WEAK REGULARITY OF PROBABILITY MEASURES

DALE SIEGEL

Kingsborough Community College
Mathematics/Computer Science Department
2001 Oriental Boulevard
Brooklyn, New York 11235

(Received September 9, 1996 and in revised form January 7, 1997)

ABSTRACT. This paper examines smoothness attributes of probability measures on lattices which indicate regularity, and then discusses weaker forms of regularity; specifically, weakly regular and vaguely regular. They are obtained from commonly used outer measures, and we study them mainly for the case of $M(\mathcal{L})$ or for those components of $M(\mathcal{L})$ with added smoothness prerequisites. This is a generalization of many concepts presented in my earlier paper (see [1]).

KEY WORDS AND PHRASES: Lattice regular, σ-smooth, and outer measures. Weakly and vaguely regular measures. Normal and complement generated lattices.
1991 AMS SUBJECT CLASSIFICATION CODES: 28C15.

1. INTRODUCTION

Let X be an arbitrary set and $\mathcal{L}$ a lattice of subsets of X. $A(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$ and $M(\mathcal{L})$ those finitely additive measures on $A(\mathcal{L})$. $M_\sigma(\mathcal{L})$ denotes those elements of $M(\mathcal{L})$ that are σ-smooth on $\mathcal{L}$; while $M_R(\mathcal{L})$ denotes those elements of $M(\mathcal{L})$ that are $\mathcal{L}$-regular. To each $\mu \in M(\mathcal{L})$ we will associate a finitely subadditive outer measure μ' on $P(X)$, and to $\mu \in M_\sigma(\mathcal{L})$ is associated an outer measure μ''. The relationships between μ, μ', and μ'' on $\mathcal{L}$ and $\mathcal{L}'$ (the complementary lattice) are investigated. This leads to a consideration of weak notions of regularity, which can be expressed in terms of μ' and μ''. In this respect the normal lattices are particularly important since for such lattices regularity of μ coincides with weak regularity. We show that if $\mu \in N(\mathcal{L})$, those $\mu \in M(\mathcal{L})$ such that for $L_n \downarrow L$, $L_n, L \in \mathcal{L}$, $\mu(L) = \inf_n \mu(L_n)$ and if $\mathcal{L}$ is complement generated then μ is weakly regular. Combining these results gives conditions for certain measures to be regular. We adhere to standard lattice and measure terminology which will be used throughout the paper (see e.g. [2-6]) and review some of this in section two for the reader's convenience.

2. DEFINITIONS AND NOTATIONS

Let X be an abstract set. Let $\mathcal{L}$ be a lattice of subsets of X. We assume throughout that $\emptyset$ and X are in $\mathcal{L}$. If $A \subset X$, then we will denote the complement of A by A' (i.e. $A' = X - A$). If $\mathcal{L}$ is a lattice of subsets of X, then $\mathcal{L}' = \{L' | L \in \mathcal{L}\}$ is the complementary lattice of $\mathcal{L}$.

LATTICE TERMINOLOGY

DEFINITION 2.1. Let $\mathcal{L}$ be a lattice of subsets of X. We say that:

1. $\mathcal{L}$ is a δ-lattice if it is closed under countable intersections; $\delta(\mathcal{L})$ is the lattice of countable intersections of sets of $\mathcal{L}$.
2. $\mathcal{L}$ is disjunctive if and only if $x \in X$, $L \in \mathcal{L}$, and $x \notin L$ imply there exists $A \in \mathcal{L}$ such that $x \in A$ and $A \cap L = \emptyset$.

3. $\mathcal{L}$ is complement generated if $L \in \mathcal{L}$ implies $L = \bigcap_{n=1}^{\infty} L'_n$, where $L_n \in \mathcal{L}$.
4. $\mathcal{L}$ is compact if and only if $X = \bigcup_{\alpha} L'_\alpha$, $L_\alpha \in \mathcal{L}$, implies there exists a finite number of L'_α that cover X.
5. $\mathcal{L}$ is countably compact if and only if $X = \bigcup_{i=1}^{\infty} L'_i$, $L_i \in \mathcal{L}$, implies there exists a finite number of the L'_i that cover X.
6. $\mathcal{L}$ is countably paracompact if, for every sequence $\{L_n\}$ in $\mathcal{L}$ such that $L_n \downarrow \emptyset$, there exists a sequence $\{\tilde{L}_n\}$ in $\mathcal{L}$ such that $L_n \subset \tilde{L}'_n$ and $\tilde{L}'_n \downarrow \emptyset$.
7. $\mathcal{L}$ is normal if and only if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$ imply there exists $C, D \in \mathcal{L}$ such that $A \subset C'$, $B \subset D'$, and $C' \cap D' = \emptyset$.

