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ON THE GROWTH OF THE SPECTRAL MEASURE

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ABSTRACT. We are concerned with the asymptotics of the spectral measure associated with a self-adjoint operator. By using comparison techniques we shall show that the eigenfunctionals of L_2 are close to the eigenfunctionals L_1 if and only if $d\Gamma_1 \asymp d\Gamma_2$ as $\lambda \to \infty$.

KEY WORDS AND PHRASES: Spectral asymptotics, spectral function, Sturm-Liouville operators.
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1 INTRODUCTION

We would like to obtain a relation between the growth of the spectral measure of a self-adjoint operator and the behaviour of its eigenfunctionals. In this study we shall assume that we have two "close" self-adjoint operators acting in the same separable Hilbert space, H say. Without loss of generality we can assume that both operators have simple spectra. To this end, let us denote by $\varphi(\lambda)$ and $y(\lambda)$ the eigenfunctionals of L_1 and L_2 respectively. Recall that the spectrum of a self-adjoint operator is defined by

$$\forall \lambda \in \sigma_i \;\; \exists \; \varphi_{i,n} \in D_{\{L_i\}} \;\; / \;\; \|\varphi_{i,n}\| = 1 \;\; \text{and} \;\; \|L_i\varphi_{i,n} - \lambda\varphi_{i,n}\| \overset{n\to\infty}{\longrightarrow} 0$$

where $i = 1, 2$. In case λ is in the continuous spectrum the sequence is not compact in the Hilbert space H. For this we can assume the existence of a countably normed perfect space Φ, such that

$$\Phi \hookrightarrow H \hookrightarrow \Phi'$$

where the embeddings are compact, for further details see [1] and [2]. For the sake of simplicity we shall assume that the embeddings are given by the identities and so

$$f \in \Phi \quad \psi \in H \quad (f, \psi) \equiv \langle f, \psi \rangle_{\Phi \times \Phi'} .$$

Since the sequence φ_n is bounded in H, it is then compact in Φ', which implies

$$\varphi_n \overset{\Phi'}{\to} \varphi(\lambda) \in \Phi'$$

and similarly for the operator L_2; Since both operators are acting in the same Hilbert space H, we shall assume that the space Φ' contains both systems of eigenfunctionals; i.e.,

$$\{y(\lambda)\} \subset \Phi' \quad \text{and} \quad \{\varphi(\lambda)\} \subset \Phi'.$$

Recall that the system $\{y(\lambda)\}$ helps define an isometry for L_2

$$\forall\, f \in \Phi \quad f \longrightarrow \hat{f}^2(\lambda) \equiv \langle f, y(\lambda) \rangle_{\Phi \times \Phi'}$$

$$f = \int \overline{\hat{f}^2(\lambda)} y(\lambda) d\Gamma_2(\lambda) \quad \text{where} \quad \hat{f}^2(\lambda) \in L^2_{d\Gamma_2(\lambda)}$$

Similarly for $\varphi(\lambda)$;

$$\forall f \in \Phi \quad f \longrightarrow \hat{f}^1(\lambda) \equiv < f, \varphi(\lambda) >_{\Phi \times \Phi'},$$
$$f = \int \overline{\hat{f}^1(\lambda)} \varphi(\lambda) d\Gamma_1(\lambda) \ \text{ where } \ \hat{f}^1(\lambda) \in L^2_{d\Gamma_1(\lambda)}$$

These transforms define isometries, and Parseval equality yields

$$\int_{\sigma_1} \hat{f}^1(\lambda) \overline{\hat{\psi}^1(\lambda)} d\Gamma_1(\lambda) = (f, \psi)_H = \int_{\sigma_2} \hat{f}^2(\lambda) \overline{\hat{\psi}^2(\lambda)} d\Gamma_2(\lambda).$$

where the nondecreasing functions $\Gamma_1(\lambda)$ and $\Gamma_2(\lambda)$ are called the spectral measures associated with L_1 and L_2, respectively. It is these functions that we would like to estimate as $\lambda \longrightarrow \infty$.

In all that follows $y(\lambda) \sim \varphi(\lambda)$ as $\lambda \rightarrow \infty$ means $\forall f \in \Phi$,

$$\hat{f}^1(\lambda) \asymp \hat{f}^2(\lambda) \quad \text{as} \quad \lambda \rightarrow \infty$$

and $d\Gamma_1(\lambda) \sim d\Gamma_2(\lambda)$ as $\lambda \rightarrow \infty$ means that $\forall F \in L^1_{d\Gamma_1(\lambda)} \cap L^1_{d\Gamma_2(\lambda)}$

$$\int_\lambda^\infty F(\eta) d\Gamma_1(\eta) \asymp \int_\lambda^\infty F(\eta) d\Gamma_2(\eta) \quad \text{as} \quad \lambda \rightarrow \infty.$$

In this work, we shall try to answer the following problem:

Statement of the Problem: under what conditions

$$y(\lambda) \sim \varphi(\lambda) \ \text{ as } \lambda \rightarrow \infty \ \Leftrightarrow \ d\Gamma_1(\lambda) \sim d\Gamma_2(\lambda) \ \text{ as } \ \lambda \rightarrow \infty.$$

