + 2.$
- (iv) $\sigma = 2m - n + 2, \eta = 2n - m + 2$ and $\rho = \xi = 0.$

In the case $m < n$, the above possibilities are mutually exclusive.

PROOF. Suppose that $c_2 \neq 0$ and $c_1 \neq 0$. If there is a total ramification point in $C \cap (D \setminus L_\infty)$, then Remark 2.1(ii) implies that, for some $b \in k$, we have $(X - b) \mid c_0(X)$, $(X - b) \mid c_1(X)$ and $(X - b) \mid c_2(X)$. From Remarks 2.2 and 2.1(iii), we have that $D \cap L_\infty$ is a total (respectively, an ordinary) ramification point of C if and only if $\deg c_0(X) = 2n - m + 1$, $\deg c_1(X) \leq n + 1$, and $\deg c_2(X) \leq m + 1$ (respectively, $\deg c_2(X) = m + 2$). From all this, we may conclude that $0 \leq \sigma + \rho \leq 2m - n + 2$. The other inequalities in (i) follow similarly.

To establish the inequalities in (ii), we suppose that $c_2(X) = 0$ and $c_1(X) \neq 0$. Then by Remark 2.1(iv), the roots of $c_3(X)$ are in a one-to-one correspondence with the ramification points of C_{n+1} of the form $(a, 0)$, with $a \in k$ (i.e., the roots of $c_3(X)$ are in a one-to-one correspondence with the ramification points of $C \cap (E \setminus L_\infty)$). From Remark 2.3, it follows that $c_3(X)$ does not have a multiple root. By Remark 2.2, since $c_2(X) = 0$, we must have $\deg c_3(X) = 2m - n + 2$ or $\deg c_3(X) = 2m - n + 1$, but either way we have $\sigma + \rho = 2m - n + 2$. Obviously, we have $\xi = 0$ and $0 \leq \eta + \sigma \leq n + 2$. Now, we do not have the restriction $0 \leq \sigma + \rho + \eta \leq m + 2$ and to avoid an overlapping with the conditions in (i), we take $\eta > m + 2 - (\sigma + \rho) = n - m$ and, thus, $n - m + 1 \leq \eta \leq n + 2 - \sigma$.

Using Remarks 2.1, 2.2, and 2.3 in the same way as above, one may prove (iii) and (iv), the former by assuming $c_1(X) = 0$, and $c_2 \neq 0$ and the latter by assuming $c_1(X) = c_2(X) = 0$. □

We will show constructively that all the possibilities in Proposition 3.1 in fact occur, and now we establish some results and notation that are needed for that. If $f, h \in k[X, Y]$, we denote by $\text{Res}_X(f, h)$ the resultant of f and h as elements of $(k[Y])[X]$ and $\text{Res}_Y(f, h)$ denotes the resultant of f and h as elements of $(k[X])[Y]$.

LEMMA 3.2. *Let $P = (a_0, \dots, a_{2m-n+2}, b_0, \dots, b_{2n-m+2})$ be a point of k^{n+m+4} , and let $p(X) = \sum_{i=0}^{2m-n+2} a_i X^{2m-n+2-i}$, $q(X) = \sum_{j=0}^{2n-m+2} b_j X^{2n-m+2-j}$, and $f(X) = p(X)Y^3 + q(X)$. Suppose that p and q do not have common roots neither multiple roots. Then the set of points P such that $\text{Res}_X(\text{Res}_Y(f, \partial f / \partial X), \text{Res}_Y(f, \partial f / \partial Y)) \neq 0$ contains an open set of k^{m+n+4} .*

PROOF. We have $\text{Res}_Y(f, \partial f / \partial Y) = 27p^3q^2$ and $\text{Res}_Y(f, \partial f / \partial X) = p^3(dq/dX)^3 - 3qp^2(dq/dX)^2(dp/dX) + 3q^2pdq/dX(dp/dX)^2 - (dp/dX)^3q^3$. From the hypothesis on $p(X)$ and $q(X)$, we conclude that if f is such that $\text{Res}_Y(f, \partial f / \partial X) \neq 0$, then there is no $a \in k$ satisfying both $\text{Res}_Y(f, \partial f / \partial Y)(X) = 0$ and $\text{Res}_Y(f, \partial f / \partial X)(X) = 0$, i.e., $\text{Res}_X(\text{Res}_Y(f, \partial f / \partial X), \text{Res}_Y(f, \partial f / \partial Y)) \neq 0$. Calculating $\text{Res}_Y(f, \partial f / \partial X)$, we get

$$\begin{aligned} \text{Res}_Y\left(f, \frac{\partial f}{\partial X}\right) &= (n - m)^3 A_0 X^{3m+3n+9} + (n - m)^2 A_1 X^{3m+3n+8} \\ &\quad + (n - m) A_2 X^{3m+3n+7} + A_3 X^{3m+3n+6} + \dots, \end{aligned} \tag{10}$$

