Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 417-422

ON STRONG FORM OF ARZELA CONVERGENCE

JANINA EWERT

Department of Mathematics
Pedagogical University
Arciszewskiego 22 b
76-200 Slupsk, POLAND

(Received November 30, 1995 and in revised form April 23, 1996)

ABSTRACT. We define some new type of convergence of nets of functions which is formulated in terms of open covers. It preserves continuity and under some assumptions implies (or coincides with) the Arzela quasi-uniform convergence. Furthermore, the introduced strong convergence is used for characterization of compactness and regularity of a topological space.

KEY WORDS AND PHRASES: Arzela convergence, almost compact space, uniform space
1991 AMS SUBJECT CLASSIFICATION CODES: 54A20.

1. INTRODUCTION

Let $f_n : X \to Y$, $n \geq 1$, be continuous functions and let $f : X \to Y$ be a limit of the sequence $\{f_n : n \geq 1\}$. Which assumptions on the convergence guarantee the continuity of f? This question led to defining various types of convergences for nets of functions with values in metric or uniform spaces [1,2,3,4,5]. But initial notions in this problem—continuity and pointwise convergence—are depending on topologies only and they can be considered even then if spaces are not uniformizable. This is the motivation of our paper. We define some new form of convergence formulated in terms of open covers; it preserves continuity and under some assumptions implies (or coincides with) the Arzela quasi-uniform convergence.

Let X, Y be topological spaces; the symbols $F(X,Y)$ and $C(X,Y)$ are used to denote the class of all functions or all continuous functions from X to Y, respectively. For any set A its closure is denoted by ClA.

A net $\{f_j : j \in J\} \subset F(X,Y)$ is said to be strongly convergent to $f \in F(X,Y)$ if:

- this net is pointwise convergent to f, and
- for each open cover $\mathcal{A}$ of Y and each $j_0 \in J$ there exists a finite set $J_0 \subset J$ such that $j \geq j_0$ for $j \in J_0$ and for each $x \in X$ there are $j \in J_0$ and $W \in \mathcal{A}$ with $\{f(x), f_j(x)\} \subset W$

THEOREM 1. Let X be a topological space and Y a regular one. If a net $\{f_j : j \in J\} \subset C(X,Y)$ is strongly convergent to a function $f : X \to Y$, then $f \in C(X,Y)$.

PROOF. Let $x_0 \in X$ and let W, V be open sets in Y such that $f(x_0) \in V \subset Cl\,V \subset W$, then $\mathcal{A} = \{W, Y \backslash Cl\,V\}$ is an open cover of Y. Now we can choose $j_0 \in J$ and a finite subset $J_0 \subset J$ such that

$$\begin{aligned} j \geq j_0 \quad &\text{for} \quad j \in J_0, \\ f_j(x_0) \in V \quad &\text{for} \quad j \in J,\ j \geq j_0, \end{aligned}$$

and for each $z \in X$ there is $j_z \in J_0$ with $\{f(z), f_{j_z}(z)\} \subset W$ or $\{f(z), f_{j_z}(z)\} \subset Y \backslash Cl\,V$ There exists a neighborhood U of x_0 such that $f_j(U) \subset V$ for $j \in J_0$, thus $f_j(U) \cap (Y \backslash Cl\,V) = \emptyset$ for $j \in J_0$

Hence for each $x \in U$ we have $\{f(x), f_{j_x}(x)\} \subset W$ which implies $f(U) \subset V$ and the proof is completed

Now we are going to compare the strong convergence with other ones. To begin with let us observe that the uniform convergence and the strong one are independent as shown in the following.

EXAMPLES. In the space R of real numbers with the usual metric let us consider functions $f_n, f : R \to R$ defined by $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ for each $x \in R, n \geq 1$; so the sequence $\{f_n : n \geq 1\}$ is uniformly convergent to f. Now, taking open intervals $U_n = \left(n - \frac{1}{2n}, n + \frac{1}{2n}\right)$ and $V_n = \left(n + \frac{1}{3n}, n + 1 - \frac{1}{3n}\right)$ for $n \geq 1$ we obtain the open cover $\mathcal{A} = \{U_n, V_n : n \geq 1\} \cup \{(-\infty, 3)\}$ of R. For fixed n, k with $k > n > 1$ and for each j with $n < j < k$ we have that U_k is the unique set in $\mathcal{A}$ containing $f(k)$ and $|f_j(k) - f(k)| = \frac{1}{j} > \frac{1}{2k}$, so $f_j(k) \notin U_k$. Hence the sequence $\{f_n : n \geq 1\}$ is not strongly convergent to f.

