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THE NEUTRIX CONVOLUTION PRODUCT IN $Z'(m)$ AND THE EXCHANGE FORMULA

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ABSTRACT. One of the problems in distribution theory is the lack of definition for convolutions and products of distribution in general. In quantum theory and physics (see e.g. [1] and [2]), one finds that some convolutions and products such as $\frac{1}{x}\cdot\delta$ are in use. In [3], a definition for product of distributions and some results of products are given using a specific delta sequence $\delta_n(x) = C_m n^m \rho(n^2 r^2)$ in an m-dimensional space. In this paper, we use the Fourier transform on $D'(m)$ and the exchange formula to define convolutions of ultradistributions in $Z'(m)$ in terms of products of distributions in $D'(m)$. We prove a theorem which states that for arbitrary elements $\tilde{f}$ and $\tilde{g}$ in $Z'(m)$, the neutrix convolution $\tilde{f}\otimes\tilde{g}$ exists in $Z'(m)$ if and only if the product $f\circ g$ exists in $D'(m)$. Some results of convolutions are obtained by employing the neutrix calculus given by van der Corput [4].

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1. INTRODUCTION

In the following, let $\rho(x)$ be a fixed infinitely differentiable function with the properties

(i) $\rho(x) = 0, \quad |x| \geq 1,$

(ii) $\rho(x) \geq 0,$

(iii) $\rho(x) = \rho(-x),$

(iv) $\int_{-1}^{1} \rho(x)dx = 1.$

We define the function $\delta_n(x)$ by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \cdots$. It is clear that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function δ.

Now let D be the space of infinitely differentiable functions with compact support. If f is an arbitrary distribution in D', we define the function f_n by $f_n = f * \delta_n$. It follows that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to f.

The following definition was given by B. Fisher [5].

DEFINITION 1. Let f and g be distributions in D' and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and equals h if

$$N - \lim_{n\to\infty} (fg_n, \phi) = (h, \phi)$$

for all ϕ in D, where N is the neutrix (see van der Corput [4]) having domain $N' = \{1, 2, \cdots, n, \cdots\}$ and range N'' the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ell n^{r-1} n, \quad \ell n^r n \,(\lambda > 0, r = 1, 2, \cdots)$$

and all functions of n which converge to zero as n tends to infinity.

Let $D'(m)$ be the space of distributions defined on the space $D(m)$ of infinitely differentiable functions of the variable $x = (x_1, x_2, \cdots, x_m)$ with compact support.

In order to give a definition for the neutrix product $f \circ g$ of two distributions f and g in $D'(m)$, we attempt to define a δ-sequence in $D(m)$ by putting

$$\delta_n(x_1, x_2, \cdots, x_m) = \delta_n(x_1) \cdots \delta_n(x_m),$$

where δ_n is defined as above. However, this definition is very difficult to use for distributions in $D'(m)$ which are functions of r, where $r = (x_1^2 + \cdots + x_m^2)^{1/2}$. Therefore an alternative definition was introduced in [3].

From now on we let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the properties

$$\text{(i)} \quad \rho(s) = 0, \quad s \geq 1, \qquad \text{(ii)} \quad \rho(s) \geq 0.$$

Define the function $\delta_n(x)$, with $x \in R^m$, by

$$\delta_n(x) = C_m n^m \rho(n^2 r^2)$$

for $n = 1, 2, \cdots$, where C_m is a constant such that

$$\int_{R^m} \delta_n(x) dx = 1.$$

DEFINITION 2. Let f and g be distributions in $D'(m)$ and let

$$g_n(x) = (g * \delta_n)(x) = (g(x - t), \delta_n(t))$$

where $t = (t_1, t_2, \cdots, t_m)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a, b), where $a = (a_1, \cdots, a_m)$ and $b = (b_1, \cdots, b_m)$, if

$$N - \lim_{n \to \infty} (f g_n, \phi) = (h, \phi)$$

for all test functions ϕ is $D(m)$ with support contained in the interval (a, b).