MEASURE TERMINOLOGY

Let $\mathcal{L}$ be a lattice of subsets of X. $M(\mathcal{L})$ will denote the set of finite-valued, bounded, finitely additive measures on $A(\mathcal{L})$. We may clearly assume throughout that all measures are non-negative.

DEFINITION 2.2.

1. A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{L}$ if $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$ imply $\mu(L_n) \to 0$.
2. A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $A(\mathcal{L})$ if $A_n \in A(\mathcal{L})$ and $A_n \downarrow \emptyset$ imply $\mu(A_n) \to 0$.
3. A measure $\mu \in M(\mathcal{L})$ is said to be $\mathcal{L}$-regular if, for any $A \in A(\mathcal{L})$, $\mu(A) = \sup\{\mu(L) : L \subset A, L \in \mathcal{L}\}$.

NOTATION 2.3. If $\mathcal{L}$ is a lattice of subsets of X, then we will denote by:

$M_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$
$M^\sigma(\mathcal{L})$ = the set of σ-smooth measures on $A(\mathcal{L})$ of $M(\mathcal{L})$
$M_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$
$M_R^\sigma(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$

DEFINITION 2.4.

1. Let $\mu \in M(\mathcal{L})$. Then $\mu \in N(\mathcal{L})$ if $L_n \in \mathcal{L}$ and $\bigcap_{n=1}^{\infty} L_n = L \in \mathcal{L}$ (in particular, if $\mathcal{L}$ is δ), $L_n \downarrow$, imply $\mu(L) = \inf \mu(L_n)$.
2. If $\mu \in M(\mathcal{L})$, then the support of μ is $S(\mu) = \cap\{L \in \mathcal{L} \mid \mu(L) = \mu(X)\}$.

REMARK 2.5. Listed below are a few basic important facts that will be used throughout the paper (see [7,8] for further details):

1. $M_R^\sigma(\mathcal{L}) = M_R(\mathcal{L}) \cap M_\sigma(\mathcal{L})$
2. $M_\sigma(\mathcal{L}) \supset N(\mathcal{L}) \supset M^\sigma(L)$
3. If $\mu \in M(\mathcal{L})$, then there exists $\nu \in M_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$ (i.e. $\mu(L) \leq \nu(L)$, all $L \in \mathcal{L}$) and $\mu(X) = \nu(X)$.

3. REGULAR PROBABILITY MEASURES

Discussion of $\mathcal{L}$-regular measures ($\mu \in M_R(\mathcal{L})$) takes place in this section. Conditions for regularity and various resulting properties are examined.

THEOREM 3.1. Let $\mathcal{L}$ be a lattice of subsets of X. Then $\mu \in M_R(\mathcal{L})$ if and only if $\mu \in M(\mathcal{L})$ and $\mu(A) = \inf\{\mu(L') : A \subset L', L \in \mathcal{L}\}$, $A \in A(\mathcal{L})$.

PROOF. 1. Suppose $\mu \in M_R(\mathcal{L})$. Then $\mu(A') = \sup\{\mu(L) : L \subset A', L \in \mathcal{L}\}$. Hence

$$\begin{aligned}\mu(A) = \mu(X) - \mu(A') &= \mu(X) - \sup\{\mu(L) : L \subset A', L \in \mathcal{L}\} \\ &= \mu(X) - \sup\{\mu(L) : L' \supset A, L \in \mathcal{L}\} \\ &= \mu(X) - \sup\{\mu(X) - \mu(L') : L' \supset A, L \in \mathcal{L}\} \\ &= \mu(X) - \mu(X) - \sup\{ - \mu(L') : L' \supset A, L \in \mathcal{L}\}.\end{aligned}$$

Therefore $\mu(A) = \inf\{\mu(L') : A \subset L', L \in \mathcal{L}\}$.

