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ON STRICT AND SIMPLE TYPE EXTENSIONS

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ABSTRACT. Let (Y,τ) be an extension of a space (X,τ'). $p \in Y$, let $\mathcal{O}_y^p = \{W \cap X : W \in \tau, p \in W\}$. For $U \in \tau'$, let $o(U) = \{p \in Y : U \in \mathcal{O}_y^p\}$. In 1964, Banaschweski introduced the strict extension $Y^\#$, and the simple extension Y^+ of X (induced by (Y,τ)) having base $\{o(U) : U \in \tau'\}$ and $\{U \cup \{p\} : p \in Y, \text{and } U \in \mathcal{O}_y^p\}$, respectively. The extensions $Y^\#$ and Y^+ have been extensively used since then. In this paper, the open filters $\mathcal{L}^p = \{W \in \tau' : W \supseteq \operatorname{int}_X \operatorname{cl}_X(U) \text{ for some } U \in \mathcal{O}_y^p\}$, and $\mathcal{U}^p = \{W \in \tau' : \operatorname{int}_X \operatorname{cl}_X(W) \in \mathcal{O}_y^p\} = \{W \in \tau' : \operatorname{int}_X \operatorname{cl}_X(W) \in \mathcal{L}^p\} = \cap\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on $X, \mathcal{O}_X^p \subset \mathcal{U}\}$ on X are used to define some new topologies on Y. Some of these topologies produce nice extensions of (X,τ'). We study some interrelationships of these extensions with $Y^\#$, and Y^+ respectively.

KEY WORDS AND PHRASES: Extension, simple extension, strict extension, H-closed, s-closed, almost realcompact, near compact.
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1. INTRODUCTION

A topological space Y is an *extension* of a space X if X is a dense subspace of Y. If Y_1 and Y_2 are two extensions of a space X, then Y_2 is said to be *projectively larger* than Y_1, written $Y_2 \geq Y_1$ (or $Y_1 \leq Y_2$), provided that there exists a continuous map $f : Y_2 \to Y_1$ such that $f|_X = i_X$, the identity map on X. Two extensions Y_1 and Y_2 of X are called *equivalent* if $Y_1 \leq Y_2$ and $Y_2 \leq Y_1$. We shall identify two equivalent extensions of X. With this convention, the class $E(X)$ of all the Hausdorff extensions of a Hausdorff space X is a set. Let $(Y,\tau) \in E(X)$ and let $p \in Y$. If N_p is the open neighborhood filter of p in Y, the set $\mathcal{O}_Y^p = \{N \cap X : N \in N_p\}$ (called the *trace* of N_p on X) is an open filter on X. If U is open in X, denote

$$o_Y(U) = \{p \in Y : U \in \mathcal{O}_Y^p\}.$$

In 1964 Banaschewski [1] introduced the extensions $Y^{\#}$ (resp. Y^{+}) the *strict extension* (resp. the *simple extension*) of X induced by Y satisfying $Y^{\#} \le Y \le Y^{+}$. The topology $\tau^{\#}$ on $Y^{\#}$ (resp. τ^{+} on Y^{+}) has for an open base the collection $\{o_Y(U): U \text{ open in } X\}$ (resp., the collection $\{U \cup \{p\}: p \in Y$, and $U \in \mathcal{O}_Y^p\}$). The extensions $Y^{\#}$, and Y^{+} have been studied extensively and have proved extremely useful regarding some properties weaker than compactness, such as nearly compact, almost realcompact, feebly compact, H-closed, s-closed, etc.. In this paper we introduce new extensions Y^l, Y^u, Y^{l*}, and Y^{u*}, study some of their properties, and compare them with Y, $Y^{\#}$, and Y^{+}. All spaces under consideration are Hausdorff.

2. THE EXTENSIONS Y^l AND Y^u.

In this section, we introduce several topologies on Y, and compare them with τ. Some of these topologies yield interesting extensions of (X, τ').

DEFINITION 2.1. Let (Y, τ) be an extension of a space (X, τ'). For $p \in Y$ define

$$\mathcal{U}^p = \{W: W \in \tau', \operatorname{int}_X \operatorname{cl}_X W \in \mathcal{O}_Y^p\}, \tag{2.1}$$

$$\mathcal{L}^p = \{W: W \in \tau', W \supseteq \operatorname{int}_X cl_X U \text{ for some } U \in \mathcal{O}_Y^p\}. \tag{2.2}$$

LEMMA 2.1.