2. FOURIER TRANSFORM ON $D'(m)$

As in Gel'fand and Shilov [6], we define the Fourier transform of a function ϕ in $D(m)$ by

$$F(\phi)(\sigma) = \psi(\sigma) = \int_{R^m} \phi(x) e^{i(x,\sigma)} dx,$$

where (x, σ) denotes $x_1\sigma_1 + \cdots + x_m\sigma_m$.

The bounded support of $\phi(x)$ makes it possible for ψ to be continued to complex values of its argument $s = (s_1, \cdots, s_m) = (\sigma_1 + i\tau_1, \cdots, \sigma_n + i\tau_m)$:

$$\psi(s) = \int_{R^m} \phi(x) e^{i(x,s)} dx.$$

Our new function $\psi(s)$, defined on C^m, in the space of functions of m complex variables, is continuous and analytic in each of its variable s_k. If $\phi(x)$ vanishes for $|x_k| > a_k$, $k = 1, \cdots, m$, then $\psi(s)$ satisfies the inequality

$$|s_1^{q_1} \cdots s_m^{q_m} \psi(\sigma_1 + i\tau_1, \cdots, \sigma_m + i\tau_m)| \leq C_q \exp(a_1|\tau_1| + \cdots + a_m|\tau_m|). \tag{1}$$

Conversely, every entire function $\psi(s_1, \cdots, s_m)$ satisfying the above inequality is the Fourier transform of some $\phi(x_1, \cdots, x_m)$ in $D(m)$ which vanishes for $|x_k| > a_k$, $k = 1, 2, \cdots, m$.

The set of all entire analytic functions $Z(m)$ with the property (1) is in fact the space

$$F(D(m)) = \{\psi : \exists \phi \in D(m) \text{ such that } F(\phi) = \psi\}.$$

Convergence in $Z(m)$ is defined via convergence in $D(m)$: a sequence $\{\psi_n\}$ tends to zero in $Z(m)$ if the sequence $\{\phi_n\}$ tends to zero in $D(m)$, where $F(\phi_n) = \psi_n$. The Fourier transform $\tilde{f}$ of a distribution in $D'(m)$ is an ultradistribution in $Z'(m)$, i.e., a continuous linear functional on $Z(m)$. It is defined by Parseval's equation

$$(\tilde{f}, \tilde{\phi}) = 2\pi(f, \phi), \quad \phi \in D(m).$$

3. CONVOLUTION IN $Z'(m)$

In order to define a convolution product in $Z'(m)$, we introduce the Fourier transform $F(\delta_n)$ of δ_n (where $\delta_n(x) = C_m n^m \rho(n^2 r^2)$) and write

$$\tau_n(\sigma) = F(\delta_n)(\sigma)$$

which is a function in $Z(m)$ for $n = 1, 2, \cdots$.

From Parseval's equation

$$\begin{aligned}(\tau_n, \psi) = 2\pi(\delta_n, \phi) \xrightarrow{n\to\infty} 2\pi(\delta, \phi) = 2\pi\phi(0) &= 2\pi \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\sigma)d\sigma \\ &= (1, \psi)\end{aligned}$$

where $\psi = \tilde{\phi}$.

Therefore $\{\tau_n\}$ is a sequence in $Z(m) \subset Z'(m)$ converging to 1 in $Z'(m)$.

Now let $\tilde{f}$ be an arbitrary ultradistribution in $Z'(m)$. Then there exists a distribution f in $D'(m)$ such that $\tilde{f} = F(f)$. Setting $\tilde{f}_n = F(f * \delta_n) = F(f_n)$, we have

$$(\tilde{f}_n, \psi) = 2\pi(f_n, \phi) \to 2\pi(f, \phi) = (\tilde{f}, \psi) \quad n \to \infty$$

where $\psi = \tilde{\phi}$ in $Z(m)$.

LEMMA 1. Let $\tilde{g}$ be an arbitrary ultradistribution in $Z'(m)$ and let $\tilde{g}_n = F(g * \delta_n)$. Then the function

$$\Theta_n(\nu) = (\tilde{g}_n(\sigma), \psi(\sigma + \nu))$$

is in $Z(m)$ for all ψ in $Z(m)$.