2. Reverse of 1. is sufficient proof.

THEOREM 3.2. Let $\mathcal{L}$ be a lattice of subsets of X. Suppose $\mu \in M(\mathcal{L})$ and $\mu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. Then $\mu \in M_R(\mathcal{L})$.

PROOF. Suppose $\mu \in M(\mathcal{L})$ and $\mu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. This implies $\mu(L) = \inf\left\{\mu(\tilde{L}') : L \subset \tilde{L}', \tilde{L} \in \mathcal{L}\right\}$, by Theorem 3.1. Let $A \in A(\mathcal{L})$. By definition, $A = \bigcup_{i=1}^{n} L_i \cap \widehat{L}_i'$, where L_i, $\widehat{L}_i \in \mathcal{L}$ and disjoint. Consider $L \cap \widehat{L}$; L, $\widehat{L} \in \mathcal{L}$. Since every $L \in \mathcal{L}$ is $\mathcal{L}'$-outer regular with respect to μ, there exists $\tilde{L}' \supset L$, $\tilde{L} \in \mathcal{L}$, such that $\mu(L) + \epsilon > \mu(\tilde{L}')$, $\epsilon > 0$. Then $\tilde{L}' \cap \widehat{L}' \supset L \cap \widehat{L}'$ and $\tilde{L}' \cup \widehat{L}' \supset L \cup \widehat{L}'$.

$$\mu\left(L \cap \widehat{L}'\right) = \mu(L) + \mu\left(\widehat{L}'\right) - \mu\left(L \cup \widehat{L}'\right) \geq \mu\left(\tilde{L}'\right) - \epsilon + \mu\left(\widehat{L}'\right) - \mu\left(\tilde{L}' \cup \widehat{L}'\right) = \mu\left(\tilde{L}' \cap \widehat{L}'\right) - \epsilon.$$

Consequently, $\mu\left(L \cap \widehat{L}'\right) + \epsilon \geq \mu\left(\tilde{L}' \cap \widehat{L}'\right)$. Therefore, $L \cap \widehat{L}'$ is $\mathcal{L}'$-outer regular with respect to μ. Now, in general, $A = \bigcup_{1}^{n}\left(L_i \cap \widehat{L}_i'\right)$, where L_i, $\widehat{L}_i$ disjoint and $\epsilon > 0$. There exists $\tilde{L}_i' \supset L_i \cap \widehat{L}_i'$ such that $\mu\left(L_i \cap \widehat{L}_i'\right) + \frac{\epsilon}{2^i} > \mu\left(\tilde{L}_i'\right)$. Then $\bigcup_{i=1}^{n} \tilde{L}_i' \supset \bigcup_{1}^{n}\left(L_i \cap \widehat{L}_i'\right) = A$ and $\bigcup_{1}^{n} \tilde{L}_i' \in \mathcal{L}'$.

$$\mu(A) = \mu\left(\bigcup_{i}^{n}\left(L_i \cap \widehat{L}_i'\right)\right) = \sum_{i=1}^{n} \mu\left(L_i \cap \widehat{L}_i'\right) \geq \sum_{1}^{n} \mu\left(\tilde{L}_i'\right) - \sum_{1}^{n} \frac{\epsilon}{2^i} \geq \mu\left(\bigcup_{1}^{n} \tilde{L}_i'\right) - \epsilon.$$

Hence $\mu(A) = \inf\{\mu(L') : A \subset L', L \in \mathcal{L}\}$. Therefore $\mu \in M_R(\mathcal{L})$, by 3.1.

THEOREM 3.3 Let $\mu_1 \in M_R(\mathcal{L})$, $\mu_2 \in M(\mathcal{L})$, $\mu_1 \leq \mu_2(\mathcal{L})$, and $\mu_1(X) = \mu_2(X)$. Then $\mu_1 = \mu_2$.

