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RESEARCH NOTES

A NEW PROOF OF MONOTONICITY FOR EXTENDED MEAN VALUES

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ABSTRACT. In this article, a new proof of monotonicity for extended mean values is given.

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1. Introduction. Stolarsky [14] first defined the extended mean values $E(r,s;x,y)$ and proved that it is continuous on the domain $\{(r,s;x,y) : r,s \in R,\ x,y > 0\}$ as follows

$$E(r,s;x,y) = \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0; \tag{1.1}$$

$$E(r,0;x,y) = \left(\frac{y^r - x^r}{\ln y - \ln x} \cdot \frac{1}{r}\right)^{1/r}, \quad r(x-y) \neq 0; \tag{1.2}$$

$$E(r,r;x,y) = e^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \quad r(x-y) \neq 0; \tag{1.3}$$

$$E(0,0;x,y) = \sqrt{xy}, \quad x \neq y; \tag{1.4}$$

$$E(r,s;x,x) = x, \quad x = y. \tag{1.5}$$

It is convenient to write $E(r,s;x,y) = E(r,s) = E(x,y) = E$.

Several authors including Leach and Sholander [2, 3], Páles [6] and Yao and Cao [15] studied the basic properties, monotonicity and comparability of the mean values E. Feng Qi [9] and in collaboration with Qiu-mig Luo [7] further investigated monotonicity of E from new viewpoints. Recently, Feng Qi [7] generalized the extended mean values and the weighted mean values [1, 4, 5] as a new concept of generalized weighted mean values with two parameters, and studied its monotonicity and other properties.

In this note, a new proof of monotonicity for extended mean values is given.

2. Lemmas. Let

$$\begin{gathered} g = g(t) - g(t;x,y) = y^t - x^t/t, t \neq 0; \\ g(0;x,y) = \ln y - \ln x. \end{gathered} \tag{2.1}$$

It is easy to see that g can be expressed in integral form as

$$g(t;x,y) = \int_x^y u^{t-1}du, \quad t \in R, \tag{2.2}$$

and

$$g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1}du, \quad t \in R. \tag{2.3}$$

Therefore, the extended mean values can be represented in terms of g by

$$\begin{aligned} E(r,s;x,y) &= \left(\frac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0; \\ E(r,r;x,y) &= \exp\left(\frac{g'_r(r;x,y)}{g(r;x,y)}\right), \quad x-y \neq 0. \end{aligned} \tag{2.4}$$

Set $F = F(r,s) = F(x,y) = F(r,s;x,y) = \ln E(r,s;x,y)$, then F also can be expressed as

$$\begin{aligned} F(r,s;x,y) &= \frac{1}{s-r}\int_r^s \frac{g'_t(t;x,y)}{g(t;x,y)}dt, \quad r-s \neq 0; \\ F(r,r;x,y) &= \frac{g'_r(r;x,y)}{g(r;x,y)}. \end{aligned} \tag{2.5}$$

Lemma 2.1. *Assume that the derivative $f''(t)$ exists on an interval I. If $f(t)$ is an increasing or convex downward function respectively on I, then the arithmetic mean of $f(t)$,*

$$\begin{aligned} \phi(r,s) &= \frac{1}{s-r}\int_r^s f(t)dt, \\ \phi(r,r) &= f(r), \end{aligned} \tag{2.6}$$

is also increasing or convex downward respectively with r and s on I.

Proof. Direct calculation yields

$$\begin{aligned} \frac{\partial\phi(r,s)}{\partial s} &= \frac{1}{(s-r)^2}\left[(s-r)f(s) - \int_r^s f(t)dt\right], \\ \frac{\partial^2\phi(r,s)}{\partial s^2} &= \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2\int_r^s f(t)dt}{(s-r)^3} \equiv \frac{\varphi(r,s)}{(s-r)^3}, \\ \frac{\partial\varphi(r,s)}{\partial s} &= (s-r)^2 f''(s). \end{aligned} \tag{2.7}$$

In the case of $f'(t) \geq 0$, $\partial\phi(r,s)/\partial s \geq 0$, thus, $\phi(r,s)$ increases with r and s, since $\phi(r,s) = \phi(s,r)$.

In the case of $f''(t) \geq 0$, $\varphi(r,s)$ increases with s. Since $\varphi(r,r) = 0$, it is easy to see that $\partial^2\phi(r,s)/\partial s^2 \geq 0$ holds. Therefore, $\phi(r,s)$ is convex downward with respect to either r or s, since $\phi(r,s) = \phi(s,r)$. □

Lemma 2.2. *Let $f,h:[a,b]\to R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p:[a,b]\to R$ be an integrable and nonnegative function. Then*

$$\int_a^b p(u)f(u)du\int_a^b p(u)h(u)du \le \int_a^b p(u)du\int_a^b p(u)f(u)h(u)du. \tag{2.8}$$

If one of the functions of f or h is nonincreasing and the other nondecreasing, then the inequality in (2.8) is reversed.

The inequality (2.8) is called Tchebycheff's integral inequality; for details, see [1, 4].