Indeed,

$$\begin{aligned}\Theta_n(\nu) &= \left(F(g_n), F\left(e^{ix\nu}\phi(x)\right)(\sigma)\right) \\ &= 2\pi\left(g_n, e^{ix\nu}\phi(x)\right) = 2\pi F(g_n\phi)(\nu).\end{aligned}$$

Now the result of the lemma follows on noting that $g_n\phi$ is in $D(m)$.

We now modify the definition for the convolution product of two distributions in $D'(m)$ given by Gel'fand and Shilov with

DEFINITION 3. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ such that the function $(\tilde{g}(\sigma), \psi(\sigma + \nu))$ is in $Z(m)$ for all ψ in $Z(m)$. Then the convolution product $\tilde{f} * \tilde{g}$ is defined by

$$\left((\tilde{f} * \tilde{g})(\sigma), \psi(\sigma)\right) = \left(\tilde{f}(\nu), (\tilde{g}(\sigma), \psi(\sigma + \nu))\right)$$

for all ψ in $Z(m)$.

It follows that $\tilde{f} * \tilde{g}$ exists if $g\phi$ is in $D(m)$. (This condition is not always true for all $g \in D'(m)$. If $\tilde{g} \in Z(m)$, then $g\phi \in D(m)$.) Indeed

$$(\tilde{g}(\sigma), \psi(\sigma + \nu)) = 2\pi\left(g, e^{ix\nu}\phi(x)\right) = 2\pi F(g\phi)(\nu),$$

where $\tilde{g} = F(g)$ and $\psi = F(\phi)$.

The following theorem then holds:

THEOREM 1. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and suppose that the convolution product $\tilde{f} * \tilde{g}$ exists. Then

$$(\tilde{f} * \tilde{g})' = \tilde{f} * \tilde{g}', \tag{2}$$

$$(\tilde{f} * \tilde{g})' = \tilde{f}' * \tilde{g}. \tag{3}$$

PROOF. If $F(\phi) = \psi$, we have

$$\psi'(\sigma) = F(ix\phi(x))(\sigma).$$

Hence $Z'(m)$ is closed under differentiation.

Certainly

$$\left((\tilde{f} * \tilde{g})', \psi\right) = -(\tilde{f} * \tilde{g}, \psi') = -(\tilde{f}(\nu), (\tilde{g}(\sigma), \psi'(\sigma + \nu)))$$
$$= (\tilde{f}(\nu), (\tilde{g}'(\sigma), \psi(\sigma + \nu))) = (\tilde{f} * \tilde{g}', \psi)$$

for all ψ in $Z(m)$. Equation (2) follows.

From the fact that if $F(\phi)$, we get

$$\psi'(\sigma + \nu) = F\left(ix\phi(x)e^{ix\nu}\right)(\sigma).$$

It follows that

$$(\tilde{g}(\sigma), \psi'(\sigma + \nu)) = 2\pi\left(g(x), ix\phi(x)e^{ix\nu}\right)$$
$$= 2\pi \frac{d}{d\nu}\left(g(x), \phi(x)e^{ix\nu}\right)$$
$$= \frac{d}{d\nu}(\tilde{g}(\sigma), \psi(\sigma + \nu)).$$

Hence

$$\left((\tilde{f} * \tilde{g})', \psi\right) = \left(\tilde{f}'(\nu), (\tilde{g}(\sigma), \psi(\sigma + \nu))\right) = \left(\tilde{f}' * \tilde{g}, \psi\right)$$

for all ψ in $Z(m)$ and Equation (3) follows.

Note that $\tilde{f}' \neq F(f')$ is general.

We now note that if $\tilde{f}$ and $\tilde{g}$ are arbitrary ultradistributions in $Z'(m)$, then the convolution product $\tilde{f} * \tilde{g}_n$ always exists by the above definition (3) since by Lemma 1, $(\tilde{g}_n(\sigma), \psi(\sigma + \nu))$ is in $Z(m)$ for all ψ in $Z(m)$. This leads us to the following definition.

