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ON THE RITT ORDER AND TYPE OF A CERTAIN CLASS OF FUNCTIONS DEFINED BY *BE*-DIRICHLETIAN ELEMENTS

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Abstract. We introduce the notions of Ritt order and type to functions defined by the series

$$\sum_{n=1}^{\infty} f_n(\sigma+i\tau_0)\exp(-s\lambda_n), \quad s=\sigma+i\tau,\ (\sigma,\tau)\in\mathbf{R}\times\mathbf{R} \qquad (*)$$

indexed by τ_0 on $\mathbf{R}$, where $(\lambda_n)_1^\infty$ is a D-sequence and $(f_n)_1^\infty$ is a sequence of entire functions of bounded index with at most a finite number of zeros. By definition, the series are *BE*-Dirichletian elements. The notions of order and type of functions, defined by *B*-Dirichletian elements, are considered in [3, 4]. In this paper, using a technique similar to that used by M. Blambert and M. Berland [6], we prove the same properties of Ritt order and type for these functions.

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1. Preliminary lemmas

Definition 1.1 (B. Lepson [10]). An entire function f is said to be of bounded index if there exists a nonnegative integer ν such that

$$\max\left\{\frac{|f^{(k)}(s)|}{k!}\,\middle|\, k\in\{0,1,\dots,\nu\}\right\}\geq\frac{|f^{(j)}(s)|}{j!}, \quad (f^{(0)}(s)=f(s)) \qquad (1.1)$$

for all j and for all s. The least such integer ν is called the index of f.

Theorem A (F. Gross [8]). *An entire function with at most a finite number of zeros is of bounded index if and only if it is of the form $P(s)\exp(\alpha s)$, where $P(s)$ is polynomial and α is a complex constant.*

Theorem B (S. M. Shah [16]). *Let $f(s)=P(s)\exp(\alpha s)$, where α is any complex number and $P(s)$ is a polynomial of degree less than n. Then f is of bounded index and the index $\nu\leq p$, where p is any integer such that $p\geq n-1$ and*

$$\frac{n|\alpha|}{p+1}+\left(\frac{n(n-1)}{2!}|\alpha|^2\right)\frac{1}{p(p+1)}+\cdots+\frac{|\alpha|^n}{(p-n+2)\cdots(p+1)}\leq 1. \qquad (1.2)$$

Let $(\lambda_n)_1^\infty$ be a D-sequence (that is a positive strictly increasing unbounded sequence) and $(f_n)_1^\infty$ be a sequence of entire functions f_n of bounded index ν_n with at most a finite number of zeros from Theorem A. As a result of the two theorems, we have $\forall s \in \mathbf{C}$, $\forall n \in \mathbf{N}\setminus\{0\}$

$$f_n(s) = P_n(s)\exp(\alpha_n s), \tag{1.3}$$

where $P_n(s)$ is a polynomial of degree m_n and α_n is a complex constant, that is,

$$s \longmapsto P_n(s) = \sum_{j=0}^{m_n} a_{n,j}\, s^j \quad \text{with } a_{n,m_n} \neq 0 \text{ and } s \in \mathbf{C}. \tag{1.4}$$

Let us suppose that $\exists k \in]0,\lambda_1[$, $\forall n \in \mathbf{N}\setminus\{0\}$

$$\alpha_n \in \overline{d_{(0,k)}}, \tag{1.5}$$

where $\overline{d_{(0,k)}}$ is the closed disc centered at 0 and of radius k.

Consider the space of elements

$$\{f_{\tau_0}\} : \sum_{n=1}^{\infty} f_n(\sigma + i\tau_0)\exp(-s\lambda_n), \quad s = \sigma + i\tau,\ (\sigma,\tau) \in \mathbf{R}\times\mathbf{R} \tag{1.6}$$

indexed by τ_0 on $\mathbf{R}$. By definition, $\{f_{\tau_0}\}$ is the *BE*-Dirichletian element. Let

$$\beta = \limsup_{n\to\infty}\left\{\frac{m_n}{\lambda_n}\right\}, \tag{1.7}$$

$$A_n = \max\left\{|a_{nj}| \mid j \in \{0,1,\dots,m_n\}\right\} \quad \forall n \in \mathbf{N}\setminus\{0\}. \tag{1.8}$$

Consider the associated Dirichletian element

$$\{f_A\} : \sum_{n=1}^{\infty} A_n \exp(-s\lambda_n), \tag{1.9}$$

whose coefficients are strictly positive and denote, by $\sigma_c^{f_A}$, the abscissa of convergence of $\{f_A\}$.

Let us state three lemmas due to M. Blambert and M. Berland [6] which we use later. These demonstrations are obvious because this sequence $(\alpha_n)_1^{\infty}$ is bounded.

Lemma 1.1. *If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N}\setminus\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, then $\{f_{\tau_0}\}$ converges absolutely on $\mathbf{C}$ for any arbitrary τ_0 in $\mathbf{R}$.*

Lemma 1.2. *If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N}\setminus\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, we have $\forall \tau_0 \in \mathbf{R}$, $\forall \tau \in \mathbf{R}$*

$$\lim_{\sigma\to\infty} f_{\tau_0}(\sigma + i\tau) = 0. \tag{1.10}$$

Lemma 1.3. *If $\sigma_c^{f_A} = -\infty$, $\forall n \in \mathbf{N}\setminus\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, we have $\forall \tau_0 \in \mathbf{R}$, $\forall \sigma \in \mathbf{R}$*

$$\begin{aligned} P_n(\sigma + i\tau_0)&\exp[\alpha_n(\sigma + i\tau_0) - \sigma\lambda_n] \\ &= \lim_{\tau_2\to\infty}\left\{\frac{1}{\tau_2}\int_{\tau_1}^{\tau_2} f_{\tau_0}(\sigma + i\tau)\exp(i\tau\lambda_n)\,d\tau\right\}, \end{aligned} \tag{1.11}$$

where τ_1 is any arbitrary real number.

