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GRAPHS AND COMMON FIXED POINT THEOREMS FOR SEQUENCES OF MAPS

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ABSTRACT. We demonstrate a usefulness of the notion of a connected graph for obtaining some common fixed point theorems. In particular, we establish two theorems of this type involving one, two and four sequences of maps. This generalizes among others the recent results of S. Chang [1], J. Jachymski [2], S. Sessa, R. N. Mukherjee and T. Som [3], and T. Taniguchi [4].

KEY WORDS AND PHRASES. Common fixed point, contractive gauge function, compatible maps, connected graph.
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1. INTRODUCTION.

There are a number of papers dealing with fixed points for sequences of maps. B. E. Rhoades [5] divided these results into four categories (for the details, see [5], p. 10). In this paper we extend the fourth class of theorems discussed in [5]. In this situation maps A_i and A_j or A_i and B_j satisfy pairwise a contractive type condition involving eventually other maps, with contractive gauge functions, which may depend on i and j. To get a common fixed point theorem, most of the authors use an iteration procedure involving all of the maps considered (see, e.g., [3], [4], [6]). Then, however, the same contractive gauge function is employed, or some additional hypothesis on gauge functions are imposed (see, e.g., the comments in [5], p. 13-14).

Recently, G. Jungck et al. [7] and B. E. Rhoades [5] obtained common fixed point theorems for a sequence of maps using earlier results involving a *finite* number of maps. As has been pointed out in [5], such a way of treatment enables one to use different contractive gauge functions, without any additional hypotheses (see Corollary 1 [5]). A theorem of this type has been also established in the recent article [2].

In this paper we give a precise description of a set of positive integers (i,j) for which a contractive condition is to be satisfied in order to guarantee the existence of a common fixed point. Most of the authors assume that a contractive condition is to hold for all (i,j) with $i \neq j$

(see, e.g., [1], [3], [7], [8]). However, H. Chatterji has observed that it suffices to use a contractive conditions for pairs

$$(i,j) \in J := \{(1,n+1) : n \in \mathbf{N}\}$$

only ($\mathbf{N}$ denotes the set of all positive integers). The same set J is employed in Theorem 5.1 [2]. All of the theorems mentioned here deal with a single sequence of maps. On the other hand, T. Taniguchi [4] establishing a fixed point theorem for two sequences of maps, has assumed that a contractive condition holds for all pairs

$$(i,j) \in J_T := \{(2n-1,2n) : n \in \mathbf{N}\} \cup \{2n,2m+1) : m \geq n \geq 1\}. \tag{1.1}$$

In the next section we show that, thanks to the notion of a connected graph, it is possible to unify and extend all of the above results. Moreover, our conditions imposed on a set J involving a graph appear to be necessary and sufficient for the existence of a common fixed point (see Theorems 2.2 and 2.3).

2. A FIXED POINT THEOREM INVOLVING GRAPH.

We start by recalling a common fixed point theorem for four maps ([2], Theorem 3.3). It can be deduced from Lemma 1 [9], that though this theorem involves a contractive gauge function, it yields the recent result of Jungck et al. [7] involving (ϵ,δ)-type conditions. For the definition of compatible maps, a generalization of the commutative map concept, see [10]. The letter $\mathbf{R}_+$ denotes the set of all nonegative reals.

THEOREM 2.1. *Let A, B, S and T be selfmaps of a complete metric space (X,d), and let $\Phi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be an upper semicontinuous (not necessarily monotonic) function such that $\Phi(t) < t$ for $t > 0$. Let (A,B,S,T) satisfy the following conditions:*

$$\begin{aligned} &d(Ax,By) \leq \Phi(max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \\ &\tfrac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}), \ \textit{for all } x,y \in X; \end{aligned} \tag{2.1}$$

$$A(X) \subseteq T(X) \ \textit{and} \ B(X) \subseteq S(X); \tag{2.2}$$

$$\textit{the pairs } A,\ S \textit{ and } B,\ T \textit{ are compatible}; \tag{2.3}$$

$$\textit{one of } A,\ B,\ S \textit{ and } T \textit{ is continuous}. \tag{2.4}$$

Then A, B, S and T have a unique common fixed point.