DEFINITION 4. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and let $\tilde{g}_n = \tilde{g}\tau_n$. Then the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ is defined to be the neutrix limit of the sequence $\{\tilde{f} * \tilde{g}_n\}$, provided the neutrix limit $\tilde{h}$ exists in the sense that

$$N - \lim_{n \to \infty} (\tilde{f} * \tilde{g}_n, \psi) = (\tilde{h}, \psi) \quad \text{for all } \psi \text{ in } Z(m).$$

Definition 4 is indeed a generalization of Definition 3, since if the convolution product $\tilde{f} * \tilde{g}$ exists by Definition 3, then $(\tilde{g}(\sigma), \psi(\sigma + \nu)) \in Z(m)$, i.e., $g\phi \in D(m)$ for all $\phi \in D(m)$. This implies $g \in C^\infty(m)$.

Therefore $(\tilde{g}_n(\sigma), \psi(\sigma + \nu)) = 2\pi F(g_n\phi)(\nu)$ converges to $(\tilde{g}(\sigma), \psi(\sigma + \nu))$ in $Z(m)$. This is because $g_n\phi \to \phi$ (if $f \in C^\infty$, then $f_n\phi$ (where $f_n = f * \delta_n$) converges to f_ϕ uniformly on the support of ϕ) in $D(m)$, and $N - \lim_{n \to \infty} (\tilde{f} * \tilde{g}_n, \psi) = (\tilde{f} * \tilde{g}, \psi)$ for all ψ in $Z(m)$.

The following theorem holds for the neutrix convolution product.

THEOREM 2. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$ and suppose that their neutrix convolution product exists. Then the neutrix convolution product $\tilde{f} \otimes \tilde{g}$ exists and

$$(\tilde{f} \otimes \tilde{g})' = \tilde{f}' \otimes \tilde{g}.$$

PROOF. We have

$$\left((\tilde{f} * \tilde{g}_n)', \psi\right) = \left(\tilde{f}' * \tilde{g}_n, \psi\right) = -(\tilde{f} * \tilde{g}_n, \psi')$$

and it follows that

$$N-\lim_{n\to\infty}\left(\tilde{f}'*\tilde{g}_n,\psi\right) = -N-\lim_{n\to\infty}(\tilde{f}*\tilde{g}_n,\psi) = -(\tilde{f}\otimes\tilde{g},\psi')$$

for arbitrary ψ in $Z(m)$. The result of the theorem follows.

Note that $(\tilde{f}\otimes\tilde{g})' = \tilde{f}\otimes\tilde{g}'$ iff $N-\lim_{n\to\infty}(\tilde{f}*(\tilde{g}\tau_n),\psi) = 0$ for all ψ in $Z(m)$.

We now prove our main result, the exchange formula.

THEOREM 3. Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $Z'(m)$. Then the neutrix convolution product $\tilde{f}\otimes\tilde{g}$ exists in $Z'(m)$ iff the neutrix product $f\circ g$ exists in $D'(m)$ and the exchange formula

$$\tilde{f}\otimes\tilde{g} = 2\pi F(f\circ g)$$

is then satisfied.

PROOF. Let $\psi = F(\phi)$ be an arbitrary function in $Z(m)$ and let

$$\Theta_n(\nu) = (\tilde{g}_n(\sigma),\psi(\sigma+\nu)) = 2\pi F(g_n\phi)(\nu).$$

Then on using Parseval's equation we have

$$\left(\tilde{f}(\nu),\Theta_n(\nu)\right) = 2\pi\left(\tilde{f}(\nu),F(g_n\phi)(\nu)\right) = (2\pi)^2(fg_n,\phi).$$

If the neutrix convolution product $\tilde{f}\otimes\tilde{g}$ exists then

$$\begin{aligned}\left(\tilde{f}\otimes\tilde{g},\phi\right) &= N-\lim_{n\to\infty}\left(\tilde{f}(\nu),\Theta_n(\nu)\right) = (2\pi)^2 N-\lim_{n\to\infty}(fg_n,\phi)\\ &= (2\pi)^2(f\circ g,\phi) = 2\pi(F(f\circ g),F(\phi)).\end{aligned}$$

The neutrix product $f\circ g$ therefore exists and the exchange formula is satisfied.

Conversely, the existence of the neutrix product $f\circ g$ implies the existence of the neutrix convolution product and the exchange formula.

