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FIXED POINTS OF A CERTAIN CLASS OF MAPPINGS IN SPACES WITH UNIFORMLY NORMAL STRUCTURE

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ABSTRACT. A fixed point theorem is proved in a Banach space E which has uniformly normal structure for asymptotically regular mapping T satisfying:

for each x, y in the domain and for $n = 1, 2, \cdots$,

$$\|T^n x - T^n y\| \le a_n \|x - y\| + b_n(\|x - T^n x\| + \|y - T^n y\|) + c_n(\|x - T^n y\| + \|y - T^n y\|),$$

where a_n, b_n, c_n are nonnegative constants satisfying certain conditions. This result generalizes a fixed point theorem of Górnicki [1].

KEY WORDS AND PHRASES: Uniformly normal structure, asymptotic regularity, fixed point.
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1. INTRODUCTION

Let E be a Banach space and K a nonempty, bounded, closed and convex subset of E. A mapping $T: K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all x, $y \in K$. Browder [2], Göhde [3] and Kirk [4] proved independently that if E is uniformly convex, then T always has a fixed point in K (see also Goebel [5]). Now, it is important (cf. [4]) that if one assumes T to be Lipschitzian with Lipschitz constant $k > 1$, then T need not have a fixed point, even if E is a Hilbert space and k is an arbitrary near 1. However, there are classes of transformations which lie between the nonexpansive transformation and those with Lipschitz constant $k > 1$ for which fixed point theorems do exist; in particular, the asymptotically nonexpansive mappings (cf. [6]) form such a class. These are mappings $T: K \to K$ having the property that T^n has Lipschitz constant k_n with $k_n \to 1$ as $n \to \infty$.

In this paper, we obtain a fixed point theorem for the class of mappings whose nth iterate T^n satisfy:

$$\|T^n x - T^n y\| \le a_n \|x - y\| + b_n(\|x - T^n x\| + \|y - T^n y\|) + c_n(\|x - T^n y\| + \|y - T^n x\|) \quad (1)$$

for each x, $y \in K$ and $n = 1, 2, \cdots$, where a_n, b_n, c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$ for all $n \ge n_0$. This class of mappings are more general than nonexpansive mappings. Also by taking $b_n = c_n = 0$ it will be seen that this class of mappings are more general than asymptotically nonexpansive mappings. Our result improves and extends the results of Górnicki [1] and others.

2. PRELIMINARIES

The concept of uniformly normal structure is due to Gillespie and Williams [7]. A Banach space E has uniformly normal structure if

$$N(E) = \sup\{r_K(K) : K \subset E \text{ is convex and } diam\, K = 1\} < 1,$$

where

$$r_K(K) = \inf\{\sup\{\|x - y\| : y \in K\} : x \in K\}.$$

It was proved in [8], [9] that $N(E) \leq 1 - \delta_E(1)$; thus $\epsilon_0(E) < 1$ implies uniformly normal structure, where $\delta_E(\,\cdot\,)$ is the modulus of convexity of E and $\epsilon_0(E)$ is the characteristic of convexity of E. Yu [10] proved that if E is a uniformly smooth space, then E has a uniformly normal structure. Also, in [11] it was proved that uniformly normal structure does not necessarily imply that the space has good geometric properties.

The following lemma is needed to prove our main result:

LEMMA 1 [12]. Let K be a nonempty closed convex subset of a Banach space E and let $\{n_i\}$ be an increasing sequence of natural numbers. Assume that $T : K \to K$ is an asymptotically regular mapping such that for some $m \in \mathbb{N}$, T^m is continuous. If

$$\lim_{i\to\infty} \|z - T^{n_i}x\| = 0$$

for some $x \in K$ and $z \in K$, then $Tz = z$.

3. MAIN RESULTS

Now we state and prove our main result:

THEOREM 1. Let K be a nonempty closed convex subset of a Banach space E which has uniformly normal structure, i.e. $N(E) < 1$. Let $T : K \to K$ be as asymptotically regular mapping which holds the inequality (1) such that $(\alpha + \beta) \cdot \gamma \cdot N(E) < 1$, where

$$\alpha = \liminf_{n\to\infty} \frac{a_n + c_n}{1 - c_n}$$

$$\beta = \liminf_{n\to\infty} \frac{b_n}{1 - c_n}$$

and

$$\gamma = \liminf_{n\to\infty} \frac{a_n + c_n}{1 - c_n - b_n}.$$

Suppose that there is a z_0 in K for which $\{T^n z_0\}$ is bounded. Then T has a fixed point in K.

PROOF. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\alpha = \liminf_{n\to\infty} \frac{a_n + c_n}{1 - c_n} = \lim_{i\to\infty} \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}}$$

$$\beta = \liminf_{n\to\infty} \frac{b_n}{1 - c_n} = \lim_{i\to\infty} \frac{b_{n_i}}{1 - c_{n_i}}$$

and

$$\gamma = \liminf_{n\to\infty} \frac{a_n + c_n}{1 - c_n - b_n} = \lim_{i\to\infty} \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i} - b_{n_i}}.$$

