Internat. J. Math. & Math. Sci.
Vol. 22, No. 3 (1999) 605–610
S 0161-1712(99)22605-1

ON A CLASS OF UNIVALENT FUNCTIONS

DINGGONG YANG and JINLIN LIU

(Received 15 September 1997)

ABSTRACT. We consider the class of univalent functions defined by the conditions $f(z)/z \neq 0$ and $|(z/f(z))''| \leq \alpha, |z| < 1$, where $f(z) = z + \cdots$ is analytic in $|z| < 1$ and $0 < \alpha \leq 2$.

Keywords and phrases. Univalent functions, subordination.

1991 Mathematics Subject Classification. 30C45.

1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be star-like in $|z| < r\,(r \leq 1)$ if and only if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad (|z| < r). \tag{1.2}$$

In [2], Nunokawa, Obradovic, and Owa proved the following theorem:

Theorem A. *Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let*

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1, \quad (z \in E). \tag{1.3}$$

Then $f(z)$ is univalent in E.

For $0 < \alpha \leq 2$, let $S(\alpha)$ denote the class of functions $f(z) \in A$ which satisfy the conditions

$$f(z) \neq 0 \quad \text{for } 0 < |z| < 1 \tag{1.4}$$

and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha, \quad (z \in E). \tag{1.5}$$

In this paper, we give an extension of Theorem A and obtain some results for the class $S(\alpha)$.

By virtue of a result due to Ozaki and Nunokawa [4], Obradovic et al. [3] considered a class of univalent functions.

2. A criterion for univalence

Theorem 1. *Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let $g(z) \in A$ be bounded* *in E and satisfy*

$$m = \inf\left\{ \left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| : z_1, z_2 \in E \right\} > 0. \tag{2.1}$$

If

$$\left| \left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)'' \right| \le K, \quad (z \in E), \tag{2.2}$$

where

$$K = \frac{2m}{M^2} \qquad \text{and} \qquad M = \sup\{ |g(z)| : z \in E \}, \tag{2.3}$$

then $f(z)$ is univalent in E.

Proof. If we put

$$h(z) = \left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)'', \tag{2.4}$$

then the function $h(z)$ is analytic in E and, by integration from 0 to z, we get

$$\left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)' = b_2 - a_2 + \int_0^z h(u)\,du \tag{2.5}$$

and

$$\frac{z}{f(z)} - \frac{z}{g(z)} = (b_2 - a_2)z + \int_0^z dv \int_0^v h(u)\,du, \tag{2.6}$$

where $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = z + b_2 z^2 + \cdots$.

Thus, we have

$$f(z) = \frac{g(z)}{1 + (b_2 - a_2)g(z) + g(z)(\psi(z)/z)}, \tag{2.7}$$

where

$$\psi(z) = \int_0^z dv \int_0^v h(u)\,du. \tag{2.8}$$

Since

$$\left(\frac{\psi(z)}{z} \right)' = \frac{1}{z^2} \int_0^z u\psi''(u)\,du = \frac{1}{z^2} \int_0^z u h(u)\,du, \tag{2.9}$$

from (2.2) and (2.4), we get

$$\left| \left(\frac{\psi(z)}{z} \right)' \right| \le \int_0^1 t|h(zt)|\,dt \le \frac{K}{2}, \tag{2.10}$$

and so

$$\left| \frac{\psi(z_2)}{z_2} - \frac{\psi(z_1)}{z_1} \right| = \left| \int_{z_1}^{z_2} \left(\frac{\psi(z)}{z} \right)' dz \right| \le \frac{K}{2} |z_2 - z_1| \tag{2.11}$$

for $z_1, z_2 \in E$ and $z_1 \ne z_2$.

If $z_1 \ne z_2$ then $g(z_1) \ne g(z_2)$ and it follows, from (2.7) and (2.11), that

$$|f(z_1)-f(z_2)| = \frac{\left|g(z_1)-g(z_2)+g(z_1)g(z_2)\left(\frac{\psi(z_2)}{z_2}-\frac{\psi(z_1)}{z_1}\right)\right|}{\left|1+(b_2-a_2)g(z_1)+g(z_1)\frac{\psi(z_1)}{z_1}\right|\left|1+(b_2-a_2)g(z_2)+g(z_2)\frac{\psi(z_2)}{z_2}\right|}$$

$$> \frac{|g(z_1)-g(z_2)|-M^2K\frac{|z_1-z_2|}{2}}{\left|1+(b_2-a_2)g(z_1)+g(z_1)\frac{\psi(z_1)}{z_1}\right|\left|1+(b_2-a_2)g(z_2)+g(z_2)\frac{\psi(z_2)}{z_2}\right|} \geq 0. \tag{2.12}$$

Hence, $f(z)$ is univalent in E. □

Corollary 1. *Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$. If*

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2, \quad (z \in E), \tag{2.13}$$

then $f(z)$ is univalent in E. The bound 2 in (2.13) is best possible.