2. Main theorems. Let us define the following quantities. For each σ on $\mathbf{C}$,

$$M(\sigma;f_{\tau_0}) = \sup\{|f_{\tau_0}(\sigma'+i\tau')| \mid \sigma' \geq \sigma, \tau' \in \mathbf{R}\}, \tag{2.1}$$

$$M_{n'}(\sigma;f_{\tau_0}) = \sup\{|f_{\tau_0,n'}(\sigma'+i\tau')| \mid \sigma' \geq \sigma, \tau' \in \mathbf{R}\}, \tag{2.2}$$

$$\mu(\sigma;f_{\tau_0}) = \sup\{|f_n(\sigma+i\tau_0)|\exp(-\sigma\lambda_n) \mid n \in \mathbf{N}\backslash\{0\}\}, \tag{2.3}$$

$$\mu_{n'}(\sigma;f_{\tau_0}) = \sup\{|f_n(\sigma+i\tau_0)|\exp(-\sigma\lambda_n) \mid n \geq n'\}; \tag{2.4}$$

where

$$f_{\tau_0,n'}(s) = \sum_{n=n'}^{\infty} f_n(\sigma+i\tau_0)\exp(-s\lambda_n). \tag{2.5}$$

The quantities defined above are finite.

Remark. The function $\sigma \mapsto M(\sigma;f_{\tau_0})$ is decreasing onto $\mathbf{R}$.

Theorem 2.1. *If* $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N}\backslash\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, *we have*

$$\lim_{\sigma\to\infty}\{M(\sigma;f_{\tau_0})\} = 0 \qquad \textit{and} \qquad \lim_{\sigma\to-\infty}\{M(\sigma;f_{\tau_0})\} = \infty. \tag{2.6}$$

Proof. We have $\forall\varepsilon\in]0,1[$, $\exists n'(=n'_\varepsilon) \in \mathbf{N}\backslash\{0\}$, $\forall n \geq n'$, $m_n/\lambda_n < \beta+\varepsilon$ and $\exists n''(=n''_\varepsilon) \in \mathbf{N}\backslash\{0\}$, $\forall n \geq n''$, $k/\lambda_n < \varepsilon$, $\forall\sigma > 0$ such that

$$\begin{aligned}\sum_{n=n_1(=\max\{n',n''\})}^{\infty} & |f_n(\sigma+i\tau_0)|\exp(-\sigma\lambda_n)\\ &\leq \sum_{n=n_1}^{\infty} A_n \exp\left\{-\sigma\lambda_n\left[1-\left(\frac{(\beta+\varepsilon)\log(1+|\sigma|+|\tau_0|)}{\sigma}+\varepsilon\left(1+\frac{|\tau_0|}{\sigma}\right)\right)\right]\right\},\end{aligned} \tag{2.7}$$

$\forall\varepsilon' \in]0,1-\varepsilon[$, $\exists\sigma'(=\sigma_{\varepsilon'}) > 0$, $\forall\sigma > \sigma'$,

$$\frac{(\beta+\varepsilon)\log(1+|\sigma|+|\tau_0|)+\varepsilon|\tau_0|}{\sigma} < \varepsilon' \tag{2.8}$$

and

$$\begin{aligned}\sigma&\left[1-\left(\frac{(\beta+\varepsilon)\log(1+|\sigma|+|\tau_0|)+\varepsilon|\tau_0|}{\sigma}+\varepsilon\right)\right]\\ &> \sigma[(1-\varepsilon)-\varepsilon'] > \sigma'[(1-\varepsilon)-\varepsilon'](>0).\end{aligned} \tag{2.9}$$

Therefore, $\exists n_1 \in \mathbf{N}\backslash\{0\}$ such that

$$M_{n_1}(\sigma;f_{\tau_0}) < f_{A,n_1}(\sigma'[(1-\varepsilon)-\varepsilon']) = \sum_{n=n_1}^{\infty} A_n\exp[-\sigma'((1-\varepsilon)-\varepsilon')\lambda_n], \tag{2.10}$$

where

$$\lim_{\sigma\to\infty}\{M_{n_1}(\sigma;f_{\tau_0})\} = 0. \tag{2.11}$$

On the other hand, we have, $\forall n \in \{1,2,\ldots,n_1-1\}$

$$\lim_{\sigma\to\infty}\{P_n(\sigma+i\tau_0)\exp[\alpha_n(\sigma+i\tau_0)-\sigma\lambda_n]\}=0 \quad (\text{with } \lambda_1>k \text{ and } \alpha_n\in\overline{d_{(0,k)}}). \tag{2.12}$$

We have

$$\lim_{\sigma\to\infty}\{M(\sigma;f_{\tau_0})\}=0. \tag{2.13}$$

On the other hand, $\forall\sigma<0$, if $M(\sigma;f_{\tau_0})$ is bounded onto $\mathbf{R}$ implies that (from Lemma 1.3)

$$\forall n\in\mathbf{N}\backslash\{0\}:\{j\in\{0,1,\ldots,m_n\}\Longrightarrow a_{n,j}=0\}. \tag{2.14}$$

Or, thus, we get the contradiction that

$$a_{n,m_n}\neq 0 \quad \forall n\in\mathbf{N}\backslash\{0\} \tag{2.15}$$

and

$$\lim_{\sigma\to-\infty}\{M(\sigma;f_{\tau_0})\}=\infty. \tag{2.16}$$

Thus, (2.13) and (2.16) prove the theorem. □

Furthermore, let

$$\rho_R^{f_{\tau_0}}=\limsup_{\sigma\to-\infty}\left\{\frac{\log^+\left(\log^+\left(M(\sigma;f_{\tau_0})\right)\right)}{-\sigma}\right\}, \tag{2.17}$$

$$\lambda_R^{f_{\tau_0}}=\liminf_{\sigma\to-\infty}\left\{\frac{\log^+\left(\log^+\left(M(\sigma;f_{\tau_0})\right)\right)}{-\sigma}\right\}. \tag{2.18}$$