4. SOME RESULTS

The following Fourier transforms of the functions r^λ and $\Delta^k\delta(x)$ were given in [6]

$$F\left(r^\lambda\right) = 2^{\lambda+m}\pi^{m/2}\frac{\Gamma\left(\frac{\lambda+m}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\rho^{-\lambda-m}$$

where $\lambda \neq -m, -m-2,\cdots$ and $\rho = \sqrt{\Sigma_{i=1}^m\sigma_i^2}$, and

$$F\left[P\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_m}\right)f(x)\right] = P(-is_1,\cdots,-is_m)F(f).$$

Hence it follows that

$$F\left(\Delta^k\delta(x)\right) = \rho^{2k}F(\delta) = \rho^{2k},$$

where Δ denotes the Laplace operator.

THEOREM 4. The neutrix convolution products $\rho^{2k-m}\otimes 1$ and $\rho^{2k-1-m}\otimes 1$ exist and

$$\rho^{2k-m}\otimes 1 = \frac{\Gamma(k)2^{k-m+1}\rho^{2k}}{\Gamma\left(\frac{m-2k}{2}\right)\pi^{m/2-1}k!m(m+2)\cdots(m+2k-2)}$$

for $k = 1,2,\cdots,\left[\frac{(m-1)}{2}\right]$ and

$$\rho^{2k-1-m}\otimes 1 = 0$$

for $k = 1,2,\cdots,\left[\frac{m}{2}\right]$.

PROOF. We have the following neutrix product (see [3]),

$$r^{-2k} \cdot \delta(x) = \frac{\Delta^k \delta(x)}{2^k k! m(m+2) \cdots (m+2k-2)}$$

for $k = 1, 2, \cdots, \left[\frac{(m-1)}{2}\right]$ and

$$r^{1-2k} \cdot \delta(x) = 0$$

for $k = 1, 2, \cdots, \left[\frac{m}{2}\right]$.

By the exchange formula

$$\begin{aligned} F(r^{-2k}) \otimes F(\delta) &= 2\pi F(r^{-2k} \cdot \delta) \\ &= 2\pi \frac{F(\Delta^k \delta)}{2^k k! m(m+2) \cdots (m+2k-2)} \\ &= 2\pi \frac{\rho^{2k}}{2^k k! m(m+2) \cdots (m+2k-2)}. \end{aligned}$$

Thus

$$2^{-2k+m} \pi^{m/2} \frac{\Gamma\left(\frac{m-2}{2}\right)}{\Gamma\left(\frac{2k}{2}\right)} \rho^{2k-m} \otimes 1 = \frac{2\pi \rho^{2k}}{2^k k! m(m+2) \cdots (m+2k-2)}.$$

It follows that

$$\rho^{2k-m} \otimes 1 = \frac{\Gamma(k) 2^{k-m+1} \rho^{2k}}{\Gamma\left(\frac{m-2k}{2}\right) \pi^{m/2-1} k! m(m+2) \cdots (m+2k-2)}.$$

The second equation follows easily.

The following neutrix product is also given in [3]

$$r^{-2k} \cdot \Delta\delta(x) = \frac{\Delta^{k+1} \delta(x)}{2^k (k+1)! (m+2) \cdots (m+2k)}$$

for $k = 1, 2, \cdots, \left[\frac{(m-1)}{2}\right]$ and

$$r^{1-2k} \cdot \Delta\delta(x) = 0$$

for $k = 1, 2, \cdots, \left[\frac{m}{2}\right]$.

Hence we obtain

THEOREM 5. The neutrix convolution product $\rho^{2k-m} \otimes \rho^2$ and $\rho^{2k-1-m} \otimes \rho^2$ exist and

$$\rho^{2k-m} \otimes \rho^2 = \frac{\Gamma(k) 2^{k-m+1}}{\Gamma\left(\frac{m-2k}{2}\right) \pi^{m/2-1} (k+1)! (m+2) \cdots (m+2k)}$$

for $k = 1, 2, \cdots, \left[\frac{(m-1)}{2}\right]$ and

$$\rho^{2k-1-m} \otimes \rho^2 = 0$$

for $k = 1, 2, \cdots, \left[\frac{m}{2}\right]$.

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