(a) Both $\mathcal{U}^p$ and $\mathcal{L}^p$ are open filters on X such that $\mathcal{L}^p \subseteq \mathcal{O}_Y^p \subseteq \mathcal{U}^p$.

(b) $\mathcal{U}^p = \{W: W \in \tau', \operatorname{int}_X \operatorname{cl}_X W \in \mathcal{L}^p\}$
$= \cap\{\mathcal{U}: \mathcal{U} \text{ is an open ultrafilter on } X, \mathcal{O}_Y^p \subset \mathcal{U}\}$

PROOF. We prove (b). Let $\mathcal{W} = \{W: W \in \tau', \operatorname{int}_X \operatorname{cl}_X W \in \mathcal{L}^p\}$. If $W \in \mathcal{W}$, then $W \in \tau'$ and $\operatorname{int}_X \operatorname{cl}_X W \supseteq \operatorname{int}_X \operatorname{cl}_X U$ for some $U \in \mathcal{O}_Y^p$. Therefore, $\operatorname{int}_X \operatorname{cl}_X W \in \mathcal{O}_Y^p$, whence $W \in \mathcal{U}^p$. Thus, $\mathcal{W} \subseteq \mathcal{U}^p$. To prove the reverse inequality, let $W \in \mathcal{U}^p$. Then $\operatorname{int}_X \operatorname{cl}_X W \in \mathcal{O}_Y^p$. Since $\operatorname{int}_X \operatorname{cl}_X W \supseteq \operatorname{int}_X \operatorname{cl}_X(\operatorname{int}_X \operatorname{cl}_X W)$ it follows that $\operatorname{int}_X \operatorname{cl}_X W \in \mathcal{L}^p$. Hence $W \in \mathcal{W}$. This proves the first equality in (b). The second equality follows from [9], completing the proof of the lemma.

REMARK 2.1. Since $\mathcal{O}_Y^p = \mathcal{O}_Y^p \# = \mathcal{O}_Y^p +$[9,10,11], it follows that each one of Y, Y^{+} and $Y^{\#}$ yield the same $\mathcal{L}^p$ (resp., $\mathcal{U}^p$) for all $p \in Y$. Moreover, if $Z \in E(X)$ has the same underlying set as Y, and is such that $Y^{\#} \le Z \le Y^{+}$, then Y and Z induce the same $\mathcal{L}^p$ (resp., $\mathcal{U}^p$) for all $p \in Y$. Also, if $p \ne q$ are distinct elements of Y then $\mathcal{L}^p \ne \mathcal{L}^q$ and $\mathcal{U}^p \ne \mathcal{U}^q$. Obviously, if $U \in \mathcal{O}_Y^p$, then $\operatorname{int}_X \operatorname{cl}_X(U) \in \mathcal{L}^p$. Moreover, $U \in \mathcal{U}^p$ if and only if $\operatorname{int}_X \operatorname{cl}_X(U) \in \mathcal{U}^p$.

DEFINITION 2.2. Let (Y, τ) be an extension of (X, τ'). For $G \in \tau'$, define

$$o_l(G) = G \cup \{p: p \in Y \setminus X, G \in \mathcal{L}^p\} \tag{2.3}$$

$$o_u(G) = G \cup \{p: p \in Y \setminus X, G \in \mathcal{U}^p\} \tag{2.4}$$

$$a_l(G) = \{p \in Y: G \in \mathcal{L}^p\} \tag{2.5}$$

$$a_u(G) = \{p \in Y: G \in \mathcal{U}^p\} \tag{2.6}$$

The proof of the Propositions 2.1, and 2.2 is straightforward.

PROPOSITION 2.1. Let (Y,τ) be an extension of (X,τ'). Then for all $U,V \in \tau'$

(a) $o_l(\varnothing) = \varnothing, o_l(X) = Y$,

(b) $o_l(U) \cap X = U$,

(c) $o_l(U \cap V) = o_l(U) \cap o_l(V)$,

(d) The family $\{o_l(G): G \in \tau'\}$ is an open base for a Hausdorff topology τ_l on Y and (Y,τ_l) is an extension of X.

PROPOSITION 2.2. Let (Y,τ) be an extension of (X,τ'). Then for all $U,V \in \tau'$,

(a) $o_u(\varnothing) = \varnothing$ and $o_u(X) = Y$,

(b) $o_u(U) \cap X = U$,

(c) $o_u(U \cap V) = o_u(U) \cap o_u(V)$,

(d) The family $\{o_u(G): G \in \tau'\}$ is an open base for a Hausdorff topology τ_u on Y and (Y,τ_u) is an extension of X.