Now let $g_n, g : R \to R$ be functions given by:

$$g(x) = 0 \quad \text{for each} \quad x \in R, \quad \text{and}$$

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in (-\infty, n-1] \cup [n+1, \infty) \\ \left(1 - \frac{1}{n}\right)x - n + 2, & \text{if } x \in [n-1, n] \\ \left(\frac{1}{n} - 1\right)x + n, & \text{if } x \in [n, n+1]. \end{cases}$$

The sequence $\{g_n : n \geq 1\}$ strongly converges to g but not uniformly.

In a uniform space $(Y, \mathcal{V})$ for any $V \in \mathcal{V}$ and $y \in Y$ we will write $V[y] = \{z \in Y : (y, z) \in V\}$. A net $\{f_j : j \in J\} \subset F(X, Y)$ is said to be $\mathcal{V}$-quasi-uniformly in the sense of Arzela convergent to $f \in F(X, Y)$ if

- it is pointwise convergent to f, and
- for each $V \in \mathcal{V}$ and $j_0 \in J$ there exists a finite set $J_0 \subset J$ with $j \geq j_0$ for $j \in J_0$ and for each $x \in X$ there is $j \in J_0$ such that $(f(x), f_j(x)) \in V$ [1,2].

Let us remark that the quasi-uniform convergence is strictly connected with a uniformity (or metric) For instance let $X = Y = (0, \infty)$, $d_1(x, y) = |x - y|$ and $d_2(x, y) = |x^2 - y^2|$; then d_1 and d_2 are topologically equivalent metrics on Y. Putting $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ for $x \in X$, $n \geq 1$ we have the sequence $\{f_n : n \geq 1\}$ which converges to f d_1-uniformly but it is not d_2-quasi-uniformly convergent to f.

PROPOSITION. Yet Y be a completely regular space. If a net $\{f_j : j \in J\} \subset F(X, Y)$ is strongly convergent to a function $f : X \to Y$, then for each compatible uniformity $\mathcal{V}$ on Y this net $\mathcal{V}$-quasi-uniformly converges to f

PROOF. Let $\mathcal{V}$ be a compatible uniformity on Y and $V \in \mathcal{V}$ We take an open set $W \in \mathcal{V}$ with $W = W^{-1}$ and $W^2 \subset V$. Then it suffices to consider the open cover $\mathcal{A} = \{W[y] : y \in Y\}$.

A uniform space $(Y, \mathcal{V})$ (or simply a uniformity $\mathcal{V}$), is said to have the Lebesgue property if for each open cover $\mathcal{A}$ of Y there is $V \in \mathcal{V}$ such that $\{V[y] : y \in Y\}$ is a refinement of $\mathcal{A}$ [6].

It is easy to see that if $\mathcal{V}$ has the Lebesgue property, then for each X the strong convergence in $F(X, Y)$ coincides with the $\mathcal{V}$-quasi-uniform one.

The Lebesgue property is closely related to the paracompactness, namely we have the following

THEOREM 2 [6]. Let Y be a Tychonoff space Then Y possesses a uniformity $\mathcal{V}$ compatible with the given topology for which $(Y, \mathcal{V})$ has the Lebesgue property if and only if Y is paracompact. Under this condition, this uniformity is the finest of all compatible uniformities, and is unique.

So, as a simple consequence we have

COROLLARY. Let Y be a paracompact space and let $\mathcal{V}$ be the finest compatible uniformity on Y. Then for each X the strong convergence in $F(X, Y)$ is equivalent to the $\mathcal{V}$-quasi-uniform one

A topological space X is called almost compact if for each open cover $\mathcal{A}$ of X there is a finite number $A_1, A_2, \dots, A_n \in \mathcal{A}$ such that $X = \bigcup_{i=1}^{n} Cl\, A_i$ [7, p. 239].