where the dots indicate the sum of powers of X lesser than $3m + 3n + 6$, $A_0 = 27a_0^3b_0^3$ and A_3 is a polynomial expression in $a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3, m$, and n . More specifically, A_3 is a linear combination over k of monomials of the type $a_\alpha a_\beta a_\gamma b_\delta b_\epsilon b_\phi$, with $\alpha, \beta, \gamma, \delta, \epsilon, \phi \in \{0, 1, 2, 3\}$ and $\alpha + \beta + \gamma + \delta + \epsilon + \phi = 3$, and calculating the coefficient of $a_1^3 b_0^3$, we get $(3n - 3m + 1)^3$. So, $\text{Res}_Y(f, \partial f / \partial X) \neq 0$ for P in an open set of k^{n+m+4} . □

REMARK 3.3. Let $C_0 \subset k^2$ be a nonsingular plane curve, defined by the equation $f_0(X, Y) = c_\ell(X)Y^\ell + \dots + c_0(X)$, and let C be the curve in S_{mn} , obtained by taking the closure in the Zariski topology of the image of C_0 in U_0 under the parametrization $(a, b) \mapsto (a^0 : \dots : a^n : a^0 b : \dots : a^m b)$. In [4, p. 67], it was shown that if $\ell > 0$ and C is nonsingular, then C is also irreducible. This is done by considering that if C has more than one (nonsingular) component, then they cannot intersect one another, but this is contradicted if one calculates their intersection numbers.

PROPOSITION 3.4. *Let σ, ρ, η , and ξ be nonnegative integers satisfying the conditions in any of the items in Proposition 3.1, and let $m \leq n$ be positive integers such that $(n + 2)/2 \leq m \leq 2n + 2$ and $m + n + 2 \leq 5$. Then there exists a trigonal nonsingular irreducible curve $C \subset S_{mn}$ of genus $g = m + n + 2$ such that, in $C \cap E$, there are exactly σ total ramification points and ρ ordinary ramification points of C and, in $C \cap D$, there are exactly ξ total ramification points and η ordinary ramification points of C .*

PROOF. We begin by assuming that σ, ρ, η and ξ satisfy the conditions (i) in Proposition 3.1. Let r, s, u, v, a, b, c , and d be nonzero polynomials of degrees $\sigma, \rho, \eta, \xi, 2m - n - \sigma - \rho, m + 2 - \sigma - \rho - \eta, n + 2 - \sigma - \eta - \xi$, and $2n - m + 2 - \xi - \eta$, respectively, and let $f = rsaY^3 + rsubY^2 + ruvcY + uvd$. Let C_0 be the plane curve, defined by $f(X, Y) = 0$, and let C be Zariski closure in S_{mn} of the curve in $U_0 \xrightarrow{\sim} k^2$ that corresponds to C_0 . We impose a series of open conditions on the coefficients of the polynomials r, s, u, v, a, b, c , and d to grant that C is a nonsingular curve. We begin by assuming that these polynomials do not have a common root when taken two at a time and that none of them have a multiple root. We also assume that the coefficient of X^{2m-n+1} in the polynomial rsa is different from zero. We have

$$\begin{aligned} \text{Res}_Y\left(f, \frac{\partial f}{\partial Y}\right) &= 27a^3d^2r^3s^3u^2v^2 - 18a^2bcd r^4s^3u^3v^2 - ab^2c^2r^5s^3u^4v^2 \\ &\quad + 4a^2c^3r^5s^2u^3v^3 + 4ab^3dr^4s^4u^4v. \end{aligned} \tag{11}$$

Thus, $\text{deg Res}_Y(f, \partial f / \partial Y) \leq 4m + n + 10$ and we assume that the coefficients of the polynomials r, s, u, v, a, b, c , and d are such that $\text{Res}_Y(f, \partial f / \partial Y)$ is a polynomial in X of degree $4m + n + 10$. Observe that if we substitute $b \mapsto 0$ and $c \mapsto 0$, then

$$\text{Res}_Y\left(f \Big|_{b=0, c=0}, \left(\frac{\partial f \Big|_{b=0, c=0}}{\partial Y}\right)\right) = \text{Res}_Y\left(f, \frac{\partial f}{\partial Y}\right) \Big|_{b=0, c=0} = 27(rsa)^3(uvd)^2 \tag{12}$$