PROOF. Suppose $\mu_1 \leq \mu_2(\mathcal{L})$ and let $L \subset \tilde{L}'$, L, $\tilde{L} \in \mathcal{L}$. This implies $\mu_2 \leq \mu_1(\mathcal{L}')$ and $\mu_2(L) \leq \mu_2(\tilde{L}') \leq \mu_1(\tilde{L}')$. If $\mu_2(L) \leq \mu_1(\tilde{L}')$, for any $\tilde{L}' \supset L$, then $\mu_2(L) \leq \inf\{\mu_1(\tilde{L}') : L \subset \tilde{L}'\} = \mu_1(L)$, since $\mu \in M_R(\mathcal{L})$. Hence $\mu_2 \leq \mu_1(\mathcal{L})$ and, consequently, $\mu_1 = \mu_2(\mathcal{L})$. Therefore $\mu_1 = \mu_2$ since $\mu_1(X) = \mu_2(X)$.

THEOREM 3.4. Let $\mathcal{L}$ be a lattice of subsets of X. Suppose $\mu \in M_R(\mathcal{L})$ and $\mu \in M_\sigma(\mathcal{L})$. Then $\mu \in M^\sigma(\mathcal{L})$.

PROOF. Given $\mu \in M_R(\mathcal{L})$ and $\mu \in M_\sigma(\mathcal{L})$. Let $\{A_n\}$ be in $A(\mathcal{L})$ and $A_n \downarrow \emptyset$. Then there exists $L_n \subset A_n$, $L_n \in \mathcal{L}$, and $\mu(A_n) - \frac{\epsilon}{2^n} < \mu(L_n)$, since $\mu \in M_R(\mathcal{L})$. Now, $L_1, L_1 \cap L_2, L_1 \cap L_2 \cap L_3, \ldots$ are in $\mathcal{L}$ and $\downarrow \emptyset$. So $\mu(L_1 \cap L_2) \leq \mu(A_1 \cap A_2) = \mu(A_2) \leq \mu(L_1 \cap L_2) + \frac{\epsilon}{2} + \frac{\epsilon}{4}$. By induction, $\mu\left(\bigcap_{i=1}^{n} A_i\right) \leq \mu\left(\bigcap_{1}^{n} L_i\right) + \sum_{i=1}^{n} \frac{\epsilon}{2^i}$ for all n. Consequently, we may assume $L_n \downarrow \emptyset$ and $\mu(A_n) < \mu(L_n) + \epsilon$, all n. Then $\lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \mu(L_n) + \epsilon$; and $\lim_{n\to\infty} \mu(L_n) = 0$ since $\mu \in M_\sigma(\mathcal{L})$. This implies $\lim_{n} \mu(A_n) \leq \epsilon$ and $\epsilon > 0$. Hence $\lim_{n} \mu(A_n) = 0$. Therefore $\mu \in M^\sigma(\mathcal{L})$, since μ is countably additive on $A(\mathcal{L})$.

THEOREM 3.5. Let $\mu \leq \nu(\mathcal{L})$, where $\mu \in M(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$, and $\mu(X) = \nu(X)$. If $\mathcal{L}$ is normal, then $\nu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\}$.

PROOF. Since $\nu \in M_R(\mathcal{L})$, $\nu(L') = \sup\{\nu(\tilde{L}) : \tilde{L} \subset L; L, \tilde{L} \in \mathcal{L}\}$. This implies $\nu(L') - \epsilon < \nu(\tilde{L}), \epsilon > 0$, for some $\tilde{L} \in \mathcal{L}$ where $\tilde{L} \subset L$. By normality, $\tilde{L} \subset A' \subset B \subset L'$, where $A, B \in \mathcal{L}$.

Then $\nu(L') - \epsilon < \nu(\tilde{L}) \leq \nu(A') \leq \mu(A') \leq \mu(B) \leq \nu(B) \leq \nu(L)$. Hence $\nu(L') = \sup\{\mu(B) : B \subset L', B \in \mathcal{L}\}$, and $\mu(B) \sim \mu(\tilde{L})$ by an ϵ-argument. Therefore $\nu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$.

THEOREM 3.6. Suppose $\mu \in M_R(\mathcal{L})$ and $\lambda \in M_R(\mathcal{L}')$ such that $\mu \leq \lambda(\mathcal{L}')$. Then $\mathcal{L}$ is normal if and only if $\mu(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$.

PROOF.