Further, let us recall that an undirected graph is a pair $< V, E >$, where V is a set and E is a family of two-element subsets of V. A graph $< V, E >$ is said to be connected if for each $x, y \in V$, there exists a finite sequence $\{x_i\}_{i=0}^n$ such that $x_0 = x$, $x_n = y$ and $\{x_{i-1}, x_i\} \in E$, for $i = 1, ..., n$ (see, e.g., [11]). The letters Π_1 and Π_2 denote the projections of $\mathbf{N}^2$ onto $\mathbf{N}$, i.e.,

$$\Pi_1(n,m) := n, \text{ and } \Pi_2(n,m) := m, \text{ for } (n,m) \in \mathbf{N}^2.$$

THEOREM 2.2. *Let A, B, S, T and A_n, B_n, S_n $(n \in \mathbf{N})$ be selfmaps of a complete metric space (X,d). Let $J \subseteq N^2$ and for $(i,j) \in J$, $\Phi_{ij} : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be an upper semicontinuous function such that $\Phi_{ij}(t) < t$ for $t > 0$. Assume that one of the following conditions hold.*

$$\begin{aligned}&\text{For each } (i,j) \in J,\ (A,B,S_i,S_j) \text{ satisfies (2.1) with } \Phi := \Phi_{ij},\\ &A(X) \cup B(X) \subseteq \textstyle\bigcap_{n=1}^{\infty} S_n(X),\ (A,S_n) \text{ and } (B,S_n) \text{ are}\\ &\text{compatible } (n \in \mathbf{N}), \text{ and all the } S_n \text{ are continuous.}\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\text{For each } (i,j) \in J,\ (A_i,A_j,S,T) \text{ satisfies (2.1) with } \Phi := \Phi_{ij},\\ &A_n(X) \subseteq S(X) \cap T(X) \text{ for } n \in \mathbf{N},\ (A_n,S) \text{ and } (B,S_n) \text{ are}\\ &\text{compatible } (n \in \mathbf{N}), \text{ and } S \text{ or } T \text{ is continuous.}\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\text{For each } (i,j) \in J,\ (A_i,A_j,B_i,B_j) \text{ satisfies (2.1) with } \Phi := \Phi_{ij},\\ &\text{(2.2), (2.3), and all the } B_n \text{ are continuous.}\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\text{For each } (i,j) \in J,\ (A_i,B_j,B_i,A_j) \text{ satisfies (2.1) with } \Phi := \Phi_{ij},\\ &\text{(2.2), } (A_n,B_n) \text{ are compatible } (n \in \mathbf{N}), \text{ and for each } n \in \mathbf{N},\\ &A_n \text{ or } B_n \text{ is continuous.}\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\text{For each } (i,j) \in J,\ (A_i,B,S,A_j) \text{ satisfies (2.1) with } \Phi := \Phi_{ij},\\ &\text{(2.2), (2.3), and } B \text{ or } S \text{ is continuous.}\end{aligned} \tag{2.9}$$

Further, define the family E_J of two-element subsets of J by

$$\{(i,j),(k,l)\} \in E_J \quad iff \quad (i,j),(k,l) \in J \text{ and } \{i,j\} \cap \{k,l\} \neq \emptyset.$$

If the graph $< J, E_J >$ is connected and $\Pi_1(J) \cup \Pi_2(J) = \mathbf{N}$ then the following families of maps have a unique common fixed point.

(1) *All the S_n $(n \in \mathbf{N})$, A and B if (2.5) holds.*
(2) *All the A_n $(n \in \mathbf{N})$, S and T if (2.6) holds.*
(3) *All the A_n and B_n $(n \in \mathbf{N})$ if (2.7) or (2.8) holds.*
(4) *All the A_n $(n \in \mathbf{N})$, B and S if (2.9) holds.*