By definition, $\rho_R^{f_{\tau_0}}$ and $\lambda_R^{f_{\tau_0}}$ are the Ritt-order and the lower Ritt-order of function f_{τ_0} defined by *BE*-Dirichletian element $\{f_{\tau_0}\}$. Also, $M(\sigma;f_A)$ is defined in a similar manner with f_A in the place of f_{τ_0}. It is trivial that

$$\rho_R^{f_A}=\limsup_{\sigma\to-\infty}\left\{\frac{\log^+\left(\log^+(f_A(\sigma))\right)}{-\sigma}\right\}, \tag{2.19}$$

$$\lambda_R^{f_A}=\liminf_{\sigma\to-\infty}\left\{\frac{\log^+\left(\log^+(f_A(\sigma))\right)}{-\sigma}\right\}. \tag{2.20}$$

Theorem 2.2. *If* $\sigma_c^{f_A}=-\infty$, $\beta<\infty$, $\forall n\in\mathbf{N}\backslash\{0\}$, $\alpha_n\in\overline{d_{(0,k)}}$, *and* $L(=\limsup_{n\to\infty}\{\log n/\lambda_n\})<\infty$, *we have* $\forall\tau_0\in\mathbf{R}$,

$$\rho_R^{f_{\tau_0}}=\rho_R^{f_A} \qquad \textit{and} \qquad \lambda_R^{f_{\tau_0}}=\lambda_R^{f_A}. \tag{2.21}$$

Proof. (1) We get the inequalities, $\forall\tau_0\in\mathbf{R}$,

$$\rho_R^{f_A}\le\rho_R^{f_{\tau_0}} \qquad \text{and} \qquad \lambda_R^{f_A}\le\lambda_R^{f_{\tau_0}}. \tag{2.22}$$

τ_0 is any arbitrary real number. Consider the closed interval $I(s,\lambda)=\{s'\in\mathbf{C}\mid|\sigma'-\sigma|\le\lambda>0,\tau'=\tau_0\}$, where $\sigma'=\mathrm{Re}(s')$, $\tau'=\mathrm{Im}(s')$, and $s=\sigma+i\tau_0$. Let $\forall n\in\mathbf{N}\backslash\{0\}$,

$$p_n(s,\lambda) = \sup\left\{ |P_n(s')| \mid s' \in \overline{d_{(s,\lambda)}} \right\}, \tag{2.23}$$

$$p_n^*(s,\lambda) = \sup\left\{ |P_n(s')| \mid s' \in I(s,\lambda) \right\}. \tag{2.24}$$

Using Lemma 1.3, we have $\forall \sigma' \in [\sigma-\lambda, \sigma+\lambda], \forall n \in \mathbf{N}\backslash\{0\}$,

$$|P_n(\sigma' + i\tau_0)| \exp\left[(\operatorname{Re}(\alpha_n) - \lambda_n)\sigma' - \operatorname{Im}(\alpha_n)\tau_0\right] \le M(\sigma-\lambda; f_{\tau_0}), \tag{2.25}$$

and then (M. Blambert and M. Berland [6])

$$p_n^*(s,\lambda) \exp\left[-(\sigma+\lambda)(\lambda_n - \operatorname{Re}(\alpha_n)) - \operatorname{Im}(\alpha_n)\tau_0\right] \le M(\sigma-\lambda; f_{\tau_0}), \tag{2.26}$$

$$6^{-m_n} p_n(s,\lambda) \le p_n^*(s,\lambda), \tag{2.27}$$

$$A_n(1+|s|)^{-m_n} \le p_n(s,\lambda) \quad \forall \lambda \ge 1, \quad \text{(M. Berland [1]).} \tag{2.28}$$

Therefore, we have $\forall \lambda \ge 1,\ \forall \sigma \in \mathbf{R},\ \forall n \in \mathbf{N}\backslash\{0\}$

$$A_n \le M(\sigma-\lambda; f_{\tau_0}) \exp\left\{\left[\sigma + \lambda + \frac{m_n}{\lambda_n'} \log(6(1+|\sigma|+|\tau_0|)) + \frac{\operatorname{Im}(\alpha_n)}{\lambda_n'}\tau_0\right]\lambda_n'\right\}, \tag{2.29}$$

where $\lambda_n' = \lambda_n - \operatorname{Re}(\alpha_n)$.

We have $\forall \varepsilon \in]0,1[,\ \exists n_1 \in \mathbf{N}\backslash\{0\},\ \forall n \ge n_1$

$$A_n \le M(\sigma-\lambda; f_{\tau_0}) \exp\left\{[\sigma + \lambda + (\beta' + \varepsilon)\log(6(1+|\sigma|+|\tau_0|)) + \varepsilon]\lambda_n'\right\}, \tag{2.30}$$

and $\forall \varepsilon_1 \in]0,1[,\ \exists \sigma_1 (= \sigma_{\varepsilon_1}) > 0,\ \forall \sigma < -\sigma_1$

$$\frac{\lambda + (\beta' + \varepsilon)\log(6(1+|\sigma|+|\tau_0|)) + \varepsilon}{-\sigma} < \varepsilon_1, \tag{2.31}$$

where $\beta' = \limsup_{n\to\infty}\{m_n/\lambda_n'\} (< \infty)$ which implies that, $\forall \varepsilon_1 \in]0,1[,\ \forall n \ge n_1,\ \forall \sigma < \sigma_1$

$$A_n \le M(\sigma-\lambda; f_{\tau_0}) \exp[\sigma(1-\varepsilon_1)\lambda_n']. \tag{2.32}$$

Now, $\forall n \in \mathbf{N}\backslash\{0\}, \operatorname{Re}(\alpha_n) \le k$ (because $\alpha_n \in \overline{d_{(0,k)}}$)

$$\lambda_n' = \lambda_n\left(1 - \frac{\operatorname{Re}(\alpha_n)}{\lambda_n}\right) \ge \lambda_n\left(1 - \frac{k}{\lambda_n}\right). \tag{2.33}$$

We have, $\forall \varepsilon_2 \in]0,1[,\ \exists n' \in \mathbf{N}\backslash\{0\},\ \forall n \ge n'$,

$$\frac{k}{\lambda_n} < \varepsilon_2 \qquad \text{and} \qquad \lambda_n' \ge \lambda_n(1-\varepsilon_2) \quad (\Longrightarrow \beta' = \beta) \tag{2.34}$$

which implies that, $\forall \lambda \ge 1,\ \forall n \ge \max\{n_1, n'\} (= n_2),\ \forall \sigma < -\sigma_1$

$$A_n \exp[-\sigma(1-\varepsilon_1)(1-\varepsilon_2)\lambda_n] \le M(\sigma-\lambda; f_{\tau_0}). \tag{2.35}$$