1. $\mu \leq \lambda(\mathcal{L}')$ implies $\lambda \leq \mu(\mathcal{L})$, by regularity. Therefore, if $\mathcal{L}$ is normal, then $\mu(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$ by 3.5.
2. Suppose $\mu(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$. Let $\mu_1, \mu_2 \in M_R(\mathcal{L})$ such that $\mu \leq \mu_1(\mathcal{L})$ and $\mu \leq \mu_2(\mathcal{L})$. Then $\mu_1 \leq \mu \leq \lambda(\mathcal{L}')$ and $\mu_2 \leq \mu \leq \lambda(\mathcal{L}')$. This implies $\mu_1(L') = \mu_2(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$. Hence $\mu_1 = \mu_2$. Therefore, $\mathcal{L}$ is normal.

THEOREM 3.7. Suppose $\mathcal{L}$ is normal and complement generated. Then $\mu \in N(\mathcal{L})$ implies $\mu \in M_R^{\sigma}(\mathcal{L})$.

PROOF. Since $\mathcal{L}$ is complement generated, there exists $L, L_n \in \mathcal{L}$ such that $L = \bigcap_{n=1}^{\infty} L'_n$, where $L_n \downarrow$. By normality, $L \subset A'_n \subset B_n \subset L'_n$, where $A_n, B_n \in \mathcal{L}$, and we may assume that $A_n \downarrow$, $B_n \downarrow$. Then $L = \bigcap_n B_n = \bigcap_n L'_n$. Now let $\mu \in N(\mathcal{L})$. This implies $\mu(L) = \inf_n \mu(B_n) = \inf_n \mu(A'_n)$. Hence $\mu \in M_R(\mathcal{L})$ by 3.1, and $N(\mathcal{L}) \subset M_\sigma(\mathcal{L})$ by 2.5. Therefore $\mu \in M_R^{\sigma}(\mathcal{L})$.

4. OUTER MEASURES

In this section we consider $\mu \in M(\mathcal{L})$, and associate with it certain "outer measures" μ' and μ''. In general, they differ from the customary induced "outer measures" $\mu^{\cdot}$ and μ^{*}. We seek to investigate the interplay of these outer measures on the lattice $\mathcal{L}$ and, conversely, the effect of $\mathcal{L}$ on them.

DEFINITION 4.1. Let $\mu \in M(\mathcal{L})$ such that $\mu \geq 0$ and let E be a subset of X.

1. $\mu'(E) = \inf\{\mu(L') : E \subset L', L \in \mathcal{L}\}$ is a finitely-subadditive outer measure.
2. $\mu''(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(L'_n) : E \subset \bigcup_{n=1}^{\infty} L'_n, L_n \in \mathcal{L}\right\}$ is a countably-subadditive outer measure.
3. $\mu^{\cdot}(E) = \inf\{\mu(A) : E \subset A, A \in A(\mathcal{L})\}$ is a finitely-subadditive outer measure.
4. $\mu^{*}(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : E \subset \bigcup_{i=1}^{\infty} A_i, A_i \in A(\mathcal{L})\right\}$ is a countably-subadditive outer measure.

DEFINITION 4.2.

1. Suppose ν is an outer measure and let E be a subset of X. Then $E \in \mathcal{S}_\nu$, the set of ν-measurable sets, if $\nu(A) = \nu(A \cap E) + \nu(A \cap E')$ for all $A \subset X$.
2. ν is said to be a regular outer measure if, for $A, E \subset X$, there exists $E \in \mathcal{S}_\nu$ such that $A \subset E$ and $\nu(A) = \nu(E)$.

PROPERTY 4.3. Proofs will be omitted.

1. If $\mathcal{L}$ is countably compact and $\mu \in M(\mathcal{L})$, then $\mu' = \mu''(\mathcal{L})$.
2. If $\mu \in N(\mathcal{L})$, then $\mu' = \mu''(\mathcal{L}')$.
3. $\mu \in M_\sigma(\mathcal{L})$ and $\mu' = \mu''(\mathcal{L}')$, where μ'' is regular, imply $\mu \in N(\mathcal{L})$.
4. If $\mu \in N(\mathcal{L})$ and $\mathcal{L}$ is δ, then $\mu' = \mu''(\mathcal{L})$.
5. Suppose $\mu \in N(\mathcal{L})$, $\mathcal{L}$ is δ, and $\mathcal{L} \subset \mathcal{S}_{\mu''}$. Then $\mu \in M_R^{\sigma}(\mathcal{L})$.