PROPOSITION 2.3. Let (Y,τ) be an extension of (X,τ'). Then for all $U,V \in \tau'$

(a) $a_l(\varnothing) = \varnothing, a_l(X) = Y$,

(b) $a_l(U) \cap X \subseteq U$,

(c) $a_l(U \cap V) = a_l(U) \cap a_l(V)$,

(d) $a_l(U) = \cup\{W: W \in \tau \text{ and } \operatorname{int}_X \operatorname{cl}_X(W \cap X) \subseteq U\}$

(e) The family $\{a_l(G): G \in \tau'\}$ is an open base for a coarser Hausdorff topology τ_{al} on Y, X is dense in (Y,τ_{al}), but (Y,τ_{al}) may not be an extension of X.

PROOF. We prove (d). The rest is straightforward. Let $p \in a_l(U)$. Then $U \in \mathcal{L}^p$. Therefore, $U \supseteq \operatorname{int}_X \operatorname{cl}_X V$ for some $V \in \mathcal{O}_Y^p$. Therefore, there exists $W \in \tau$ such that $p \in W$ and $W \cap X = V$. It follows that $\operatorname{int}_X \operatorname{cl}_X(W \cap X) \subseteq U$. Conversely, if $W \in \tau$ is such that $\operatorname{int}_X \operatorname{cl}_X(W \cap X) \subseteq U$ and $p \in W$, then $W \cap X \in \mathcal{O}_Y^p$. So, $\operatorname{int}_X \operatorname{cl}_X(W \cap X) \in \mathcal{L}^p$. This implies that $U \in \mathcal{L}^p$ and hence $p \in a_l(U)$. The proof of the proposition is now complete.

PROPOSITION 2.4. Let (Y,τ) be an extension of (X,τ'). Then for all $U,V \in \tau'$,

(a) $a_u(\varnothing) = \varnothing$ and $a_u(X) = Y$,

(b) $a_u(U) \cap X = \text{int}_X\, \text{cl}_X(U)$,

(c) $a_u(U \cap V) = a_u(U) \cap a_u(V)$,

(d) $a_u(U) = \cup\{W: W \in \tau \text{ and } W \cap X \subseteq \text{int}_X\, \text{cl}_X(U)\}$

(e) The family $\{a_u(G): G \in \tau'\}$ is an open base for a coarser Hausdorff topology τ_{au} on Y, X is dense in (Y, τ_{au}), but (Y, τ_{au}) may not be an extension of X.

PROOF. We prove (d). The rest is straightforward. Let $p \in a_u(U)$. Then $U \in \mathcal{U}^p$. Therefore, $\text{int}_X\, \text{cl}_X U \in \mathcal{O}_Y^p$. It follows that there exists $W \in \tau$ such that $p \in W$ and $W \cap X \subseteq \text{int}_X\, \text{cl}_X U$. Conversely, if $W \in \tau$ is such that $W \cap X \subseteq \text{int}_X\, \text{cl}_X U$ and $p \in W$, then $W \cap X \in \mathcal{O}_Y^p$. So, $\text{int}_X\, \text{cl}_X U \in \mathcal{O}_Y^p$. Therefore, $U \in \mathcal{U}^p$ and $p \in a_u(U)$.

DEFINITION 2.3. The spaces (Y, τ_l), (Y, τ_u), (Y, τ_{al}), and (Y, τ_{au}) described in propositions 2.1-2.4 will, henceforth, be denoted by Y^l, Y^u, Y^{al}, and Y^{au} respectively. If $A \subseteq Y$, then $\text{int}_{Y^l}(A)$ (resp. $\text{cl}_{Y^l}(A)$) will be denoted by $\text{int}_l(A)$ (resp., $\text{cl}_l(A)$). Likewise, $\text{int}_u(A), \text{cl}_u(A), \text{int}_{al}(A), \text{cl}_{al}(A), \text{int}_{au}(A)$, and $\text{cl}_{au}(A)$ are defined in an analogous manner.

LEMMA 2.2. If $U \in \tau'$, then

(a) $a_l(U) \subseteq o_l(U) \subseteq o_Y(U) \subseteq o_u(U) \subseteq o_u(\text{int}_X\, \text{cl}_X U) = a_u(U) = a_u(\text{int}_X\, \text{cl}_X U)$,

(b) $a_l(U) \setminus X = o_l(U) \setminus X$, and $a_u(U) \setminus X = o_u(U) \setminus X$

(c) $o_l(\text{int}_X\, \text{cl}_X U) \setminus X = o_u(U) \setminus X$, and

(d) if U is regular open (*i.e.* $U = \text{int}_X\, \text{cl}_X U$), then $a_u(U) = a_l(U)$, and the equality holds in (a).