Put, $\forall n \in \mathbf{N}\backslash\{0\}$

$$\mu_{n_{(1,2)}} = (1-\varepsilon_1)(1-\varepsilon_2)\lambda_n \quad (>0), \tag{2.36}$$

$(\mu_{n_{(1,2)}})$ is a D-sequence. Consider the Dirichletian element

$$\left\{f_{A_{(1,2)}}\right\} : \sum_{n=1}^{\infty} A_n \exp\left(-s\mu_{n_{(1,2)}}\right) \tag{2.37}$$

indexed by the couple $(1,2)$ and denote, by $\sigma_c^{f_{A_{(1,2)}}}$, the abscissa of convergence of $\{f_{A_{(1,2)}}\}$. We have, $\sigma_c^{f_{A_{(1,2)}}} = -\infty$ (since $\sigma_c^{f_A} = -\infty$), $f_{A_{(1,2)}}$ is an entire function and, its Ritt-order is

$$\rho_R^{f_{A_{(1,2)}}} = \limsup_{\sigma\to-\infty}\left\{\frac{\log^+\left(\log^+\left(f_{A_{(1,2)}}(\sigma)\right)\right)}{-\sigma}\right\}, \tag{2.38}$$

and its lower Ritt-order is

$$\lambda_R^{f_{A_{(1,2)}}} = \liminf_{\sigma\to-\infty}\left\{\frac{\log^+\left(\log^+\left(f_{A_{(1,2)}}(\sigma)\right)\right)}{-\sigma}\right\}. \tag{2.39}$$

Now, $\forall\sigma\in\mathbf{R}$,

$$f_{A_{(1,2)}}(\sigma) = f_A\left(\sigma(1-\varepsilon_1)(1-\varepsilon_2)\right) \tag{2.40}$$

and

$$\rho_R^{f_{A_{(1,2)}}} = (1-\varepsilon_1)(1-\varepsilon_2)\rho_R^{f_A}, \tag{2.41}$$

$$\lambda_R^{f_{A_{(1,2)}}} = (1-\varepsilon_1)(1-\varepsilon_2)\lambda_R^{f_A}. \tag{2.42}$$

Put (Q. S. Liu [11]) $\forall\sigma\in\mathbf{R}$,

$$\mu_{n_2}\left(\sigma;f_{A_{(1,2)}}\right) = \sup\left\{A_n\exp\left(-\sigma\mu_{n_{(1,2)}}\right) \mid n\geq n_2\right\}. \tag{2.43}$$

We have $\forall\varepsilon>0, \forall\sigma\in\mathbf{R}$

$$\begin{aligned} f_{A_{(1,2)},n_2}(\sigma) &\leq \sum_{n=n_2}^{\infty}\left(A_n\exp\left[-(\sigma-L-\varepsilon)\mu_{n_{(1,2)}}\right]\right)\exp\left[-(L+\varepsilon)\mu_{n_{(1,2)}}\right] \\ &\leq \mu_{n_2}\left(\sigma-L-\varepsilon;f_{A_{(1,2)}}\right)K_{n_2}(\varepsilon) \end{aligned} \tag{2.44}$$

with $K_{n_2}(\varepsilon) = \sum_{n=n_2}^{\infty}\exp[-(L+\varepsilon)\mu_{n_{(1,2)}}]$.

Hence, $\forall\varepsilon>0$, $\forall\sigma\in\mathbf{R}$

$$f_{A_{(1,2)},n_2}(\sigma) \leq \mu_{n_2}\left(\sigma-L-\varepsilon;f_{A_{(1,2)}}\right)K_{n_2}(\varepsilon). \tag{2.45}$$

Then we have, $\forall\lambda\geq 1$, $\exists n_2=\max\{n_1,n'\}$, $\forall n\geq n_2$, $\forall\sigma<-\sigma_1$

$$A_n\exp\left(-\sigma\mu_{n_{(1,2)}}\right) \leq M(\sigma-\lambda;f_{\tau_0}). \tag{2.46}$$

This implies that

$$\mu_{n_2}\left(\sigma;f_{A_{(1,2)}}\right) \leq M(\sigma-\lambda;f_{\tau_0}). \tag{2.47}$$

From (2.45) and (2.47), we have

$$f_{A_{(1,2)},n_2}(\sigma+L+\varepsilon) \leq M(\sigma-\lambda;f_{\tau_0})K_{n_2}(\varepsilon), \tag{2.48}$$

$$\rho_R^{f_{A_{(1,2)},n_2}} \leq \rho_R^{f_{\tau_0}}\left(\lim_{\sigma\to-\infty}\left(\frac{\sigma-\lambda}{\sigma+L+\varepsilon}\right)\right) = \rho_R^{f_{\tau_0}}. \tag{2.49}$$

Or

$$\rho_R^{f_{A_{(1,2)},n_2}} = \rho_R^{f_{A_{(1,2)}}} \quad \text{(M. Blambert [5]).} \tag{2.50}$$

We have, $\forall \varepsilon_1 \in]0,1[$, $\forall \varepsilon_2 \in]0,1[$

$$\rho_R^{f_{A_{(1,2)}}} = (1-\varepsilon_1)(1-\varepsilon_2)\rho_R^{f_A} \le \rho_R^{f_{\tau_0}}, \tag{2.51}$$

$$\lambda_R^{f_{A_{(1,2)}}} = (1-\varepsilon_1)(1-\varepsilon_2)\lambda_R^{f_A} \le \lambda_R^{f_{\tau_0}}. \tag{2.52}$$

As ε_1 and ε_2 are arbitrary, we have

$$\rho_R^{f_A} \le \rho_R^{f_{\tau_0}}, \quad \text{and then} \quad \lambda_R^{f_A} \le \lambda_R^{f_{\tau_0}}. \tag{2.53}$$