PROOF. Assume that one of conditions (2.5)-(2.9) holds. By Theorem 2.1, for each $(i,j) \in J$ there is a unique common fixed point z_{ij} for the suitable quaternion of maps. Let $(i,j),(k,l) \in J$ and $\{i,j\} \cap \{k,l\} \neq \emptyset$. Assume that $j = k$ or $j = l$. Then, by putting in (2.1), $\Phi := \Phi_{ij}$, $x := z_{ij}$ and $y := z_{kl}$, and replacing (A,B,S,T) by the suitable quaternion of maps, we obtain that $d(x,y) \leq \Phi_{ij}(d(x,y))$. Hence, $x = y$ since $\Phi_{ij}(t) < t$ for $t > 0$. Assume that $i = k$ or $i = l$. Similarly, by putting in (2.1), $x := z_{kl}$ and $y := z_{ij}$ we obtain that $d(x,y) \leq \Phi_{ij}(d(x,y))$. Therefore, $z_{ij} = z_{kl}$.

Now, let (i,j) and (k,l) be arbitrary elements of J. By the connectivity of $< J, E_J >$, there exists an $n \in \mathbf{N}$ and a sequence $\{(i_k,j_k)\}_{k=0}^{n}$ in J such that $(i_0,j_0) = (i,j)$, $(i_n,j_n) = (k,l)$, and $\{i_{k-1},j_{k-1}\} \cap \{i_k,j_k\} \neq \emptyset$, for $k = 1,...,n$. Then, by the preceding part of the proof, $z_{i_{k-1}j_{k-1}} = z_{i_kj_k}$ for $k = 1,...,n$, which immediately gives $z_{ij} = z_{kl}$. This means, there is a $z_0 \in X$ such that $z_{ij} = z_0$, for all $(i,j) \in J$.

Now, fix an $n \in \mathbf{N}$ and assume that (2.5) holds. By hypothesis, there is a $k \in \mathbf{N}$ such that $(k,n) \in J$ or $(n,k) \in J$. For example, assume $(k,n) \in J$. Then z_{kn} is the common fixed point of A, B, S_k and S_n; in particular, $z_0 (= z_{kn})$ is the common fixed point of A, B and S_n. The same argument may be used in the cases, in which any one of conditions (2.6)-(2.9) holds instead of (2.5). □

REMARK 2.1. It is easy to verify that, for each of the sets J_k $(k = 1,2,3,4)$ defined below, the graph $< J_k, E_{J_k} >$ is connected and $\Pi_1(J_k) \cup \Pi_2(J_k) = \mathbf{N}$.

(1) $J_1 := \{(i,j) : i,j \in \mathbf{N} \text{ and } i \neq j\}$.

(2) $J_2 := \{(n, n+1) : n \in \mathbf{N}\}$.

(3) $J_3 := \{(1, n+1) : n \in \mathbf{N}\}$.

(4) $J_4 := \{(1, 2n) : n \in \mathbf{N}\} \cup \{(2n, 2n+1) : n \in \mathbf{N}\}$.

REMARK 2.2. By putting $J := J_1$ and assigning the Φ_{ij} to be the same function, and assuming that (2.5) or (2.6) holds, Theorem 2.2 yields Theorem 2 of Chang [1] and Theorem 3 of Sessa et al. [3], respectively. Clearly, we may also put here $J := J_k$ for any $k \in \{2,3,4\}$ in order to obtain the essential extensions of the above theorems.

REMARK 2.3. By putting in Theorem 2.2, $J := J_2$ and

$$\Phi_{ij} := kt \text{ for some } k \in (0,1),\ t \in \mathbf{R}_+ \text{ and } (i,j) \in J,$$

and assuming that (2.7) holds, we obtain the result generalizing Theorem A of Taniguchi [4] who has employed the condition

$$d(A_i x, A_j y) \leq k d(B_i x, B_j y), \text{ for } x, y \in X \text{ and } (i,j) \in J_T$$

(J_T is defined by (1.1)), which is less general than (2.1).

REMARK 2.4. By putting in Theorem 2.2, $J := J_3$ and assuming that (2.6) holds, we obtain Theorem 5.1 of Jachymski [2], the generalization of an earlier result due to Chatterji [6].

REMARK 2.5. Clearly, conditions (2.5)-(2.9) may be weakened. It suffices to have the suitable quaternions of maps satisfy the assumptions of Theorem 2.1. We have slightly strengthened them for aesthetic reasons.