PROOF. Part (a): We show that $o_u(\text{int}_X\, \text{cl}_X U) = a_u(U)$, the rest being straightforward. Certainly, $o_u(\text{int}_X\, \text{cl}_X U) \cap X = \text{int}_X\, \text{cl}_X U = a_u(U) \cap X$. Let $p \in o_u(\text{int}_X\, \text{cl}_X U) \setminus X$. Then $\text{int}_X\, \text{cl}_X U \in \mathcal{U}^p$. Therefore, $U \in \mathcal{U}^p$, and $p \in a_u(U) \setminus X$. Conversely, let $p \in a_u(U) \setminus X$. Then, $U \in \mathcal{U}^p$. So, $p \in o_u(U) \setminus X \subseteq o_u(\text{int}_X\, \text{cl}_X U) \setminus X$. The above arguments prove (a).

To prove (c), let $q \in o_l(\text{int}_X\, \text{cl}_X G) \setminus X$. Then, $\text{int}_X\, \text{cl}_X G \in \mathcal{L}^q$ whence, $G \in \mathcal{U}^q$. Therefore, $q \in o_u(G) \setminus X$. Thus, $o_l(\text{int}_X\, \text{cl}_X G) \setminus X \subseteq o_u(G) \setminus X$. To prove the reverse inequality, let $q \in o_u(G) \setminus X$. Then, $G \in \mathcal{U}^q$, whence $\text{int}_X\, \text{cl}_X G \in \mathcal{L}^q$. Therefore, $q \in o_l(\text{int}_X\, \text{cl}_X G) \setminus X$ and $o_u(G) \setminus X \subseteq o_l(\text{int}_X\, \text{cl}_X G) \setminus X$. Hence, $o_l(\text{int}_X\, \text{cl}_X G) \setminus X = o_u(G) \setminus X$. The rest of the lemma is straightforward.

Given a space (X, τ'), the family $\{\text{int}_X\, \text{cl}_X U : U \in \tau'\}$ forms an open base for a coarser Hausdorff topology τ'_s on X. The space $X_s = (X, \tau'_s)$ is called the *semiregularization* of X. A space (X, τ') is called *semiregular* if $(X, \tau') = X_s$

THEOREM 2.1. If X is semiregular, and (Y, τ) (not necessarily semiregular) is an extension of X, then Y^l is an extension of X such that $Y^l \leq Y$.

PROOF. If X is semiregular, then $o_l(U) = a_l(U)$ for all $U \in \tau'$. Hence, Y^l is an extension of X such that $Y^l = Y^{al} \leq Y$.

THEOREM 2.2. The spaces Y^{al} and Y^{au} are homeomorphic.

PROOF. For all $U \in \tau'$, $a_l(\text{int}_X \text{cl}_X U) = o_u(\text{int}_X \text{cl}_X U) = a_u(U)$ implies that $\tau_{au} \subseteq \tau_{al}$. Also, if $G \in \tau'$ and $p \in a_l(G)$, then $G \supseteq \text{int}_X \text{cl}_X(U)$ for some $U \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$. Now, if $q \in a_u(U)$, then $\text{int}_X \text{cl}_X U \in \mathcal{L}^q$ which implies that $G \in \mathcal{L}^q$, or $q \in a_l(G)$. Therefore, $p \in a_u(U) \subseteq a_l(G)$. Hence, $\tau_{al} \subseteq \tau_{au}$. This proves the theorem.

LEMMA 2.3. Let (Y,τ) be an extension of (X,τ'). Then, for all $G \in \tau'$ the following are true.