(2) We get the inequalities, $\forall \tau_0 \in \mathbf{R}$,

$$\rho_R^{f_{\tau_0}} \le \rho_R^{f_A} \qquad \text{and} \qquad \lambda_R^{f_{\tau_0}} \le \lambda_R^{f_A}. \tag{2.54}$$

From Theorem 2.1, we have $\forall \varepsilon \in]0,1[$, $\forall \varepsilon' \in]0,1-\varepsilon[$, $\exists \sigma' > 0$, $\forall \sigma < -\sigma'$

$$\begin{aligned} &\sigma\left[1-\left((\beta+\varepsilon)\frac{\log(1+|\sigma|+|\tau_0|)}{|\sigma|}+\varepsilon\left(1+\frac{|\tau_0|}{|\sigma|}\right)\right)\theta_\sigma\right] \\ &\qquad = \sigma\left[1+(\beta+\varepsilon)\frac{\log(1+|\sigma|+|\tau_0|)}{|\sigma|}+\varepsilon\left(1+\frac{|\tau_0|}{|\sigma|}\right)\right] \\ &\qquad > \sigma\left[1+\left((\beta+2\varepsilon)\varepsilon'+\varepsilon\right)\right] = \sigma(1+\varepsilon_1), \end{aligned} \tag{2.55}$$

where $\varepsilon_1 = (\beta+2\varepsilon)\varepsilon'+\varepsilon$, $\theta_\sigma = 1$ if $\sigma > 0$ and $\theta_\sigma = -1$ if $\sigma < 0$.

Hence, $\forall \sigma < -\sigma'$, $\exists n_1 \in \mathbf{N}\backslash\{0\}$,

$$M_{n_1}(\sigma; f_{\tau_0}) \le f_{A,n_1}(\sigma(1+\varepsilon_1)), \tag{2.56}$$

which implies that

$$\rho_R^{f_{\tau_0,n_1}} \le \rho_R^{f_{A,n_1}}(1+\varepsilon_1) = (1+\varepsilon_1)\rho_R^{f_A}, \tag{2.57}$$

and where

$$\rho_R^{f_{\tau_0,n_1}} \le \rho_R^{f_A} \qquad \text{and} \qquad \lambda_R^{f_{\tau_0,n_1}} \le \lambda_R^{f_A}. \tag{2.58}$$

Now, $\sigma \in \mathbf{R}$,

$$M(\sigma; f_{\tau_0}) \le M_{n_1}^0(\sigma; f_{\tau_0}) + M_{n_1}(\sigma; f_{\tau_0}), \tag{2.59}$$

where

$$M_{n_1}^0(\sigma; f_{\tau_0}) = \sup\left\{ \left| f_{\tau_0,n_1}^0(\sigma'+i\tau') \right| \mid \sigma' \ge \sigma, \tau' \in \mathbf{R} \right\} \tag{2.60}$$

and

$$\left\{ f_{\tau_0,n_1}^0 \right\} : \sum_{n=1}^{n_1-1} f_n(\sigma+i\tau_0)\exp(-s\lambda_n). \tag{2.61}$$

Then $\forall \tau_0 \in \mathbf{R}$,

$$\rho_R^{f_{\tau_0}} \le \max\left\{ \rho_R^{f_{\tau_0,n_1}}, \rho_R^{f_{\tau_0,n_1}^0} \right\} = \rho_R^{f_{\tau_0,n_1}} \tag{2.62}$$

since $\rho_R^{f_{\tau_0,n_1}^0} = 0$.

Finally, we have

$$\rho_R^{f_{\tau_0}} \le \rho_R^{f_A}, \tag{2.63}$$

and, similarly, we can show that

$$\lambda_R^{f_{\tau_0}} \le \lambda_R^{f_A}. \tag{2.64}$$

Hence, (1) and (2) implies (2.21) which proves this theorem. □

If $\rho_R^{f_{\tau_0}} > 0$, we put

$$\tau_R^{f_{\tau_0}} = \limsup_{\sigma\to-\infty}\left\{\frac{\log\left(M(\sigma;f_{\tau_0})\right)}{\exp\left(-\sigma\rho_R^{f_{\tau_0}}\right)}\right\}, \tag{2.65}$$

$$\nu_R^{f_{\tau_0}} = \liminf_{\sigma\to-\infty}\left\{\frac{\log\left(M(\sigma;f_{\tau_0})\right)}{\exp\left(-\sigma\rho_R^{f_{\tau_0}}\right)}\right\}. \tag{2.66}$$

By definition, $\tau_R^{f_{\tau_0}}$ and $\nu_R^{f_{\tau_0}}$ are the Ritt-type and the lower Ritt-type of order of f_{τ_0}.

It is trivial that if $\rho_R^{f_A} > 0$,

$$\tau_R^{f_A} = \limsup_{\sigma\to-\infty}\left\{\frac{\log\left(f_A(\sigma)\right)}{\exp\left(-\sigma\rho_R^{f_A}\right)}\right\}, \tag{2.67}$$

$$\nu_R^{f_A} = \liminf_{\sigma\to-\infty}\left\{\frac{\log\left(f_A(\sigma)\right)}{\exp\left(-\sigma\rho_R^{f_A}\right)}\right\}. \tag{2.68}$$

Theorem 2.3. *If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $L(=\lim_{n\to\infty}(\log n/\lambda_n)) = 0$, $\forall n \in \mathbf{N}\backslash\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, $\rho_R^{f_{\tau_0}} > 0$, we have $\forall \tau_0 \in \mathbf{R}$,*

$$\tau_R^{f_{\tau_0}} = \tau_R^{f_A}. \tag{2.69}$$

Proof. (1) We have the inequality, $\forall \tau_0 \in \mathbf{R}$,

$$\tau_R^{f_A} \le \tau_R^{f_{\tau_0}}, \tag{2.70}$$

τ_0 is any arbitrary real number. From Theorem 2.2, we have, $\forall \varepsilon \in]0,1[$, $\exists n_1 \in \mathbf{N}\backslash\{0\}$, $\forall n \ge n_1$, $\forall \sigma \in \mathbf{R}$; $\forall \lambda \ge 1$,

$$A_n \le M(\sigma-\lambda;f_{\tau_0})\exp\left\{[\sigma+\lambda+(\beta'+\varepsilon)\log\left(6(1+|\sigma|+|\tau_0|)\right)+\varepsilon]\lambda'_n\right\}, \tag{2.71}$$

where

$$\lambda'_n = \lambda_n - \operatorname{Re}(\alpha_n) \qquad \text{and} \qquad \beta' = \limsup_{n\to\infty}\left\{\frac{m_n}{\lambda'_n}\right\} \quad (\beta' = \beta) < \infty. \tag{2.72}$$