3. A CONVERSE TO THEOREM 2.2.

The following theorem is a converse to Theorem 2.2. It appears that the connectivity of the graph $< J, E_J >$ and the condition $\Pi_1(J) \cup \Pi_2(J) = \mathbf{N}$ are necessary for the existence of a common fixed point. More precisely, we have.

THEOREM 3.1. *Let $J \subseteq \mathbf{N}^2$ and E_J be defined as in Theorem 2.2. If the graph $< J, E_J >$ is not connected or $\Pi_1(J) \cup \Pi_2(J) \neq \mathbf{N}$, then there exist a complete metric space (X,d) and selfmaps S, T, A_n $(n \in \mathbf{N})$ of X for which (2.6) is satisfied with Φ_{ij} being the same linear function, and there is no common fixed point for the family of maps.*

PROOF. Assume that $\Pi_1(J) \cup \Pi_2(J) \neq \mathbf{N}$. Then there exists an $n_0 \in \mathbf{N}$ such that, for all $k \in \mathbf{N}$, $(k, n_0) \notin J$ and $(n_0, k) \notin J$. Let $X := \mathbf{R}_+$ be endowed with the Euclidean metric, $S := I$, the identity on X, $T := S$,

$$A_n x := \frac{1}{2} x \text{ for } n \neq n_0 \text{ and } x \in X,\ \ A_{n_0} x := x + 1 \ \ (x \in X),$$

and $\Phi_{ij}(t) := \frac{1}{2}t$ for $(i,j) \in J$ and $t \in \mathbf{R}_+$. Then, for all $i,j \neq n_0$, (A_i, A_j, S, T) satisfies (2.1). In particular, by the definition of n_0, (2.1) holds for each (A_i, A_j, S, T) with $(i,j) \in J$. So it is clear that (2.6) is satisfied. But since A_{n_0} is a fixed point free map, there is no common fixed point for the family considered.

Now assume that $\Pi_1(J) \cup \Pi_2(J) = \mathbf{N}$ and that the graph $< J, E_J >$ is not connected. Then $< J, E_J >$ is the sum of its connected components, i.e., $J = \bigcup_{n=1}^{p} J_n$, where $p \in \mathbf{N} \cup \{\infty\}$, $p \geq 2$, $J_n \cap J_m = \emptyset$ if $n \neq m$, and for each of the n considered, $< J_n, E_{J_n} >$ is connected. Hence and by the definition of E_J, if $n \neq m$ and $(i,j) \in J_n$, $(k,l) \in J_m$ then $\{i,j\} \cap \{k,l\} = \emptyset$. Therefore, we infer that

$$(\Pi_1(J_n) \cup \Pi_2(J_n)) \cap (\Pi_1(J_m) \cup \Pi_2(J_m)) = \emptyset, \quad \text{for } n \neq m. \tag{3.1}$$

Let (X,d), S, T and Φ_{ij} $((i,j) \in J)$ be defined as above. Let

$$A_n x := \frac{1}{2}x \;\; for \;\; x \in X \;\text{ and }\; n \in \Pi_1(J_1) \cup \Pi_2(J_1),$$

and $A_n x := \frac{1}{2}x + \frac{1}{2}$ for $x \in X$ and the rest of n. If $(i,j) \in J_1$, then $A_i x = A_j x = \frac{1}{2}x$ $(x \in X)$. If $(i,j) \in J - J_1$ then, for some $n \geq 2$, $(i,j) \in J_n$ so $i,j \in \Pi_1(J_n) \cup \Pi_2(J_n)$. Hence and by (3.1), $i,j \notin \Pi_1(J_1) \cup \Pi_2(J_1)$. Then, by the definition of $\{A_n\}_{n=1}^{\infty}$, we get that $A_i x = A_j x = \frac{1}{2}x + \frac{1}{2}$ $(x \in X)$. Therefore, we may infer that (2.1) holds for each (A_i, A_j, S, T) with $(i,j) \in J$. Consequently, (2.6) is satisfied. Simultaneously, the above family of maps has no common fixed point. □

REMARK 3.1. It is easy to verify that for each of the sets J_2, J_3 and J_4 defined in Remark 2.1, the following condition holds:

"if for some $k \in \{2,3,4\}$, $J^* \subseteq J_k$ and $J^* \neq J_k$, then either $\Pi_1(J^*) \cup \Pi_2(J^*) \neq \mathbf{N}$, or the graph $< J^*, E_{J^*} >$ is not connected".