THEOREM 4.4. Let $\mu \in M_\sigma(\mathcal{L})$. Then

(a) $\mu(X) = \mu''(X)$,

(b) $\mu \leq \mu'' \leq \mu'(\mathcal{L})$,

(c) $\mu'' \leq \mu = \mu'(\mathcal{L}')$.

PROOF. (a) Clearly $\mu''(X) \leq \mu(X)$. If $\mu''(X) < \mu(X)$, then there exists $L'_i \in \mathcal{L}'$, $i = 1, 2, \ldots$, such that $X = \bigcup_{i=1}^{\infty} L'_i$ and $\sum_{i=1}^{\infty} \mu(L'_i) < \mu(X)$. But $\sum_{i=1}^{\infty} \mu(L'_i) = \lim_{n\to\infty} \sum_{1}^{n} \mu(L'_i) \geq \lim_{n\to\infty} \mu\left(\bigcup_{1}^{n} L'_i\right)$. Also

$\bigcup_1^n L'_i \uparrow X$ and $\bigcup_1^n L'_i \in \mathcal{L}'$. This implies that $\lim_n \mu\left(\bigcup_1^n L'_i\right) = \mu(X)$ since $\mu \in M_\sigma(\mathcal{L})$. Therefore $\mu(X) = \mu''(X)$.

(b) Suppose there exists $L \in \mathcal{L}$ such that $\mu(L) > \mu''(L)$. Then $\mu''(X) \leq \mu''(L) = \mu''(L') < \mu(L) + \mu''(L')$. Then $\mu''(X) \leq \mu''(L) + \mu''(L') < \mu(L) + \mu''(L')$, but $\mu'' \leq \mu(\mathcal{L})$. This implies $\mu''(X) < \mu(L) + \mu(L') = \mu(X)$, which contradicts (a). Hence $\mu \leq \mu''(\mathcal{L})$; and $\mu'' \leq \mu'$ everywhere clearly. Thus $\mu'' \leq \mu'(\mathcal{L})$. Therefore $\mu \leq \mu'' \leq \mu'(\mathcal{L})$.

(c) Clearly $\mu'' \leq \mu'(\mathcal{L}')$ and, by definition, $\mu = \mu'(\mathcal{L}')$. Therefore $\mu'' \leq \mu = \mu'(\mathcal{L}')$.

THEOREM 4.5. Suppose ν is a finite, regular, finitely-subadditive outer measure defined on $P(X)$, the set of all subsets of X. Then $E \in \mathcal{S}_\nu$ if and only if $\nu(X) = \nu(E) + \nu(E')$.

PROOF. 1. Suppose ν is a finitely-subadditive regular outer measure and $E \in \mathcal{S}_\nu$. Then $\nu(E) + \nu(E') = \nu(X)$, clearly.

2. Suppose ν is a finitely-subadditive regular outer measure and $\nu(X) = \nu(E) + \nu(E')$. Let $B \in \mathcal{S}_\nu$. Then, by regularity, there exists a set $F \subset X$ such that $F \subset B$ and $\nu(F) = \nu(B)$. Then, since $B \in \mathcal{S}_\nu$, $\nu(E) = \nu(E \cap B) + \nu(E \cap B')$ and $\nu(E') = \nu(E' \cap B) + \nu(E' \cap B')$. So

$$\begin{aligned}\nu(X) = \nu(E) + \nu(E') &= \nu(E \cap B) + \nu(E \cap B') + \nu(E' \cap B) + \nu(E' \cap B') \\ &\geq \nu(B) + \nu(B') = \nu(X),\end{aligned}$$

since $B \in \mathcal{S}_\nu$. Also, $\nu(B \cap E) + \nu(B \cap E') + \nu(B' \cap E) + \nu(B' \cap E') = \nu(B) + \nu(B')$ since all $\nu(X) =$ finite measure. Now subtract from the equation above $\nu(B' \cap E) + \nu(B' \cap E') \geq \nu(B')$, which is true by the finite subadditivity of ν. Then $\nu(B \cap E) + \nu(B \cap E') \leq \nu(B)$. Also, $F \cap E \subset B \cap E$ and $F \cap E' \subset B \cap E'$. This implies

$$\nu(F \cap E) + \nu(F \cap E') \leq \nu(B \cap E) + \nu(B \cap E') \leq \nu(B) = \nu(F).$$

Hence $\nu(F) = \nu(F \cap E) + \nu(F \cap E')$. Therefore $E \in \mathcal{S}_\nu$.