Now, $\forall \sigma < 0$,

$$A_n \le M(\sigma-\lambda;f_{\tau_0})\exp\left\{\left[\sigma-\lambda-\frac{\sigma\left(2\lambda+(\beta+\varepsilon)\log\left(6(1+|\sigma|+|\tau_0|)\right)+\varepsilon\right)}{-\sigma}\right]\lambda'_n\right\}. \tag{2.73}$$

Also, we have $\forall \varepsilon_1 \in]0,1[$, $\exists \sigma_1 (=\sigma_{\varepsilon_1}) > 0$, $\forall \sigma < -\sigma_1$,

$$\frac{2\lambda(\beta+\varepsilon)\log(6(1+|\sigma|+|\tau_0|))+\varepsilon}{-\sigma} < \varepsilon_1, \tag{2.74}$$

$$\begin{aligned} A_n &\le M(\sigma-\lambda; f_{\tau_0})\exp\left\{[\sigma(1-\varepsilon_1)-\lambda]\lambda'_n\right\} \\ &\le M(\sigma-\lambda; f_{\tau_0})\exp\left[\left(\sigma-\frac{\lambda}{1-\varepsilon_1}\right)\mu_{n_{(1,2)}}\right], \end{aligned} \tag{2.75}$$

where $\forall n \ge n_2 = \max\{n_1, n\}$,

$$\mu_{n_{(1,2)}} = (1-\varepsilon_1)(1-\varepsilon_2)\lambda_n, \qquad \lambda'_n \ge \lambda_n(1-\varepsilon_2) \quad (\varepsilon_2 \in]0,1[) \tag{2.76}$$

which implies that, $\forall \lambda \ge 1$, $\forall n \ge n_2$, $\forall \sigma < -\sigma_1$

$$A_n \le M\left(\sigma-\lambda; f_{\tau_0}\right)\exp\left[\sigma\left(1-\frac{\lambda}{1-\varepsilon_1}\right)\mu_{n_{(1,2)}}\right]. \tag{2.77}$$

$\tau_R^{f_{\tau_0}}$ is the Ritt-type of order of f_{τ_0}, we have, $\forall \varepsilon' > 0$, $\exists \sigma' (=\sigma_{\varepsilon'}) > 0$, $\forall \sigma < -\sigma'$

$$\log\left(M(\sigma-\lambda; f_{\tau_0})\right) \le \left(\tau_R^{f_{\tau_0}} + \varepsilon'\right)\exp\left[-(\sigma-\lambda)\rho_R^{f_{\tau_0}}\right]. \tag{2.78}$$

Hence, $\forall n \ge n_2$, $\forall \varepsilon' > 0$,

$$\log A_n \le \left(\tau_R^{f_{\tau_0}} + \varepsilon'\right)\exp\left[-(\sigma-\lambda)\rho_R^{f_{\tau_0}}\right] + \left(\sigma - \frac{\lambda}{(1-\varepsilon_1)}\right)\mu_{n_{(1,2)}}. \tag{2.79}$$

Let us consider f_n, the function defined by

$$f_n(\sigma) = a\exp[-(\sigma-\lambda)b] + \mu_{n_{(1,2)}}(\sigma+c), \tag{2.80}$$

and indexed by $n > n_2$. Choosing

$$a = \tau_R^{f_{\tau_0}} + \varepsilon' > 0, \qquad b = \rho_R^{f_{\tau_0}} > 0, \qquad c = \frac{\lambda}{\varepsilon_1 - 1}, \tag{2.81}$$

we get $\forall n \ge n_2$, $\forall \sigma \in \mathbf{R}\setminus\{\sigma_n\}$,

$$f_n(\sigma) > f_n(\sigma_n) \tag{2.82}$$

with

$$\sigma_n - \lambda = \frac{1}{b}\log\left(\frac{ab}{\mu_{n_{(1,2)}}}\right) \tag{2.83}$$

and

$$\lim_{n\to\infty}\sigma_n = -\infty \Longrightarrow \exists n_3 \in \mathbf{N}\setminus\{0\}, \quad \forall n \ge \max\{n_2, n_3\}, \tag{2.84}$$

$$\sigma_n < -\max\{\sigma_1, \sigma'\}, \tag{2.85}$$

where

$$\log A_n \leq f_n(\sigma_n) = \frac{\mu_{n_{(1,2)}}}{b} + \left(\frac{\varepsilon_1\lambda}{1-\varepsilon_1} + \frac{1}{b}\log\frac{ab}{\mu_{n_{(1,2)}}}\right)\mu_{n_{(1,2)}}$$
$$\Updownarrow \tag{2.86}$$
$$\mu_{n_{(1,2)}}\left(A_n^{b/\mu_{n_{(1,2)}}}\right) \leq eab\exp\left(\frac{\varepsilon_1\lambda b}{1-\varepsilon_1}\right).$$

Or, if $L(=\lim_{n\to\infty}(\log n/\lambda_n)) = 0$, we have

$$\tau_R^{f_{A_{(1,2)}}} e\rho_R^{f_{A_{(1,2)}}} = \limsup_{n\to\infty}\left\{\mu_{n_{(1,2)}}\left(A_n^{\rho_R^{f_{A_{(1,2)}}}/\mu_{n_{(1,2)}}}\right)\right\}, \tag{2.87}$$