In order to answer the above question, we shall compare the self-adjoint operators L_1 and L_2, see [3]. Recall that a shift operator or transmutation is defined by

$$y(\lambda) = V\varphi(\lambda) \qquad \lambda \in \sigma_1;$$

Clearly the definition of V depends on σ_2 and σ_1 and we shall agree to set

$$y(\lambda) = 0 \ \text{ if } \ \lambda \notin \sigma_2, \ \text{and} \ \varphi(\lambda) = 0 \ \text{ if } \ \lambda \notin \sigma_1$$
$$y(\lambda) = V\varphi(\lambda) \quad \lambda \in \sigma_2 \subset \sigma_1 \subset R.$$

Condition $\sigma_2 \subset \sigma_1$ insures that $V0 = 0$ and so defines an operator on the algebraic span of $\{\varphi(\lambda)\}$. Thus it is clear that in order for V and V^{-1} to exist as linear operator it is necessary that $\sigma_2 \subset \sigma_1$ and $\sigma_1 \subset \sigma_2$

$$\sigma_2 \equiv \sigma_1.$$

It is readily seen that $\{\varphi(\lambda)\}$ form a complete set in the reflexive space (perfect) Φ', and so the space generated by $\{\varphi(\lambda)\}$ is dense in Φ'. Consequently V is densely defined. This in turns allows us to define the adjoint operator $V' : \Phi \longrightarrow \Phi$.

2 MAIN RESULTS

We shall agree to say $\Gamma_1(\lambda)$ is Abs-$d\Gamma_2$ if there exists $g(\eta) \in L^{1,loc}_{d\Gamma_2}$ such that

$$\Gamma_1(\lambda) = \int_0^\lambda g(\eta) d\Gamma_2(\eta) + \Gamma_1(0)$$

This fact shall be denoted by

$$g(\lambda) \equiv \frac{d\Gamma_1}{d\Gamma_2}(\lambda) \in L^{1,loc}_{d\Gamma_2}$$

In this case the condition $d\Gamma_1(\lambda) \sim d\Gamma_2(\lambda)$ in the statement of the problem can be restated as $g(\lambda) \asymp 1$ as $\lambda \longrightarrow \infty$. Recall that due to reflexivity of the space Φ, the operator V' is defined in Φ and since $\Phi \hookrightarrow H$, V' is actually defined in H. Let us denote this extension to the space H by $\tilde{V}$. Since we are interested in the case where $y(\lambda) \sim \varphi(\lambda)$ we can expect V to be bounded. In this regard we have the following result:

Theorem 1: If the extension $\tilde{V} : H \to H$, is a bounded operator then $\Gamma_1(\lambda)$ is $d\Gamma_2$-ABS continuous.
Proof: It is clear that for $f \in D_{V'}$

$$\begin{aligned} < f, y(\lambda) >_{\Phi\times\Phi'} &= < f, V\varphi(\lambda) >_{\Phi\times\Phi'} \\ &= < V'f, \varphi(\lambda) >_{\Phi\times\Phi'} \end{aligned}$$

In other words

$$\hat{f}^2(\lambda) = \widehat{V'f}^1(\lambda). \tag{2.1}$$

Equation 2.1 obviously holds for $f \in H$. Indeed let $f_n \in D_{V'} \subset H$ such that $f_n \xrightarrow{H} f \in H$. Given that $\tilde{V}$ is a bounded operator in H, we obviously have $\tilde{V}f_n \to \tilde{V}f$. Using the fact that $\forall n, \quad \hat{f}_n^2(\lambda) = \widehat{V'f_n}^1(\lambda)$ and the isometries are bounded operators we have $\hat{f}_n^2 \to \hat{f}^2$ and $\widehat{\tilde{V}f_n}^1 \to \widehat{\tilde{V}f}^1$. Therefore

$$\hat{f}^2(\lambda) = \widehat{\tilde{V}f}^1(\lambda) \quad f \in H. \tag{2.2}$$

From which we deduce that $\forall f \in H$

$$\begin{aligned} \int \hat{f}^2(\lambda)\overline{\hat{f}^2(\lambda)}d\Gamma_1(\lambda) &= \int \widehat{\tilde{V}'f}^1\overline{\widehat{\tilde{V}'f}^1}d\Gamma_1(\lambda) \\ &= (\tilde{V}'f, \tilde{V}'f) \\ &= \|\tilde{V}'f\|^2 \\ &\leq c\|f\|^2 \\ &\leq c\int |\hat{f}^2(\lambda)|^2 d\Gamma_2(\lambda) \qquad \forall f \in H. \end{aligned}$$

Thus each $d\Gamma_2$ negligible set is a $d\Gamma_1$ negligible set. Henceforth $\Gamma_1(\lambda)$ to be $d\Gamma_2(\lambda)$-Abs continuous. The above inequality is exactly a sufficient condition for the Radon-Nikodym theorem to hold, see [4].