THEOREM 3. Let X be an almost compact space, Y a Tychonoff one and let $\{f_j : j \in J\} \subset F(X,Y)$, $f \in C(X,Y)$. Then the following conditions are equivalent:

(a) the net $\{f_j : j \in J\}$ is strongly convergent to f;

(b) for each compatible uniformity $\mathcal{V}$ on Y the net $\{f_j : j \in J\}$is $\mathcal{V}$-quasi-uniformly convergent to f,

(c) there exists a compatible uniformity $\mathcal{V}$ on Y such that the net $\{f_j : j \in J\}$ is $\mathcal{V}$-quasi-uniformly convergent on f.

PROOF. In virtue of the proposition it suffices to show the implication (c) $\Rightarrow$ (a) Let $\mathcal{V}$ be a compatible uniformity on Y such that the net $\{f_j : j \in J\}$ is $\mathcal{V}$-quasi-uniformly convergent to f We fix an open cover $\mathcal{A}$ of Y and $j_0 \in J$. Then for each $x \in X$ we choose a set $W_x \in \mathcal{A}$ and $V_x \in \mathcal{V}$ with $V_x = V_x^{-1}$ and $V_x^3[f(x)] \subset W_x$ Since f is continuous for each $x \in X$ there is a neighborhood U_x of x for which $f(U_x) \subset V_x[f(x)]$. From the assumption on X the open cover $\{U_x : x \in X\}$ contains a finite family $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$such that $X = \bigcup_{k=1}^{n} Cl\, U_{x_k}$. We put $V = \bigcap_{k=1}^{n} V_{x_k}$. Now, by the $\mathcal{V}$-quasi-uniform convergence there exists a finite set $J_1 \subset J$ such that. $j \geq j_0$ for each $j \in J_1$ and for each $z \in X$ we can take $j_z \in J_1$ with $(f(z), f_{j_z}(z)) \in V$. Furthermore, $z \in Cl\, U_{x_k}$ for some k, $k \leq n$ Hence

$$f(z) \in f(Cl\, U_{x_k}) \subset Cl\, V_{x_k}[f(x_k)] \subset V_{x_k}^2[f(x_k)] \subset W_{x_k}$$

and

$$f_{j_z}(z) \in V[f(z)] \subset V_{x_k}[f(z)] \subset V_{x_k}^3[f(x_k)] \subset W_{x_k}.$$

So we have shown that for each $z \in X$ it holds $\{f(z), f_{j_z}(z)\} \subset W_{x_k}$ for some $j_z \in J_1$ and $W_{x_k} \in \mathcal{A}$, which finishes the proof.

THEOREM 4. For a topological space X let us consider the following properties:

(a) the strong convergence in $F(X, [0,1])$ coincides with the pointwise one;

(b) X is an almost compact space;

(c) for each regular space Y the strong convergence in $C(X,Y)$ is equivalent to the pointwise one;

then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

PROOF. We will show the implication (a) $\Rightarrow$ (b). Let $\mathcal{A}$ be an open cover of X and let $\{U_s : s \in S\}$ be the family of all finite sums of sets belonging to $\mathcal{A}$. We define a relation " $\leq$ " in S assuming $s_1 \leq s_2$ iff $U_{s_1} \subset U_{s_2}$; so $(S, \leq)$ is a directed set. Now let us consider functions $f_s, f : X \to [0,1], s \in S$, given by $f(x) = 1$ for $x \in X$ and

$$f_s(x) = \begin{cases} 1, & \text{if } x \in Cl\, U_s \\ 0, & \text{if } x \in X \backslash Cl\, U_s. \end{cases}$$

It is easy to see that f is the pointwise limit of the net $\{f_s : s \in S\}$; thus, by the assumption this net strongly converges to f. Hence for the open cover $\mathcal{G} = \{[0, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4}), (\frac{1}{2}, 1]\}$ of $[0,1]$ and a fixed $s_0 \in S$ there exists a finite set $S_1 \subset S$ with $s \geq s_0$ for $s \in S_1$ and for each $x \in X$ some $s \in S_1$ can be taken such that both $f(x)$ and $f_s(x)$ are contained in the same set from $\mathcal{G}$. This implies that for each $x \in X$ there is $s \in S_1$ for which $f_s(x) = 1$, which means $X = \bigcup\{Cl\, U_s : s \in S_1\}$ From the definition of the sets U_s it follows that $\mathcal{A}$ contains a finite family $\mathcal{A}_1$ with $X = \bigcup\{Cl\, U : U \in \mathcal{A}_1\}$, so X is almost compact