(M. Berland [3], following the theorem of Lindelöf-Blambert-Yu) and

$$\rho_R^{f_{\tau_0}} = \rho_R^{f_A} = \frac{1}{(1-\varepsilon_1)(1-\varepsilon_2)}\rho_R^{f_{A_{(1,2)}}}, \quad \text{(Theorem 2.2)}, \tag{2.88}$$

$$\tau_R^{f_A} = \tau_R^{f_{A_{(1,2)}}}, \tag{2.89}$$

from which

$$(1-\varepsilon_1)(1-\varepsilon_2)\lambda_n\left(A_n^{\rho_R^{f_A}/[(1-\varepsilon_1)(1-\varepsilon_2)\lambda_n]}\right) \leq e\left(\tau_R^{f_{\tau_0}} + \varepsilon'\right)\rho_R^{f_A}\exp\left(\frac{\varepsilon_1\lambda\rho_R^{f_{\tau_0}}}{1-\varepsilon_1}\right) \tag{2.90}$$

and

$$A_n^{\rho_R^{f_A}/\lambda_n} \leq A_n^{\rho_R^{f_A}/[(1-\varepsilon_1)(1-\varepsilon_2)\lambda_n]}. \tag{2.91}$$

Then, $\forall\varepsilon_1 \in]0,1[$, $\forall\varepsilon_2 \in]0,1[$, $\forall\varepsilon' > 0$, $\rho_R^{f_A} > 0$,

$$\tau_R^{f_A} \leq \frac{\tau_R^{f_{\tau_0}} + \varepsilon'}{(1-\varepsilon_1)(1-\varepsilon_2)}\exp\left(\frac{\varepsilon_1\lambda\rho_R^{f_{\tau_0}}}{1-\varepsilon_1}\right), \tag{2.92}$$

as ε_1, ε_2, and $\varepsilon'$are arbitrary, we deduce immediately that

$$\forall\tau_0 \in \mathbf{R} : \tau_R^{f_A} \leq \tau_R^{f_{\tau_0}}. \tag{2.93}$$

(2) We get, when τ_0 is a fixed real number, $\forall\varepsilon > 0$, $\exists\sigma_\varepsilon > 0$, $\forall\sigma < -\sigma_\varepsilon$,

$$\left|f_{\tau_0}(\sigma + i\tau)\right| \leq \left|f_A(\sigma(1+\varepsilon))\right|. \tag{2.94}$$

In particular, $\forall\varepsilon' > 0$, $\exists\sigma_{\varepsilon'} > 0$, $\forall\varepsilon \in]0,\varepsilon'/\sigma_{\varepsilon'}[$, $\exists\sigma_\varepsilon > 0$, $\forall\sigma < -\max\{\sigma_\varepsilon,\sigma_{\varepsilon'}\}$

$$f_A(\sigma(1+\varepsilon)) \leq f_A(\sigma - \varepsilon'), \quad \text{(M. Berland [3])} \tag{2.95}$$

and, hence, $\forall\varepsilon' > 0$,

$$\forall\sigma < -\max\{\sigma_\varepsilon,\sigma_{\varepsilon'}\} : M\left(\sigma; f_{\tau_0}\right) \leq f_A(\sigma - \varepsilon'). \tag{2.96}$$

From $\rho_R^{f_{\tau_0}} = \rho_R^{f_A} > 0$, we get the inequality, $\forall \varepsilon' > 0$,

$$\tau_R^{f_{\tau_0}} \le \tau_R^{f_A} \exp\left(\varepsilon' \rho_R^{f_A}\right). \tag{2.97}$$

As ε' is arbitrary, we have

$$\forall \tau_0 \in \mathbf{R} : \tau_R^{f_{\tau_0}} \le \tau_R^{f_A}. \tag{2.98}$$

As a result of this theorem, we have an expression for $\tau_R^{f_{\tau_0}}$ in terms of λ_n and A_n.

If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N}\backslash\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, $\forall \tau_0 \in \mathbf{R}$,

$$L\left(= \lim_{n\to\infty}\left(\frac{\log n}{\lambda_n}\right)\right) = 0, \qquad \rho_R^{f_{\tau_0}} > 0, \tag{2.99}$$

we have

$$\tau_R^{f_{\tau_0}}\, e\, \rho_R^{f_{\tau_0}} = \limsup_{n\to\infty}\left\{\lambda_n\left(A_n^{\rho_R^{f_{\tau_0}}/\lambda_n}\right)\right\}. \tag{2.100}$$

□

Remark. The notions of Ritt-type of order of functions, defined by B−Dirichletian elements, are considered in [3] with the same result of this theorem.

Theorem 2.4. *If* $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N}\backslash\{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, $L = 0$, $\lambda_n \sim \lambda_{n+1}$, φ *defined by*

$$\varphi(n) = \frac{\log\left(A_n/A_{n+1}\right)}{\lambda_{n+1} - \lambda_n}, \tag{2.101}$$

is a nondecreasing function of $n \ge n_1$, *and* $\rho_R^{f_A} > 0$, *we have,* $\forall \tau_0 \in \mathbf{R}$,

$$\nu_R^{f_{\tau_0}} = \nu_R^{f_A}. \tag{2.102}$$

Proof. (1) We have the inequality, $\forall \tau_0 \in \mathbf{R}$,

$$\nu_R^{f_A} \le \nu_R^{f_{\tau_0}}. \tag{2.103}$$

Suppose that the inequality is false. Then

$$\exists \tau_0 \in \mathbf{R} : \nu_R^{f_{\tau_0}} < \nu_R^{f_A}. \tag{2.104}$$

Let $\varepsilon \in]0, \nu_R^{f_A} - \nu_R^{f_{\tau_0}}[$, $\varepsilon' \in]0, \varepsilon/\nu_B^{f_A}[$ and $\nu = \nu_R^{f_{A-\varepsilon}}/(1-\varepsilon')$; then $\nu_R^{f_{\tau_0}} < \nu < \nu_R^{f_A}$. Under the conditions stated in Theorem 2.4, R. K. Srivastava [17] proved that

$$\nu_R^{f_A}\, e\, \rho_R^{f_A} = \liminf_{n\to\infty}\left\{\lambda_n\left(A_n^{\rho_R^{f_A}/\lambda_n}\right)\right\}, \tag{2.105}$$

which implies that $\exists n' \in \mathbf{N}\backslash\{0\}$, $\forall n \ge n'$,

$$\nu e \rho_R^{f_A} < \lambda_n\left(A_n^{\rho_R^{f_A}/\lambda_n}\right). \tag{2.106}$$