In view of Theorem 3.1, we may infer that J_2, J_3 and J_4 are minimal subsets of $\mathbf{N}^2$, which may be used in Theorem 2.2. Obviously, there are a number of another examples of such minimal sets.

4. COMMON FIXED POINTS FOR FOUR SEQUENCES OF MAPS.

Finally, using again the concept of a connected graph, we establish a common fixed point theorem for four sequences of maps.

THEOREM 4.1. *Let $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ be sequences of selfmaps of a complete metric space (X,d). Let $J \subseteq \mathbf{N}^2$ and for $(i,j) \in J$, $\Phi_{ij} : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be an upper semicontinuous function such that $\Phi_{ij}(t) < t$ for $t > 0$. Assume that, for each $(i,j) \in J$, (A_i, B_j, S_i, T_j) satisfies conditions (2.1)-(2.4) with $\Phi := \Phi_{ij}$. Further, define the family F_J of two-element subsets of J by*

$$\{(i,j),(k,l)\} \in F_J \quad iff \quad (i,j),(k,l) \in J \; and \; i = k \; or \; j = l.$$

If the graph (J, F_J) is connected and $\Pi_1(J) = \Pi_2(J) = \mathbf{N}$, then all the A_n, B_n, S_n and T_n $(n \in \mathbf{N})$ have a unique common fixed point.

PROOF. By Theorem 2.1, for each $(i,j) \in J$ there is a common fixed point z_{ij} of maps A_i, B_j, S_i and T_j. Let $(i,j),(k,l) \in J$ and $\{(i,j),(k,l)\} \in F_J$. Let $A := A_i$, $B := B_j$, $S := S_i$, $T := T_j$ and $\Phi := \Phi_{ij}$. By the definition of F_J, we have that $i = k$ or $j = l$. In the first case, put in (2.1) for the above defined maps, $x := z_{kl}$ and $y := z_{ij}$; in the second, put there $x := z_{ij}$ and $y := z_{kl}$. In both cases we obtain that $d(x,y) \leq \Phi_{ij}(d(x,y))$. Hence, $x = y$, i.e. $z_{ij} = z_{kl}$ since $\Phi_{ij}(t) < t$ for $t > 0$.

Now, if (i,j) and (k,l) are arbitrary elements of J, we may obtain, employing the connectivity of $< J, F_J >$ and using the same argument as in the proof of Theorem 2.2, that $z_{ij} = z_{kl}$. Thus, there is a $z_0 \in X$ such that $z_{ij} = z_0$, for all $(i,j) \in J$. Now, let $n \in \mathbf{N}$. By hypothesis, there exist $i, j \in \mathbf{N}$ such that $(i,n),(n,j) \in J$. Then z_{in} is a fixed point of B_n and T_n, and z_{nj} is a fixed point of A_n and S_n. Since $z_{in} = z_{nj} = z_0$, z_0 is the common fixed point of A_n, B_n, S_n and T_n. $\square$

REMARK 4.1. It is easy to verify that for each of the sets J_k $(k = 1,2,3)$ defined below, the graph $< J_k, F_{J_k} >$ is connected and $\Pi_1(J_k) = \Pi_2(J_k) = \mathbf{N}$.

(1) $J_1 := \{(i,j) : i,j \in \mathbf{N}$ and $i \neq j\}$.

(2) $J_2 := \{(n,n+1) : n \in \mathbf{N}\} \cup \{(n,n+2) : n \in \mathbf{N}\} \cup \{(n_0+1,1)\}$, where $n_0 \in \mathbf{N}$ is fixed.

(3) $J_3 := \{(1,n+1) : n \in \mathbf{N}\} \cup \{(n,n+1) : n \in \mathbf{N}\} \cup \{(n_0+1,1)\}$, where $n_0 \in \mathbf{N}$ is fixed.

Moreover, J_2 and J_3 are minimal subsets of $\mathbf{N}^2$, which may be used in Theorem 4.1.

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