Proof. Setting $g(z) = z$ in Theorem 1, we conclude that $f(z)$ is univalent in E for $f(z)$ satisfying condition (2.13).

To show that the result is sharp, we consider

$$f(z) = \frac{z}{(1+z)^{2+\epsilon}}, \quad (\epsilon > 0). \tag{2.14}$$

Note that

$$\left|\left(\frac{z}{f(z)}\right)''\right| = (2+\epsilon)(1+\epsilon)|1+z|^{\epsilon}, \quad (z \in E) \tag{2.15}$$

and $f'(1/(1+\epsilon)) = 0$. Hence, $f(z)$ is not univalent in E and the proof is complete. □

From Corollary 1, we easily get

Corollary 2. *Let*

$$f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n} \in A \tag{2.16}$$

and

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2. \tag{2.17}$$

Then $f(z)$ is univalent in E.

3. The class $S(\alpha)$. According to Corollary 1, all the functions in $S(\alpha)(0 < \alpha \leq 2)$ are univalent in E. Let the functions $f(z)$ and $g(z)$ be analytic in E. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in E, with $w(0) = 0$ and $|w(z)| < 1 (z \in E)$, such that $f(z) = g(w(z))$ for $z \in E$.

For our next results, we need the following.

Lemma 1 [5]. *Let $f(z)$ and $g(z)$ be analytic in E with $f(0) = g(0)$. If $h(z) = zg'(z)$ is star-like in E and $zf'(z) \prec h(z)$, then*

$$f(z) \prec g(z) = g(0) + \int_0^z \frac{h(t)}{t}\,dt. \tag{3.1}$$

Theorem 2. *Let $f(z) = z + a_2z^2 + \cdots \in S(\alpha)$ with $0 < \alpha \le 2$. Then, for $z \in E$,*

$$\left| \frac{z}{f(z)} - 1 \right| \le |z| \left(|a_2| + \frac{\alpha}{2}|z| \right); \tag{3.2}$$

$$1 - |z| \left(|a_2| + \frac{\alpha}{2}|z| \right) \le \operatorname{Re} \frac{z}{f(z)} \le 1 + |z| \left(|a_2| + \frac{\alpha}{2}|z| \right); \tag{3.3}$$

$$|f(z)| \ge \frac{|z|}{1 + |a_2||z| + \frac{\alpha}{2}|z|^2}. \tag{3.4}$$

Equalities in (3.2), (3.3), and (3.4) are attained if we take

$$f(z) = \frac{z}{1 \pm az + \frac{\alpha}{2}z^2} \in S(\alpha), \quad \left(0 \le a \le \sqrt{2\alpha}\right). \tag{3.5}$$

Proof. In view of (1.5), we have

$$z \left(\frac{z}{f(z)} \right)'' \prec \alpha z. \tag{3.6}$$

Applying the lemma to (3.6), we find that

$$\left(\frac{z}{f(z)} \right)' + a_2 \prec \alpha z. \tag{3.7}$$

By using a result of Hallenbeck and Ruscheweyh [1, Thm. 1], (3.7) gives

$$\frac{1}{z} \int_0^z \left[\left(\frac{t}{f(t)} \right)' + a_2 \right] dt \prec \frac{\alpha}{2} z, \tag{3.8}$$

i.e.,

$$\frac{z}{f(z)} = 1 - a_2 z + \frac{\alpha}{2} z w(z), \tag{3.9}$$

where $w(z)$ is analytic in E and $|w(z)| \le |z| (z \in E)$ by Schwarz lemma.