(a) $\text{cl}_{al}(G) \subseteq \text{cl}_l(G) = \text{cl}_l(\text{int}_X \text{cl}_X(G))$,

(b) $\text{cl}_u(G) = \text{cl}_{au}(G) = \text{cl}_u(\text{int}_X \text{cl}_X G))$,

(c) $\text{cl}_u(G) = \text{cl}_l(G)$,

(d) $\text{cl}_Y(G) = \text{cl}_{au}(G) = \text{cl}_{al}(\text{int}_X \text{cl}_X(G))$, and

(e) $\text{cl}_u(o_u(G)) = \text{cl}_{au}(a_u(\text{int}_X \text{cl}_X(G)))$

PROOF. Part (a): Let $p \in \text{cl}_{al}(G)$, and let $o_l(U)$ be a basic open neighborhood of p in Y^l. If $p \in o_l(U) \cap X$, then $p \in U \subseteq \text{int}_X \text{cl}_X U \in \mathcal{L}^p$. Therefore, $a_l(\text{int}_X \text{cl}_X U)$ is an open neighborhood of p in Y^{al}. Consequently, $a_l(\text{int}_X \text{cl}_X U) \cap G \neq \varnothing$. By Proposition (2.7) (b), $\text{int}_X \text{cl}_X U \cap G \neq \varnothing$. Hence $U \cap G \neq \varnothing$. This in turn implies that $o_l(U) \cap G \neq \varnothing$, and $p \in \text{cl}_l(G)$. If $p \in o_l(U) \setminus X$, then $U \in \mathcal{L}^p$. Now, $a_l(U)$ is an open neighborhood of p in Y^{al}. Consequently, $a_l(U) \cap G \neq \varnothing$. Therefore, $o_l(U) \cap G \neq \varnothing$ whence $p \in \text{cl}_l(G)$.

Part (b): Let $p \in \text{cl}_{au}(G)$, and let $o_u(U)$ be a basic open neighborhood of p in Y^u. Since $o_u(U) \subseteq a_u(U), a_u(U)$ is an open neighborhood of p in Y^{au}. Hence, $a_u(U) \cap G \neq \varnothing$. Therefore, $\text{int}_X \text{cl}_X U \cap G \neq \varnothing$, whence $U \cap G \neq \varnothing$. Consequently, $o_u(U) \cap G \neq \varnothing$. Therefore, $p \in cl_u(G)$. Therefore, $\text{cl}_{au}(G) \subseteq \text{cl}_u(G)$. Conversely, let $p \in \text{cl}_u(G)$, and let $a_u(U)$ be a basic open neighborhood of p in Y^{au}. If $p \in a_u(U) \cap X = \text{int}_X \text{cl}_X U$, then $o_u(\text{int}_X \text{cl}_X U)$ is an open neighborhood of p in Y^u. Therefore, $a_u(U) \cap X = \text{int}_X \text{cl}_X U \cap G = o_u(\text{int}_X \text{cl}_X U) \cap G \neq \varnothing$. Hence, $p \in \text{cl}_{au}(G)$. Now, if $p \in a_u(U) \setminus X$, then $U \in \mathcal{U}^p$ and $o_u(U)$ is an open neighborhood of of p in Y^u. Therefore, $o_u(U) \cap G \neq \varnothing$. Consequently, $a_u(U) \cap G \neq \varnothing$, and $p \in \text{cl}_{au}(G)$. Therefore, $\text{cl}_u(G) \subseteq \text{cl}_{au}(G)$. Hence, $\text{cl}_u(G) \subseteq \text{cl}_{au}(G)$. The other half of (b) is straightforward.

The proof of (c) is straightforward.

Part (d): Let $p \in \text{cl}_{au}(G)$, and let W be an open neighborhood of p in Y. Then, $W \cap X \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$ shows that $o_u(W \cap X)$ is an open neighborhood of p in Y^{au}. Therefore, $a_u(W \cap X) \neq \varnothing$. This shows that $W \cap G \neq \varnothing$, whence $p \in \text{cl}_Y(G)$. Conversely, let $p \in \text{cl}_Y(G)$, and let $a_u(U)$ be a basic open neighborhood of p in Y^{au}. Then, $U \in \mathcal{U}^p$. So, $o_Y(\text{int}_X \text{cl}_X U)$ is an open neighborhood of p in Y such that $o_Y(\text{int}_X \text{cl}_X U) \cap G \neq \varnothing$. This implies that $a_u(U) \cap G \neq \varnothing$. Hence, $p \in \text{cl}_{au}(G)$. The rest follows from (c).

THEOREM 2.3. The spaces $Y^l \setminus X, Y^{al} \setminus X$, and $Y^u \setminus X$ are pairwise homeomorphic.