THEOREM 4.6. Suppose $\mu \leq \nu(\mathcal{L})$, where $\mu \in M(\mathcal{L})$ and $\nu \in M_R(\mathcal{L})$. Then:

(a) $\mu \leq \nu = \nu' \leq \mu'(\mathcal{L})$

(b) if $\mathcal{L}$ is normal, then $\mu' = \nu'(\mathcal{L})$.

PROOF. (a) Since $\nu \in M_R(\mathcal{L})$, $\nu(E) = \nu'(E) = \inf\{\nu(L') : E \subset L', L \in \mathcal{L}\}$. Also, $\mu \leq \nu(\mathcal{L})$ implies $\nu \leq \mu(\mathcal{L}')$, which implies $\nu' \leq \mu'(\mathcal{L})$ and $\nu' \leq \mu'(\mathcal{L}')$. Therefore $\mu \leq \nu = \nu' \leq \mu'(\mathcal{L})$.

(b) Let $L \in \mathcal{L}$. Then, by normality,

$$\begin{aligned}\nu'(L) = \nu(L) = \nu(X) - \nu(L') &= \nu(X) - \sup\{\nu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \\ &= \nu(X) - \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \\ &= \inf\left\{\mu\left(\tilde{L}'\right) : \tilde{L}' \supset L\right\} = \mu'(L).\end{aligned}$$

Therefore $\mu' = \nu'(\mathcal{L})$.

5. WEAKER NOTIONS OF REGULARITY

Previously we have considered some properties related to $\mu \in M_R(\mathcal{L})$. We now want to consider weaker notions of regularity, and see when they might coincide with regularity; and, in general, to investigate their properties and interplay with the underlying lattice.

DEFINITION 5.1. Let $L \in \mathcal{L}$, where $\mathcal{L}$ is a lattice of subsets of X.

1. A measure $\mu \in M(\mathcal{L})$ is said to be *weakly regular* if $\mu(L') = \sup\{\mu'(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$.
2. A measure $\mu \in M_\sigma(\mathcal{L})$ is said to be *vaguely regular* if $\mu(L') = \sup\{\mu''(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$.

NOTATION 5.2.

$M_W(\mathcal{L}) =$ the set of weakly regular measures of $M(\mathcal{L})$

$M_V(\mathcal{L}) =$ the set of vaguely regular measures of $M_\sigma(\mathcal{L})$

LEMMA 5.3. $M_R^\sigma(\mathcal{L}) \subset M_V(\mathcal{L}) \subset M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$

REMARK 5.4. If $\mu' = \mu''(\mathcal{L})$, then $M_V(\mathcal{L}) = M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$. This occurs if:

(a) $\mathcal{L}$ is countably compact,

(b) $\mu \in N(\mathcal{L})$ and $\mathcal{L}$ is δ,

(c) $\mathcal{L}$ is normal and complement generated,

(d) $\mathcal{L}$ is δ-normal.

THEOREM 5.5. Suppose $\mu \leq \nu(\mathcal{L})$, where $\mu \in M_W(\mathcal{L})$ and $\nu \in M_R(\mathcal{L})$. Then $\mu' = \nu(\mathcal{L})$ implies $\mu = \nu$.

PROOF. Let $M_W(\mathcal{L}) \ni \mu \leq \nu \in M_R(\mathcal{L})$ and suppose $\mu' = \nu(\mathcal{L})$. Then $\mu \leq \nu = \nu' \leq \mu'(\mathcal{L})$ by 4.6. Now, $\mu \in M_W(\mathcal{L})$ implies $\mu(L') = \sup\{\mu'(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\}$ and $\nu \in M_R(\mathcal{L})$ implies $\nu(L') = \sup\{\nu(\tilde{L}) : \tilde{L} \subset L', L, \tilde{L} \in \mathcal{L}\}$. Then, since $\mu' = \nu(\mathcal{L})$, $\mu(L') = \nu(L')$, which implies $\mu = \nu(\mathcal{L}')$. Therefore $\mu = \nu$, since $\mu(X) = \nu(X)$.