Since $\{T^n z_0\}$ is bounded (and hence $\{T^n z\}$ is bounded for any z in K), by Lemma 1, we can inductively construct a sequence $\{z_m\}$ such that z_m is the unique asymptotic center of the sequence $\{T^{n_i} z_{m-1}\}_{i\geq 1}$ with respect to the functional

$$\limsup_{n\to\infty} \|x - T^{n_i} z_{m-1}\|$$

over x in K. Now for each $m \geq 1$, we set

$$D_m = \lim_{i \to \infty} \|z_m - T^{n_i} z_m\|$$

and

$$r_m = \lim_{i \to \infty} \|z_{m+1} - T^{n_i} z_m\|.$$

Using (1), we have

$$\begin{aligned}\|T^{n_i}x - T^{n_j}y\| &\leq \|T^{n_i}x - T^{n_i+n_j}y\| + \|T^{n_i+n_j}y - T^{n_j}y\| \\ &\leq a_{n_i}\|x - T^{n_j}y\| + b_{n_i}(\|x - T^{n_i}x\| + \|T^{n_j}y - T^{n_i+n_j}y\|) \\ &\quad + c_{n_i}(\|x - T^{n_i+n_j}y\| + \|T^{n_j}y - T^{n_i}x\|) + \|T^{n_i+n_j}y - T^{n_j}y\|\end{aligned}$$

implies

$$\begin{aligned}\|T^{n_i}x - T^{n_j}y\| \leq \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|x - T^{n_j}y\| + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|x - T^{n_i}x\| \\ + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|T^{n_j}y - T^{n_i+n_j}y\|. \end{aligned} \tag{2}$$

By inequality (2), the result of Casini and Maluta [13], and the asymptotic regularity of T, we have

$$\begin{aligned} r_m &\leq N(E) \cdot \limsup_{s \to \infty} (\|T^{n_i}z_m - T^{n_j}z_m\| : n_i, n_j \geq s) \\ &\leq N(E) \cdot \limsup_{i \to \infty} \left(\limsup_{j \to \infty} \|T^{n_i}z_m - T^{n_j}z_m\| \right) \\ &\leq N(E) \cdot \limsup_{i \to \infty} \left[\limsup_{j \to \infty} \left\{ \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_j}z_m\| + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i}z_m\| \right.\right. \\ &\quad \left.\left. + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \sum_{l=0}^{n_i-1} \cdot \|T^{n_j+l+1}z_m - T^{n_j+l}z_m\| \right\} \right] \end{aligned}$$

and so

$$r_m \leq (\alpha + \beta) \cdot N(E) \cdot D_m, \quad m = 0, 1, \cdots, \tag{3}$$

where $N(E)$ is the normal structure coefficient of E. Moreover, for $i > 1$, we have

$$\begin{aligned} \|T^{n_i}z_m - z_m\| &\leq \limsup_{j \to \infty} \|T^{n_i}z_m - T^{n_j}z_{m-1}\| \leq \limsup_{j \to \infty} \left\{ \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_j}z_{m-1}\| \right. \\ &\quad \left. + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i}z_m\| + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \sum_{l=0}^{n_i-1} \|T^{n_j+l+1}z_{m-1} - T^{n_j+l}z_{m-1}\| \right\} \\ &\leq \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot r_{m-1} + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i}z_m\| \\ &\leq \frac{a_{n_i} + c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot r_{m-1}. \end{aligned}$$

Taking the limit superior as $i \to \infty$ on each side, by definition of z_m, we get

$$\begin{aligned} D_m &\leq \lim_{i \to \infty} \left(\frac{a_{n_i} + c_{n_i}}{1 - b_{n_i} - c_{n_i}} \right) \cdot r_{m-1} \\ &\leq \gamma \cdot r_{m-1}. \end{aligned} \tag{4}$$

By (3) and (4), we obtain

$$\begin{aligned} r_m &\leq (\alpha + \beta) \cdot \gamma \cdot N(E) \cdot r_{m-1} \\ &= A \cdot r_{m-1}, \end{aligned}$$

where $A = (\alpha + \beta) \cdot \gamma \cdot N(E) < 1$ by the assumption of the theorem. Since

$$\|z_{m+1} - z_m\| \le r_m + D_m \to 0$$

as $m \to \infty$, it follows that z_m is a Cauchy sequence. Let $\lim_{m\to\infty} z_m = z \in K$. Then, we have

$$\begin{aligned}\|z - T^{n_i} z\| &\le \|z - z_m\| + \|z_m - T^{n_i} z_m\| + \|T^{n_i} z_m - T^{n_i} z\| \\ &\le \|z - z_m\| + \|z_m - T^{n_i} z_m\| + a_{n_i} \|z_m - z\| \\ &\quad + b_{n_i}(\|z_m - T^{n_i} z_m\| + \|z - T^{n_i} z\|) + c_{n_i}(\|z_m - T^{n_i} z\| + \|z - T^{n_i} z_m\|)\end{aligned}$$

and so

$$\|z - T^{n_i} z\| \le \frac{1 + a_{n_i} + 2c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot \|z - z_m\| + \frac{1 + b_{n_i} + c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot \|z_m - T^{n_i} z_m\|.$$

Taking the limit superior as $i \to \infty$ on each side, we obtain

$$\limsup_{i\to\infty} \|z - T^{n_i} z\| \le \limsup_{i\to\infty} \frac{1 + a_{n_i} + 2c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot \|z - z_m\| + \limsup_{i\to\infty} \frac{1 + b_{n_i} + c_{n_i}}{1 - b_{n_i} - c_{n_i}} \cdot D_m \to 0$$

as $m \to \infty$. Therefore we have $Tz = z$ by Lemma 1. This completes the proof.

If we put $b_n = c_n = 0$ in (1), then from Theorem 1, we have the following result.

COROLLARY 1 [1, Theorem 3]. Let K be a nonempty bounded closed convex subset of a Banach space E which has uniformly normal structure, i.e. $N(E) < 1$. If $T : K \to K$ is an asymptotically regular mapping such that

$$\liminf_{n\to\infty} \|T^n\| = k < [N(E)]^{-\frac{1}{2}},$$

then T has a fixed point in K.

REMARK. In place of bounded subset of K in [1], we have weaker assumption that there is a z_0 in K for which $\{T^n z_0\}$ is bounded.

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