PROOF. To prove the continuity of the identity map $i: Y^u \setminus X \to Y^l \setminus X$, let $o_l(G) \setminus X$ be a basic open neighborhood of p in $Y^l \setminus X$. Then, $G \in \mathcal{L}^p$. Hence $G \supseteq \operatorname{int}_X \operatorname{cl}_X U$ for some $U \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$. Therefore, $o_u(U) \setminus X$ is an open neighborhood of p in Y^u such that $o_u(U) \setminus X \subseteq o_l(G) \setminus X$. To prove that the identity map $i: Y^l \setminus X \to Y^u \setminus X$ is continuous, let $o_u(G) \setminus X$ be a basic open neighborhood of p in $Y^u \setminus X$. Then $o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X$ is an open neighborhood of p in $Y^l \setminus X$ such that $o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X = o_u(G) \setminus X$. Hence, the spaces $Y^l \setminus X$, and $Y^u \setminus X$ are homeomorphic. The rest of the theorem follows directly from Lemma 2.2.

Let Z_1 and Z_2 be spaces. A map $f: Z_1 \to Z_2$ is called θ - *continuous* [3] if for every $p \in Z_1$ and for every open neighborhood V of $f(p)$ in Z_2, there exists an open neighborhood U of p in Z_1 such that $f(\operatorname{cl}_{Z_1} U) \subseteq \operatorname{cl}_{Z_2}(V)$. f is called *perfect* if f is a closed map (not necessarily continuous) such that $f^{\leftarrow}(z)$ is compact in Z_1 for every $z \in Z_2$. Also, f is called *irreducible* if f is closed and there is no proper closed subset K of Z_1 for which $f(K) = Z_2$. Two extensions Z_1, and Z_2 of a space X are called θ - *equivalent* if there exists a θ - homeomorphism f from Z_1 onto Z_2 such that $f|_X = i_X$, the identity map on X.

The next theorem depicts some of the several interrelationships between the spaces Y, $Y^\#$, Y^l, Y^u, and Y^{al}.

THEOREM 2.4. Let (Y, τ) be an extension of a space (X, τ'). The following statements are true.

(a) The identity map $i: Y^{al} \to Y$ is perfect, irreducible and θ - continuous.

(b) The identity map $i: Y^{au} \to Y^u$ is perfect, irreducible and θ - continuous.

(c) The identity map $i: Y^{al} \to Y^\#$ is θ - continuous.

(d) The identity map $i: Y^\# \to Y^l$ is θ - continuous.

(e) The identity map $i: Y^\# \to Y^u$ is θ - continuous.

(f) The identity map $i: Y^l \to Y^\#$ is θ - continuous.

(g) The identity map $i: Y^u \to Y^\#$ is θ - continuous.

(h) The identity map $i: Y^l \to Y^u$ is θ - continuous.

(i) The identity map $i: Y^u \to Y^l$ is θ - continuous.

(j) The identity map $i: Y^l \to Y$ is θ - continuous.

(k) The identity map $i: Y^u \to Y$ is θ - continuous.

(l) The identity map $i: Y^\# \to Y^{al}$ is θ - continuous.

PROOF. Below, we outline the proofs of some parts of the theorem. The rest of the proofs are analogous.

Part (a) Since $\tau_{al} \subseteq \tau, i: Y \to Y^{al}$ is continuous. Hence, $i: Y \to Y^{al}$ is irreducible and perfect. To prove the θ - continuity of $i: Y^{al} \to Y$, let V be an open neighborhood of p in Y. Then $V \cap X \in \mathcal{O}_Y^p$ and $\operatorname{int}_X \operatorname{cl}_X (V \cap X) \in \mathcal{L}^p$. Therefore, $a_l(\operatorname{int}_X \operatorname{cl}_X (V \cap X))$ is an open neighborhood of p in Y^{al} such that

$\mathrm{cl}_{al}(a_l(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) = \mathrm{cl}_Y(a_l(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) = \mathrm{cl}_Y[a_l(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) \cap X]$ $= \mathrm{cl}_Y(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) \subseteq \mathrm{cl}_Y(V)$. Hence $i: Y^{al} \to Y$ is θ-continuous.

Part (b): For all $G \in \tau', a_u(G) = o_u(\mathrm{int}_X \mathrm{cl}_X G) \in \tau_u$ shows that $i: Y^u \to Y^{au}$ is continuous. Therefore, $i: Y^{au} \to Y$ is irreducible and perfect. Let $o_u(G)$ be a basic open neighborhood of p in Y^u. Since $o_u(G) \subseteq a_u(G), a_u(G)$ is an open neighborhood of p in Y^{au} such that $\mathrm{cl}_{au}(a_u(G)) = \mathrm{cl}_u(o_u(G))$, establishing the θ-continuity of $i: Y^{au} \to Y^u$.