THEOREM 5.6. Suppose $\mathcal{L}$ is complement generated. If $\mu \in N(\mathcal{L})$ and μ'' is a regular outer-measure, then $\mu \in M_V(\mathcal{L}) \subset M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$.

PROOF. Suppose $\mathcal{L}$ is complement generated and $\mu \in N(\mathcal{L})$. Then $\mu \in M_\sigma(\mathcal{L})$ by 2.5; and $\mu = \mu' = \mu''(\mathcal{L}')$, by 4.3 and 4.4. Now let $L \in \mathcal{L}$. Then, since $\mathcal{L}$ is complement generated, $L = \bigcap_{n=1}^{\infty} L'_n$, $L_n \in \mathcal{L}$, $L'_n \downarrow$. By the regularity of μ'' and the fact that $L' \subset \bigcap_{n=1}^{\infty} L'_n$, we have $\mu''(L') = \lim_n \mu''(L_n)$. But $\mu = \mu' = \mu''(\mathcal{L}')$ since $\mu \in N(\mathcal{L})$. Thus $\mu(L') = \sup\{\mu''(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. Hence $\mu \in M_V(\mathcal{L})$. Therefore, by 5.3, $\mu \in M_V(\mathcal{L}) \subset M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$.

THEOREM 5.7. Suppose $\mathcal{L}$ is normal and $\mu \in M_W(\mathcal{L})$. Then $\mu \in M_R(\mathcal{L})$.

PROOF. Suppose $\mathcal{L}$ is normal and $\mu \in M_W(\mathcal{L})$. Let $\mu \leq \nu(\mathcal{L})$, where $\nu \in M_R(\mathcal{L})$. Then, using 4.6,

$$\begin{aligned} \nu(L') &= \sup\{\nu(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\} = \sup\{\nu'(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\} \\ &= \sup\{\mu'(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\} = \mu(L') \quad \text{since} \quad \mu \in M_W(\mathcal{L}). \end{aligned}$$

So $\mu = \nu(\mathcal{L}')$, which implies $\mu = \nu$ since $\mu(X) = \nu(X)$. Therefore $\mu \in M_R(\mathcal{L})$.

REMARK 5.8. We saw in Theorem 5.7 that if $\mathcal{L}$ is normal, then $M_W(\mathcal{L}) = M_R(\mathcal{L})$. However, the converse is not true. For example, let $\mathcal{L} = \{\emptyset, X, A, B, A, \cup B\}$, where $A, B \subset X (A, B \neq \emptyset)$ such that $A \cap B = \emptyset$ and $A \cup B \neq X$. Here $\mathcal{L}$ is clearly not normal, but $M_W(\mathcal{L}) = M_R(\mathcal{L})$.

REFERENCES

[1] SIEGEL, D., Outer measures and weak regularity of measures, *Internat. Jour. Math. & Math. Sci.* **18**, no. 1 (1995), 49-58.

[2] ALEXANDROFF, A.D., Additive set-functions in abstract spaces (Chapter I), *Mat. Sb.* **8** (1940), 307-348.

[3] ALEXANDROFF, A.D., Additive set-functions in abstract spaces (Chapters II-III), *Mat. Sb.* **9** (1941), 563-628.

[4] ALEXANDROFF, A.D., Additive set-functions in abstract spaces (Chapters IV-VI), *Mat. Sb.* **13** (1943), 169-238.

[5] FROLIK, Z., Prime filters with the C.I.P., *Comm. Math. Univ. Carolinae* **13** (1972), 553-575.

[6] SZETO, M., Measure repleteness and mapping preservations, *Jour. of the Indian Math. Soc.* **43** (1979), 35-52.

[7] CAMACHO, JR., J., On maximal measures with respect to a lattice, *Internat. Jour. Math. & Math. Sci.* **14**, no. 1 (1991), 93-98.

[8] EID, G., On maximal measures and Wallman spaces, *Internat. Jour. Math. & Math. Sci.* **13**, no. 1 (1990), 31-38.