has also degree $4m - n + 10$ as a polynomial in X . Similarly, we assume that the degrees of $\text{Res}_Y(f, \partial f / \partial X)$ and $\text{Res}_Y(f \Big|_{b=0, c=0}, \partial(f \Big|_{b=0, c=0}) / \partial X)$ as polynomials in X are equal to $3m + n + 9$ if $n \neq m$ or equal to $3m + n + 6$ if $n = m$ (as in the proof of Lemma 3.2, one may check that the coefficient of $X^{3m+3n+9}$ in $\text{Res}_Y(f, \partial f / \partial X)$ is equal

to $(n - m)$ times a nonidentically null polynomial expression in the leading coefficients of $r, s, u, v, a, b, c,$ and d that does not vanish when $b = c = 0$; the coefficients of $X^{3m+3n+8}$ and $X^{3m+3n+7}$ are zero if $n = m$ and the coefficient of $X^{3m+3n+6}$ is a nonzero polynomial expression in n, m and the coefficients of $r, s, u, v, a, b, c,$ and d that does not vanish when $n = m$ or when $b = c = 0$.

Thus, we have

$$\begin{aligned} & \operatorname{Res}_X \left(\operatorname{Res}_Y \left(f, \frac{\partial f}{\partial Y} \right), \operatorname{Res}_Y \left(f, \frac{\partial f}{\partial X} \right) \right) \Big|_{b=0, c=0} \\ &= \operatorname{Res}_X \left(\operatorname{Res}_Y \left(f|_{b=0, c=0}, \frac{\partial (f|_{b=0, c=0})}{\partial Y} \right), \operatorname{Res}_Y \left(f|_{b=0, c=0}, \frac{\partial (f|_{b=0, c=0})}{\partial X} \right) \right). \end{aligned} \quad (13)$$

Of course, we also substitute $db/dx \mapsto 0$ and $dc/dx \mapsto 0$ in the expression on the left side of the equality. This equality together with Lemma 3.2 show that we may choose coefficients for $r, s, u, v, a, b, c,$ and d such that $\operatorname{Res}_X(\operatorname{Res}_Y(f, \partial f/\partial Y), \operatorname{Res}_Y(f, \partial f/\partial X)) \neq 0$, which implies that C_0 , and a fortiori $C \cap U_0$ do not have a singular point. Observe that $C \cap L_\infty \cap E = \emptyset$ since $\deg(rsa) = 2m - n + 2$. So, to grant the smoothness of the points of C in $C \cap L_\infty$, it suffices to grant the smoothness of the points with zero abscissa of the plane curve defined by $f_n(X, Y) = 0$, where $f_n(X, Y)$ is obtained from $f_0(X, Y)$ by substituting $X \mapsto 1/X$ and multiplying the resulting coefficient of Y^i by $X^{2n-m+2-i(n-m)}$, $i = 0, 1, 2, 3$. To this end, it suffices to have $\operatorname{Res}_Y(f_n, \partial f_n/\partial Y)(0) \neq 0$ but this is just another open condition on the (leading) coefficients of $r, s, u, v, a, b, c,$ and d , and so we may take it for granted. Finally, we should check the smoothness of the points of C in $C \cap (E \setminus L_\infty)$ or, equivalently, we must check the smoothness of the points of zero ordinate of the plane curve defined by $f_{n+1}(X, Y) := uvdY^3 + ruvcY^2 + rsubY + rsa = 0$. But this is already granted because, since rsa is a polynomial without multiple roots, the curves defined by $f_{n+1}(X, Y) = 0$ and $(\partial f_{n+1}/\partial X)(X, Y) = 0$ have no points with zero ordinate in common. This shows that C is a nonsingular curve and, by Remark 3.3, C is an irreducible (trigonal, of genus $m + n + 2$) nonsingular curve. Using Remark 2.1, it is easy to see that C has exactly σ (respectively, ρ) total (respectively, ordinary) ramification points lying over E and η (respectively, ξ) total (respectively, ordinary) ramification points lying over D .

To find curves C with ramification points as described in the other items of Proposition 3.1 we proceed exactly as above, except for the starting curve C_0 . If $\sigma, \rho, \xi,$ and η satisfy condition (ii) in Proposition 3.1, then we begin with a plane curve defined by $f(X, Y) = r s Y^3 + r u c Y + u d$, where $r, s, u, c,$ and d are polynomials in X of degrees $\sigma, \rho, \eta, n + 2 - \sigma - \eta,$ and $2n - m + 2 - \eta$ respectively. If $\sigma, \rho, \xi,$ and η satisfy conditions (iii) in Proposition 3.1, then we begin with a plane curve defined by $f(X, Y) = r a Y^3 + r u b Y^2 + u v$, where $r, u, v, a,$ and b are polynomials in X of degrees $\sigma, \eta, \xi, 2m - n + 2 - \sigma,$ and $m + 2 - \sigma - \eta$ respectively. If $\sigma, \rho, \xi,$ and η satisfy conditions (iv) in Proposition 3.1, then we begin with a plane curve defined by $f(X, Y) = r Y^3 + u$, where r and u are polynomials in X of degrees $2m + n - 2$ and $2n - m + 2$ respectively. □

Observe that the above proposition and the Riemann-Hurwitz Theorem show that

there are trigonal curves C such that all the points in $C \cap D$ or in $C \cap E$ are total ramification points and these are all the ramification points of C .