Part (c): To prove the θ-continuity of $i: Y^{al} \to Y^{\#}$, let $p \in Y$ and let $o_Y(G), G \in \tau'$ be a basic open neighborhood of p in $Y^{\#}$. Then, $G \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$ implies that $a_l(\mathrm{int}_X \mathrm{cl}_X G)$ is an open neighborhood of p in Y^{al} such that $\mathrm{cl}_{al}(a_l(\mathrm{int}_X \mathrm{cl}_X G)) \subseteq \mathrm{cl}_Y\#(o_Y(G))$

Part (d): Let $o_l(G)$ be a basic open neighborhood of p in Y^l. Then, $o_Y(G)$ is an open neighborhood of p in $Y^{\#}$ such that $\mathrm{cl}_Y\#(o_Y(G)) \subseteq \mathrm{cl}_l(o_l(G))$, establishing the θ-continuity of $i: Y^{\#} \to Y^l$.

Part (h): Let $o_u(G)$ be a basic open neighborhood of p in Y^u. Then, $o_l(\mathrm{int}_X \mathrm{cl}_X G)$ is an open neighborhood of p in Y^l satisfying $\mathrm{cl}_l(o_l(\mathrm{int}_X \mathrm{cl}_X G)) \subseteq \mathrm{cl}_u(o_u(G))$

Part (l): Let $p \in Y$ and let $a_l(G), G \in \tau'$ be a basic open neighborhood of p in Y^{al}. Then, $G \supseteq \mathrm{int}_X \mathrm{cl}_X U$ for some $U \in \mathcal{O}_Y^p$. So, $p \in o_Y(U)$. Now, $\mathrm{cl}_Y(o_Y(U)) = \mathrm{cl}_Y(o_Y(U) \cap X) = \mathrm{cl}_Y(U) \subseteq \mathrm{cl}_{al}(U) \subseteq \mathrm{cl}_{al}(a_l(G))$.

We now summarize the results proved above in the following theorem.

THEOREM 2.5. The spaces Y, $Y^{\#}$, Y^l, Y^u, and Y^{al}, are pairwise θ-homeomorphic. The spaces Y^l, and Y^u are θ-equivalent extensions of X with homeomorphic remainders.

It is well known that spaces Y and Z are θ-homeomorphic if and only if their semiregularizations are homeomorphic. [11] Hence, we have the following corollary.

COROLLARY 2.1. Let (Y,τ) be an extension of a space (X,τ). Then, the spaces $Y_s, Y_s^{\#}, Y_s^l, Y_s^u$, and Y_s^{al} are pairwise homeomorphic. Moreover, Y_s, Y_s^l, and Y_s^u are equivalent extensions of X_s.

3. THE EXTENSIONS Y^{l*}, AND Y^{u*}.

In this section, we define extensions Y^{l*}, and Y^{u*}, analogous to the simple extension Y^+ of (X,τ') induced by an extension (Y,τ) of X. The spaces Y^{l*}, Y^{al}, Y^{u*}, and Y^{au*} all have the same underlying set as the set Y. An open base for the topology τ_{l*} on Y^{l*} (respectively, τ_{al*} on Y^{al*}) is the family $\tau' \cup \{G \cup \{p\} : p \in Y \setminus X, G \in \mathcal{L}^p\}$ (respectively, $\tau' \cup \{G \cup \{p\} : G \in \mathcal{L}^p\}$). An open base for the topology τ_{u*} on Y^{u*} (respectively, τ_{au*} on Y^{au*}) is the family $\tau' \cup \{G \cup \{p\} : p \in Y \setminus X, G \in \mathcal{U}^p\}$ (respectively, $\tau' \cup \{G \cup \{p\} : G \in \mathcal{U}^p\}$). For any $A \subset Y, \mathrm{cl}_{l*}(A)$ will denote the closure of A in Y^{l*}, with analogous notations in other cases. The proofs of the following statements are straightforward, and we omit the details. Obviously, the spaces $Y^{l*} \setminus X, Y^{u*} \setminus X, Y^{al*} \setminus X$, and $Y^{au*} \setminus X$ are all discrete.

THEOREM 3.1. The spaces Y^{l*}, and Y^{u*} are extensions of (X,τ') such that $Y^{u*} \geq Y^+ \geq Y^{l*}$. The set X is dense in the spaces Y^{al*}, and Y^{au*}. But, Y^{al*} and Y^{au*} may not be extensions of X.

LEMMA 3.1. For each $G \in \tau'$, $\text{cl}_{l*}(o_l(G)) = \text{cl}_l(o_l(G)$, and $\text{cl}_{u*}(o_u(G)) = \text{cl}_u(o_u(G))$.