Now, from (3.9), we can easily derive the inequalities (3.2), (3.3), and (3.4). □

Theorem 3. *Let $f(z) \in S(\alpha)$ and have the form*

$$f(z) = z + a_3 z^3 + a_4 z^4 + \cdots. \tag{3.10}$$

(a) *If $2/\sqrt{5} \le \alpha \le 2$, then $f(z)$ is star-like in $|z| < \sqrt{2/\alpha} \cdot 1/\sqrt[4]{5}$;*

(b) *If $\sqrt{3} - 1 \le \alpha \le 2$, then $\operatorname{Re} f'(z) > 0$ for $|z| < \sqrt{(\sqrt{3}-1)/\alpha}$.*

Proof. If we put

$$p(z) = \frac{z^2 f'(z)}{f^2(z)} = 1 + p_2 z^2 + \cdots, \tag{3.11}$$

then, by (1.5), we have

$$zp'(z) = -z^2\left(\frac{z}{f(z)}\right)'' \prec \alpha z, \tag{3.12}$$

and it follows, from the lemma, that

$$p(z) \prec 1+\alpha z, \tag{3.13}$$

which implies that

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \le \alpha |z|^2, \quad (z \in E). \tag{3.14}$$

(a) Let $2/\sqrt{5} \le \alpha \le 2$ and

$$|z| < r_1 = \sqrt{\frac{2}{\alpha}} \cdot \frac{1}{\sqrt[4]{5}}. \tag{3.15}$$

Then, by (3.14), we have

$$\left|\arg \frac{z^2 f'(z)}{f^2(z)}\right| < \arcsin \frac{2}{\sqrt{5}}. \tag{3.16}$$

Also, from (3.2) in Theorem 2 with $a_2 = 0$, we obtain

$$\left|\frac{z}{f(z)} - 1\right| < \frac{\alpha}{2} r_1^2, \tag{3.17}$$

and so

$$\left|\arg \frac{z}{f(z)}\right| < \arcsin \frac{1}{\sqrt{5}}. \tag{3.18}$$

Therefore, it follows, from (3.16) and (3.18), that

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \left|\arg \frac{z^2 f'(z)}{f^2(z)}\right| + \left|\arg \frac{z}{f(z)}\right| < \arcsin \frac{2}{\sqrt{5}} + \arcsin \frac{1}{\sqrt{5}} = \frac{\pi}{2} \tag{3.19}$$

for $|z| < r_1$. This proves that $f(z)$ is star-like in $|z| < r_1$.

(b) Let $\sqrt{3}-1 \le \alpha \le 2$ and

$$|z| < r_2 = \sqrt{\frac{\sqrt{3}-1}{\alpha}}. \tag{3.20}$$

Then we have

$$\begin{aligned} |\arg f'(z)| &\le \left|\arg \frac{z^2 f'(z)}{f^2(z)}\right| + 2\left|\arg \frac{z}{f(z)}\right| < \arcsin(\alpha r_2^2) + 2\arcsin\left(\frac{\alpha}{2} r_2^2\right) \\ &= \arcsin\left(\sqrt{3}-1\right) + 2\arcsin\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi}{2}. \end{aligned} \tag{3.21}$$

Thus, $\operatorname{Re} f'(z) > 0$ for $|z| < r_2$. □

Corollary 3. *Let $f(z) \in S(\alpha)$ and have the form (3.10)*

(a) *if $0 < \alpha \le 2/\sqrt{5}$, then $f(z)$ is star-like in E;*

(b) *if $0 < \alpha \le \sqrt{3}-1$, then $\operatorname{Re} f'(z) > 0$ for $z \in E$.*

References

[1] D. J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc. **52** (1975), 191–195. MR 51 10603. Zbl 311.30010.

[2] M. Nunokawa, M. Obradovic, and S. Owa, *One criterion for univalency*, Proc. Amer. Math. Soc. **106** (1989), no. 4, 1035–1037. MR 91h:30020. Zbl 672.30022.

[3] M. Obradovic, N. N. Pascu, and I. Radomir, *A class of univalent functions*, Math. Japon. **44** (1996), no. 3, 565–568. MR 97i:30016. Zbl 868.30013.

[4] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (1972), 392–394. MR 45 8821. Zbl 233.30011.

[5] T. J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. **37** (1970), 775–777. MR 42#4722. Zbl 206.36202.

Yang: Department of Mathematics, Suzhou University, Suzhou 215006, China

Liu: Water Conservancy College, Yangzhou University, Yangzhou 225009, China