Lemma 2.3. *Let $i,j,k\in N$, we have*

$$g^{(2(i+k)+1)}(t;x,y)g^{(2(j+k)+1)}(t;x,y)\le g^{(2k)}(t;x,y)g^{(2(i+j+k+1))}(t;x,y). \tag{2.9}$$

If $x,y\ge 1$, then

$$g^{(i+k)}(t;x,y)g^{(j+k)}(t;x,y)\le g^{(k)}(t;x,y)g^{(i+j+k)}(t;x,y). \tag{2.10}$$

If $0<x,\ y\le 1$, then

$$g^{(2i+k+1)}(t;x,y)g^{(2j+k+1)}(t;x,y)\le g^{(k)}(t;x,y)g^{(2(i+j+1)+k)}(t;x,y); \tag{2.11}$$

$$g^{(2i+k+1)}(t;x,y)g^{(2j+k)}(t;x,y)\ge g^{(k)}(t;x,y)g^{(2(i+j)+k+1)}(t;x,y); \tag{2.12}$$

$$g^{(2i+k)}(t;x,y)g^{(2j+k)}(t;x,y)\le g^{(k)}(t;x,y)g^{(2(i+j)+k)}(t;x,y). \tag{2.13}$$

Proof. By Tchebycheff's integral inequality (2.8) applied to the functions $p(u)=(\ln u)^{2k}u^{t-1}, f(u)=(\ln u)^{2i+1}$ and $h(u)=(\ln u)^{2j+1}$ for $i,j,k\in N$, $u\in[x,y]$, $t\in R$, inequality (2.9) follows easily.

By the same arguments, inequalities (2.10), (2.11), (2.12), and (2.13) also follow from Tchebycheff's integral inequality. □

Lemma 2.4. *The functions $g_t^{(2(k+i)+1)}(t;x,y)/g_t^{(2k)}(t;x,y)$ are increasing with respect to t,x, and y for i and k being nonnegative integers.*

Proof. By simple computation, we have

$$\left(\frac{g^{(2(k+i)+1)}(t)}{g^{(2k)}(t)}\right)'=\frac{g^{(2(i+k+1))}(t)g^{(2k)}(t)-g^{(2(i+k)+1)}(t)g^{(2k+1)}(t)}{\left[g^{(2k)}(t)\right]^2}. \tag{2.14}$$

Combining (2.9) and (2.14), we conclude that the derivative of $g^{(2(k+i)+1)}(t)/g^{(2k)}(t)$ with respect to t is nonnegative, and $g^{(2(k+i)+1)}(t;x,y)/g_t^{(2k)}(t;x,y)$ increases with t.

Differentiating directly, using the integral expression (2.3) of g and rearranging gives

$$
\begin{aligned}
&\frac{\partial}{\partial y}\left(\frac{g_t^{(2(k+i)+1)}(t;x,y)}{g_t^{(2k)}(t;x,y)}\right)\\
&\quad=\frac{\partial/\partial y\left[g_t^{(2(k+i)+1)}(t;x,y)\right]g_t^{(2k)}(t;x,y)-g_t^{(2(k+i)+1)}(t;x,y)\,\partial/\partial y\left[g_t^{(2k)}(t;x,y)\right]}{\left[g_t^{(2k)}(t;x,y)\right]^2}\\
&\quad=\frac{y^{t-1}(\ln y)^{2k}}{\left[g_t^{(2k)}(t;x,y)\right]^2}\left[(\ln y)^{2i+1}\int_x^y(\ln u)^{2k}u^{t-1}du-\int_x^y(\ln u)^{2(i+k)+1}u^{t-1}du\right]\geq 0.
\end{aligned}
\tag{2.15}
$$

Therefore, the desired monotonicity with respect to both x and y follows, for the involved functions are symmetric in x and y. This completes the proof. □

3. Proof of monotonicity

Theorem 3.1. *The extended mean values $E(r,s;x,y)$ are increasing with respect to both r and s, or to both x and y.*

Proof. This is a simple consequence of Lemma 2.1 and Lemma 2.3 in combination with its integral forms (2.4) and (2.5) of $E(r,s;x,y)$. □

Remark 1. It may be pointed out that the method used in this paper could yield more general results (see [4, 12], and so on).

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References

[1] J.-C. Kuang, *Changyong budengshi. [Applied inequalities]*, 2nd ed., Hunan Jiaoyu Chubanshe, Changsha, China, 1993 (Chinese). MR 95j:26001.

[2] E. B. Leach and M. C. Sholander, *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90. MR 58 22428. Zbl 379.26012.

[3] ______, *Extended mean values II*, J. Math. Anal. Appl. **92** (1983), 207–223. MR 85b:26007. Zbl 517.26007.

[4] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970. MR 43#448. Zbl 199.38101.

[5] D. S. Mitrinović, J. E. Pečaric, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993. MR 94c:00004. Zbl 771.26009.

[6] Z. Páles, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988), 271–281. MR 89f:26023. Zbl 649.26014.

[7] F. Qi, *Generalized weighted mean values with two parameters*, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **454** (1998), no. 1978, 2723–2732. Zbl 990.09932.

[8] ______, *On a two-parameter family of nonhomogeneous mean values*, Tamkang Journal of Mathematics **8** (1998), no. 2.

[9] F. Qi and Q.-M. Luo, *Refinement and extensions of an inequality*, Mathematics and Informatics Quarterly (1998), to appear.

[10] ______, *A simple proof of monotonicity for extended mean values*, J. Math. Anal. Appl. **224** (1999), no. 2, 356–359. Zbl 980.55030.

[11] F. Qi and S.-L. Xu, *A new proof of monotonicity for generalized weighted mean values*, submitted.

[12] ———, *Refinements and extensions of an inequality. II*, J. Math. Anal. Appl. **211** (1997), 616–620. CMP 97 15.

[13] ———, *The function* $(b^x - a^x)/x$*: inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359. CMP 97 15.

[14] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92. MR 50 10186. Zbl 302.26003.

[15] R. Yang and D. Cao, *Generalizations of the logarithmic mean*, J. Ningbo Univ. (1989), no. 2, 105–108.

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