THEOREM 3.2. Each one of the identity maps $i: Y^+ \to Y^{u*}$, and $i: Y^{l*} \to Y^+$ is θ-continuous.

THEOREM 3.3. The spaces Y^+, Y^{l*}, Y^{u*}, Y^{al*}, and Y^{au*} are θ-homeomorphic. Moreover, Y^+, Y^{l*}, and Y^{u*} are θ-equivalent extensions of X with homeomorphic remainders.

COROLLARY 3.1. If (Y,τ) is an extension of a space (X,τ'), then the spaces $Y_s^+, Y_s^{l*}, Y_s^{u*}, Y_s^{al*}$, and Y_s^{au*} are homeomorphic in pairs. Moreover, the spaces Y_s^+, Y_s^{l*}, and Y_s^{u*} are equivalent extensions of X_s

REMARKS 3.1. (a) If **P** is any property of topological spaces which is preserved under θ-continuous surjections, and if (Y,τ) is a **P**-extension of (X,τ'), then Y^l, Y^u, Y^{l*}, and Y^{u*} are also **P**-extensions of X.

(b) The extensions Y^l, Y^u, Y^{l*}, and Y^{u*} introduced above are, in general, all distinct from Y, $Y^{\#}$, and Y^+. It would be interesting to find a characterization of spaces Y for which $Y^{\#} = Y^l$. A space Z is called H-closed if it is closed in every Hausdorff space in which it is embedded [see 11 for more details]. The Katetov (respectively, Fomin) extension of a space (X,τ') is the space κX (respectively, σX) whose underlying set is the set $X \cup \{p: p \text{ is a free open ultrafilter on } X\}$, and whose topology has for an open base the family $\tau' \cup \{U \cup \{p\}: U \in p, \text{ and } p \in \kappa X \setminus X\}$ (respectively, the family $\{o_{\kappa X}(U): U \in \tau'\}$). The spaces κX, and σX are H-closed extensions of X such that $(\sigma X)^+ = \kappa X$, and $(\kappa X)^{\#} = \sigma X$ [3, 6, 11]. In general $(\sigma X)^l \neq \sigma X, (\kappa X)^u \neq \kappa X, (\sigma X)^{u*} \neq \sigma X$, and $(\kappa X)^{l*} \neq \kappa X$. Analogous remarks apply to the Banaschewski-Fomin-Shanin extension μX [13] of a Hausdorff space X

(c) A space Z is called *compact like*, or *nearly compact* if every regular open cover of Z is reducible to a finite subcover. A space X has a compactlike extension if and only if X_s is Tychonoff [14]. Compactlike extensions (=*near* compactifications) of Hausdorff almost completely regular spaces X (whence, X_s is Tychonoff) have been constructed in [2] via EF-Proximities. For a Hausdorff space X whose semiregularization X_s is Tychonoff, a maximal compactlike extension BX of X, satisfying $(BX)_s = \beta X_s$, is constructed in [14]. If (X,τ') is any Hausdorff almost complétely regular space, and if (Y,τ) is any near compactification of (X,τ'), then so are Y^l, Y^u, Y^{l*}, and Y^{u*}.

(d) A space Z is called almost real compact if every open ultrafilter on Z with countable closed intersection property in Z converges in Z[4]. A space Z is almost realcompact if and only if Z_s is almost realcompact [12]. Almost realcompactifications of a Hausdorff space have been constructed (among others) in [7], and [12]. If (X,τ') is any Hausdorff space, and if (Y,τ) is any almost realcompactification of (X,τ'), then so are Y^l, Y^u, Y^{l*}, and Y^{u*}.

(e) A Hausdorff space Z is called extremally disconnected if for each open subset U of Z, $cl_Z(U)$ is open. A space Z is extremally disconnected if and only if each dense subspace of Z [respectively, if and only if Z_s] is extremally disconnected [see 11 for more details]. A Hausdorff space Z is called *s-closed* if it is H-closed and extremally disconnected [8]. A Hausdorff space Z is s-closed if and only if Z_s is s-closed. It is shown in [8] that every extremally disconnected space X admits an s-closed extension ,viz.

κX ; moreover, an extension Y of X is s-closed if and only if X is C^*-embedded in Y. If (X,τ') is any extremally disconnected Hausdorff space, and if (Y,τ) is any s-closed extension of (X,τ'), then so are Y^l, Y^u, Y^{l*}, and Y^{u*}.

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