Now, $\forall\varepsilon_1 \in]0,1[$, $\forall\varepsilon_2 \in]0,1[$, $\exists\sigma_1 (=\sigma_{\varepsilon_1}) > 0$, $\forall\sigma < -\sigma_1$, $\forall n \geq n_2 (= \max\{n_1, n'\})$,

$$A_n \leq M\left(\sigma - \lambda; f_{\tau_0}\right) \exp\left[\left(\sigma - \frac{\lambda}{1-\varepsilon_1}\right)\mu_{n_{(1,2)}}\right], \tag{2.107}$$

where λ is a constant lying in $[1, \infty[$ and

$$\mu_{n_{(1,2)}} = (1-\varepsilon_1)(1-\varepsilon_2)\lambda_n \quad \text{(see Theorem 2.3)}, \tag{2.108}$$

which gives

$$\log A_n - \left(\sigma - \frac{\lambda}{1-\varepsilon_1}\right)\mu_{n_{(1,2)}} \leq \log\left(M\left(\sigma - \lambda; f_{\tau_0}\right)\right), \tag{2.109}$$

$$\log A_n + \frac{\lambda}{1-\varepsilon_1}\mu_{n_{(1,2)}} - \sigma\mu_{n_{(1,2)}} \leq \log\left(M\left(\sigma - \lambda; f_{\tau_0}\right)\right). \tag{2.110}$$

Let us consider φ_n, the function defined by

$$\varphi_n(\sigma) = \frac{\alpha_n - \beta_n\sigma}{\exp\left[-(\sigma-\lambda)\rho_R^{f_A}\right]}, \tag{2.111}$$

and indexed by $n \geq n_2$. Choose

$$\alpha_n = \log A_n + \frac{\lambda}{1-\varepsilon_1}\mu_{n_{(1,2)}}, \qquad \beta_n = \mu_{n_{(1,2)}} \quad (>0). \tag{2.112}$$

This takes the maximum value at

$$\sigma_n = \frac{\alpha_n}{\beta_n} - \frac{1}{\rho_R^{f_A}} \quad \left(= \frac{\log A_n}{\mu_{n_{(1,2)}}} + \frac{\lambda}{1-\varepsilon_1} - \frac{1}{\rho_R^{f_A}}\right), \tag{2.113}$$

$$\begin{aligned}\max\{\varphi_n(\sigma) \mid \sigma \in \mathbf{R}\} &= \frac{\mu_{n_{(1,2)}}}{\rho_R^{f_A} e}\left(A^{\rho_R^{f_A}/\mu_{n_{(1,2)}}}\right)\exp\left(\frac{\varepsilon_1\lambda}{1-\varepsilon_1}\rho_R^{f_A}\right) \\ &\leq \frac{\log\left(M(\sigma_n - \lambda; f_{\tau_0})\right)}{\exp\left[-(\sigma_n-\lambda)\rho_R^{f_{\tau_0}}\right]} \quad \left(\text{for } \rho_R^{f_A} = \rho_R^{f_{\tau_0}}\right).\end{aligned} \tag{2.114}$$

As $\forall n \in \mathbf{N}\backslash\{0\}$,

$$\mu_{n_{(1,2)}} < \lambda_n \Longleftrightarrow A^{\rho_R^{f_A}/\mu_{n_{(1,2)}}} > A_n^{\rho_R^{f_A}/\lambda_n} \tag{2.115}$$

which gives

$$\frac{(1-\varepsilon_1)(1-\varepsilon_2)}{\rho_R^{f_A} e}\exp\left(\frac{\varepsilon_1\lambda}{1-\varepsilon_1}\rho_R^{f_A}\right)\lambda_n\left(A_n^{\rho^{f_A}/\lambda_n}\right) \leq \frac{\log M(\sigma_n - \lambda; f_{\tau_0})}{\exp\left[(-\sigma_n-\lambda)\rho_R^{f_{\tau_0}}\right]}. \tag{2.116}$$

Finally, we have, $\forall\varepsilon_3 > 0$, $\exists(\sigma_n)_1^\infty$, $\lim_{n\to\infty}\sigma_n = -\infty$,

$$\frac{\log M(\sigma_n - \lambda; f_{\tau_0})}{\exp\left[(-\sigma_n-\lambda)\rho_R^{f_{\tau_0}}\right]} \leq \nu_R^{f_{\tau_0}} + \varepsilon_3. \tag{2.117}$$

Hence, we get, $\forall \varepsilon_3 > 0$, $\forall \varepsilon_1 \in]0,1[$, $\forall \varepsilon_2 \in]0,1[$,

$$\left((1-\varepsilon_1)(1-\varepsilon_2)\exp\left(\frac{\varepsilon_1 \lambda}{1-\varepsilon_1}\rho_R^{f_A}\right)\right)\nu \le \nu_R^{f_{\tau_0}} + \varepsilon_3. \tag{2.118}$$

Choosing $\varepsilon_3 = \varepsilon_1 = \varepsilon_2$ of $]0,1[$, we get

$$\nu \le \nu_R^{f_{\tau_0}}. \tag{2.119}$$

Thus, we get the contradiction that

$$\nu_R^{f_{\tau_0}} < (\nu \le)\nu_R^{f_{\tau_0}} \tag{2.120}$$

which proves, under the stated conditions, that it is impossible to find a τ_0 of $\mathbf{R}$ such that $\nu_R^{f_{\tau_0}} < \nu_R^{f_A}$.

(2) We have the inequality, $\forall \tau_0 \in \mathbf{R}$,

$$\nu_R^{f_{\tau_0}} \le \nu_R^{f_A} \quad \text{(see Theorem 2.3, 2).} \tag{2.121}$$

As a result of this theorem, we have an expression for $\nu_R^{f_{\tau_0}}$ in terms of λ_n and A_n, $\forall \tau_0 \in \mathbf{R}$,

$$\nu_R^{f_{\tau_0}} e \rho_R^{f_{\tau_0}} = \liminf_{n\to\infty}\left\{\lambda_n\left(A_n^{\rho_R^{f_{\tau_0}}/\lambda_n}\right)\right\}. \tag{2.122}$$

□

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