The next proposition extends a result in [2], where similar bounds were obtained but only total ramification points were considered.

PROPOSITION 3.5. *Let σ (respectively, ρ) be the number of total (respectively, ordinary) ramification points on E . Let τ (respectively, ζ) be the number of total (respectively, ordinary) ramification points in C . Then one of the following possibilities occurs.*

- (i) $\tau = 2n - m + 2$, $\sigma = 2m - n + 2$ and C has no other ramification points.
- (ii) $0 \leq \tau + \zeta + 2\sigma + \rho \leq 2m + 4$.

PROOF. Let $f_0 = c_3(X)Y^3 + c_2Y^2 + c_1(X)Y + c_0(X)$ be the equation of the plane curve C_0 which is isomorphic to $C \cap U_0$, as we have seen in the last section, and suppose that $\text{Res}_Y(\partial f_0/\partial Y, \partial^2 f_0/\partial Y^2) \neq 0$. It is easy to check that if (a, b) is a total ramification point of C_0 , then a is a root of $12c_3(X)(3c_1(X)c_3(X) - c_2(X)^2) = \text{Res}_Y(\partial f_0/\partial Y, \partial^2 f_0/\partial Y^2)$ and Remark 2.1(i) yields that a must be a root of $3c_1(X)c_3(X) - c_2(X)^2$. From the bounds on the degrees of c_1, c_2 , and c_3 , we see that this polynomial has a degree which is at most $2m + 4$. From Remark 2.1(iv), we have that if $(a', 0)$ is an ordinary (respectively, a total) ramification point of C_{n+1} , then a' is a root (respectively, is at least a double root) of $3c_1(X)c_3(X) - c_2(X)^2$. In a similar way, one may see that if there is a point of the form $(0, b)$ in C_n and it is a total ramification point, then 0 is a root of $3\tilde{c}_1(X)\tilde{c}_3(X) - \tilde{c}_2(X)^2 = (3c_1(1/X)c_3(1/X) - c_2(1/X)^2)X^{2m+4}$. Also, if the point at the origin of k^2 is an ordinary (respectively, a total) ramification point of C_{g-1} , then, by Remark 2.1(v), we must have that 0 is a root (respectively, is at least a double root) of $3\tilde{c}_1(X)\tilde{c}_3(X) - \tilde{c}_2(X)$. Note that these two last cases are mutually exclusive, for the points in the Y -axis of the plane that contains C_{g-1} as well as the points in the Y -axis of the plane that contains C_n are all mapped into the line L_∞ through the appropriate local parametrizations of S_{mn} described in the previous section and of course a line in the scroll may contain at most one ramification point of C . Moreover, for one of these two last cases to occur, we must have $\deg(3c_1(X)c_3(X) - c_2(X)^2) < 2m + 4$ so that anyway, we get $0 \leq \tau + 2\sigma + \rho \leq 2m + 4$.

Suppose now that $3c_1(X)c_3(X) - c_2(X)^2 = 0$. Then by Remarks 2.1(iv) and 2.3, we see that $c_3(X)$ does not have a multiple root and that $c_3(X) \mid c_1(X)$. Observe that, by Remark 2.2, we must have $\deg c_3(X) = 2m - n + 2$, $\deg c_2(X) = m + 2$, and $\deg c_1(X) = n + 2$ or else $\deg c_3(X) = 2m - n + 1$, $\deg c_2(X) = m + 1$, and $\deg c_1(X) = n + 1$. Either way, from Remark 2.1, we have that $\sigma = 2m - n + 2$ and $\rho = 0$. To establish the first assertion now, we use the Riemann-Hurwitz Theorem and note that it is enough to show that there is no ordinary ramification point of C in $C \setminus E$. From $3c_1(X)c_3(X) = c_2(X)^2$, it is easy to verify that $12c_3(X)(\partial f_0/\partial Y)(X, Y) = (\partial^2 f_0/\partial Y^2)^2(X, Y)$ and also that there is no ordinary ramification point in C_0 . Finally, we note that $3\tilde{c}_1(X)\tilde{c}_3(X) = \tilde{c}_2(X)^2$, hence, $12\tilde{c}_3(X)(\partial f_n/\partial Y)(X, Y) = (\partial^2 f_n/\partial Y^2)^2(X, Y)$. Then, as above, we conclude that there is no ordinary ramification point in C_n , which completes the proof. \square

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