Now, we suppose that X is almost compact, Y is a regular space and $\{f_j : j \in J\} \subset C(X,Y)$ is a net of functions which pointwise converges to some $f \in C(X,Y)$. Let $\mathcal{A}$ be an open cover of Y and $j_0 \in J$ For each set $W \in \mathcal{A}$ and each point $y \in W$ we choose an open set $V_{y,W}$ with $y \in V_{y,W} \subset Cl\, V_{y,W} \subset W$ Then $\{V_{y,W} : W \in \mathcal{A}, y \in W\}$ is an open cover of Y, so

$$\{f^{-1}(V_{y,W}) \cap f_j^{-1}(V_{y,W}) : W \in \mathcal{A}, y \in W, j \in J, j \geq j_0\}$$

is an open cover of X. Since X is almost compact there exists a finite family $\{f^{-1}(V_{y_k,W_k}) \cap f_{j_k}^{-1}(V_{y_k,W_k}) : k = 1, 2, ..., n\}$ such that

$$X = \bigcup_{k=1}^{n} Cl\big(f^{-1}(V_{y_k,W_k}) \cap f_{j_k}^{-1}(V_{y_k,W_k})\big).$$

Let us put $J_1 = \{j_1, j_2, .., j_n\}$; then $j_k \geq j_0$ for each $j_k \in J_1$. Furthermore for each $x \in X$ there is $j_k \in J_1$ and V_{y_k,W_k} for which it holds

$$x \in Cl\big(f^{-1}(V_{y_k,W_k}) \cap f_{j_k}^{-1}(V_{y_k,W_k})\big) \subset f^{-1}(Cl\,V_{y_k,W_k}) \cap f_{j_k}^{-1}(Cl\,V_{y_k,W_k}),$$

thus $f(x) \in W_k$ and $f_{j_k}(x) \in W_k$ which finishes the proof.

Let us observe that in the above theorem the implication (a) $\Rightarrow$ (b) is not reversible; moreover (b) $\Rightarrow$ (c) is not true if in (c) the $C(X,Y)$ is replaced by $F(X,Y)$. It suffices to take into account $X = Y = [0,1]$ with the usual metric and functions $f_n, f : [0,1] \to [0,1], n \geq 1$, given by $f_n(x) = x^n$ for $n \geq 1, x \in [0,1]$ and $f(1) = 1, f(x) = 0$ if $x \in [0,1)$.

THEOREM 5. For a Tychonoff space X the following conditions are equivalent:

(a) for each topological space Y the strong convergence is equivalent to the pointwise one on $C(X,Y)$;

(b) the strong convergence coincides on $C(X[0,1])$ with the pointwise one;

(c) X is a compact space.

PROOF. The implication (a) $\Rightarrow$ (b) is evident. Let (b) be satisfied and let $\mathcal{V}$ be the uniformity on $[0,1]$ determined by the usual metric. Since $([0,1],\mathcal{V})$ has the Lebesgue property, the $\mathcal{V}$-quasi-uniform convergence coincides with the strong convergence. Thus, according to (b), the $\mathcal{V}$-quasi-uniform convergence is equivalent to the pointwise one on $C(X,[0,1])$. So, from [2, Th. 6] it means that X is compact. Finally we assume that X is compact. Let Y be a topological space and let $\{f_j : j \in J\}$ be a net in $C(X,Y)$ which pointwise converges to $f \in C(X,Y)$. Now, for an open cover $\mathcal{A}$ of Y and for fixed $j_0 \in J$ it suffices to consider the open cover $\{f^{-1}(W) \cap f_j^{-1}(W) : W \in \mathcal{A}, j \in J, j \geq j_0\}$ of X, and the proof is completed.

THEOREM 6. Let Y be a Hausdorff space. If for each topological space X the strong convergence coincides with the pointwise one on $C(X,Y)$, then Y is a regular space

PROOF. Assume that Y is not regular and let T denote the topology on Y. There exists an open set U_0 and a point $a \in U_0$ such that for each neighborhood V of a we have $Cl\,v \not\subset u_0$; hence $Cl\,u_0 \neq Y$ and we cn fix a point $b \in Y \backslash Cl\,u_0$. By $N(a)$ we denote the family of all neighborhoods of a which are contained in U_0. For any $V_1, V_2 \in N(a)$ we write $V_1 \leq V_2$ iff $V_2 \subset V_1$; thus $(N(a), \leq)$ is a directed set. If $V \in N(a)$, then $\emptyset \neq (Cl\,V)\backslash U_0$; so for each $V \in N(a)$ we choose a point $x_V \in (Cl\,V)\backslash U_0$. We denote

$$A_V = \{x_W : W \in N(a), W \geq V, W \neq V\}, \quad \text{for} \quad V \in N(A).$$

Furthermore let

$$\tau = \{A \subset Y : a \notin A \quad \text{or } A \text{ contains some } V \in N(a)\}.$$

Then τ is a topology on $Y, T \subset \tau$ and for each point $y \in Y, y \neq a$ we have $\{y\} \in \tau$. Now we define functions $f_V, f : (Y,\tau) \to (Y,T)$ by the following

$$f(x) = x \quad \text{for} \quad x \in Y;$$

$$f_V(x) = \begin{cases} b, & \text{if} \quad x \in A_V; \\ x, & \text{if} \quad x \in Y \backslash A_V. \end{cases}$$

Evidently f is continuous; we are going to show that all f_V are continuous. Let f_V be fixed If $x \in A_V$, then $x = x_W$ with $V \leq W \neq V$ and $f_V(x_W) = b$. For a T-neighborhood G of $f_V(X_W)$, $G \subset Y\backslash Cl\,U_0$ we can take the τ-neighborhood $\{x_W, b\}$ of x_W for which $f_V(\{x_W,b\}) \subset G$, so f_V is continuous on

A_V. Now, let $x \in Y \backslash A_V$, $x \neq a$. If G is a T-neighborhood of $f_V(x)$, then it suffices to take the τ-neighborhood $\{x\}$ of x; thus f_V is continuous on $Y \backslash (A_V \cup \{a\})$. Finally, let $x = a$ and let G be a T-neighborhood of $f_V(a)$. Then $G \cap U_0 \in N(a)$, so $G \cap U_0$ is a τ-neighborhood of a such that $f_V(G \cap U_0) \subset G$. Hence we have shown the continuity of all functions f_V. It is easy to see that the net $\{f_V : V \in N(a)\}$ pointwise converges to f For each point $x \in (Cl\, U_0) \backslash U_0$ we choose a T-neighborhood U_x of x such that $\{a, b\} \cap U_x = \emptyset$; then $\mathcal{A} = \{U_0, Y \backslash Cl\, U_0\} \cup \{U_x : x \in (Cl\, U_0) \backslash U_0\}$ is a T-open cover of Y. For a finite set $\{V_1, V_2, ..., V_n\} \subset N(a)$ with $V_i \neq V_j$ for $i \neq j$; $i, j \leq n$ we can take $V \in N(a)$ such that $V_j \leq V$ and $V_j \neq V$ for $j \leq n$. Then we have

$$f(x_V) = x_V \in U_{x_V}, \qquad f(x_V) \notin Y \backslash Cl\, U_0,$$

$$f_{V_j}(x_V) = b \in Y \backslash Cl\, U_0 \quad \text{and} \quad f_{V_j}(x_V) \notin U_{x_V} \quad \text{for} \quad j \leq n.$$

This means that the sets of the cover $\mathcal{A}$ do not contain none of sets $\{f_{V_j}(x_V), f(x_V)\}$ for $j \leq n$, hence the net $\{f_V : V \in N(a)\}$ is not strongly convergent to f which finishes the proof.

REFERENCES

[1] ARZELÀ, C., Sulle serie di funzioni, *Mem. R. Accad. Sci. Inst. Bologna* ser. 5 (8), (1899-1900), 130-186, 701-744.

[2] BUKOVSKÁ, Z., BUKOVSKÝ, L. and EWERT, J., Quasi-uniform convergence and L-spaces, *Real Anal. Exchange* 18 (1992-93), 321-329.

[3] EWERT, J., Almost uniform convergence, *Periodica Math. Hungarica* 26 (1) (1993), 77-84

[4] POPPE, H., Compactness in function spaces with a generalized uniform structure II, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 18 (1970), 567-573.

[5] PREDOI, M., Sur la convergence quasi-uniforme, *Periodica Math. Hungarica* 10 (1979), 31-40.

[6] KASAHARA, S. and KASAHARA, K., Note on the Lebesgue property in uniform spaces, *Proc. Japan Acad.* 31 (1955), 1615-1617.

[7] CSÁSZÁR, A., *General Topology*, Budapest, 1978.