In all that follows we shall assume that $d\Gamma_1(\lambda)$ is $d\Gamma_2 - Abs$ continuous which is denoted by

$$g(\lambda) \equiv \frac{d\Gamma_1}{d\Gamma_2}(\lambda).$$

We now need to define a function of an operator, namely $g(L_2)$ for the next result:

$$\begin{aligned} \Phi &\xrightarrow{g(L_2)} H \\ f &\longrightarrow g(L_2)f \equiv \int g(\lambda)\overline{\hat{f}^2(\lambda)}y(\lambda)d\Gamma_2(\lambda). \end{aligned}$$

Theorem 2: Assume that V admits closure in Φ' and Γ_1 is Abs-$d\Gamma_2(\lambda)$ then

$$\forall\psi \in D_{V'} \subset \Phi \qquad \left(\sqrt{\frac{d\Gamma_2}{d\Gamma_1}}(L_2)\right)' \left(\sqrt{\frac{d\Gamma_2}{d\Gamma_1}}(L_2)\right)\psi = \overline{V}V'\psi \quad \text{in } \Phi'.$$

Proof: From equation 2.1 and the fact that the embeddings are defined by identities, we deduce that $\forall f, \psi \in D_{V'} \subset \Phi$

$$\begin{aligned} \int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}d\Gamma_1(\lambda) &= \int \widehat{V'f}^1\overline{\widehat{V'\psi}^1}d\Gamma_1(\lambda) \qquad (2.3) \\ &= (V'f, V'\psi) \qquad (2.4) \\ &= < V'f, V'\psi >_{\Phi\times\Phi'} \\ &= < f, \overline{V}V'\psi >_{\Phi\times\Phi'}. \end{aligned}$$

However the left handside of equation 2.3 can rewritten as

$$\begin{aligned}\int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}d\Gamma_1(\lambda) &= \int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}g(\lambda)d\Gamma_2(\lambda)\\ &= \int \sqrt{g(\lambda)}\hat{f}^2(\lambda)\overline{\sqrt{g(\lambda)}\hat{\psi}^2}d\Gamma_2(\lambda)\\ &= \int \widehat{\sqrt{g(L_2)}f}^{\,2}\,\overline{\widehat{\sqrt{g(L_2)}\psi}^{\,2}}\,d\Gamma_2(\lambda)\\ &= (\sqrt{g(L_2)}f, \sqrt{g(L_2)}\psi) \qquad (2.5)\\ &= < f, \sqrt{g(L_2)}'\sqrt{g(L_2)}\psi >_{\Phi\times\Phi'} .\end{aligned}$$

Observe that if we set $f = \psi$ in equations 2.4 and 2.5 then we would obtain

$$||\sqrt{g(L_2)}f|| = ||V'f|| \tag{2.6}$$

from which we deduce that $D_{V'} \subset D_{\sqrt{g(L_2)}} \subset \Phi$, from we obtain

$$\forall\psi \in D_{V'} \quad \sqrt{g(L_2)}'\sqrt{g(L_2)}\psi = \overline{V}V'\psi. \tag{2.7}$$

Remark: Observe that both operators $\sqrt{g(L_2)}'\sqrt{g(L_2)}$ and $\overline{V}V'$ are mappings from $\Phi \longrightarrow \Phi'$.

It is easy to see that if we restrict equation 2.7 to

$$f \in D_{g(L_2)} \equiv \{f \in \Phi \;/\; g(\lambda)\hat{f}^2(\lambda) \in L^2_{d\Gamma_2}\}$$

then it reduces to

$$\forall f \in D_{V'} \cap D_{g(L_2)} \quad \frac{d\Gamma_1}{d\Gamma_2}(L_2) = g(L_2) = \overline{V}V' \;\text{ in }\; \Phi' \tag{2.8}$$

The next result describes the domain of $\tilde{V}'$.

Theorem 3: $\tilde{V}$ is densely defined if and only if $L^2_{d\Gamma_1(\lambda)} \cap L^2_{d\Gamma_2(\lambda)}$ is dense in $L^2_{d\Gamma_2(\lambda)}$.

Proof: From equation 2.2 it is readily seen that

$$f \in D_{\tilde{V}} \Leftrightarrow \hat{f}^2(\lambda) \in L^2_{d\Gamma_1(\lambda)} \cap L^2_{d\Gamma_2(\lambda)}$$

Then use the fact that $f \longrightarrow \hat{f}^2$ is an isometry between H and $L^2_{d\Gamma_2(\lambda)}$.

This work is based on the following result.

Theorem 4: Assume that

- V admits closure in Φ'
- Γ_1 is Abs- $d\Gamma_2(\lambda)$
- $\overline{V}^{-1}$ exists
- $\overline{V} : \Phi \longrightarrow \Phi$ is a bounded operator

then

$$g(\lambda)\varphi(\lambda) - y(\lambda) = (V' - 1)y(\lambda) \quad \text{in } \Phi'.$$

Proof: Notice that conditions of Theorem 2 hold and so it follows that

$$\sqrt{g(L_2)}'\sqrt{g(L_2)} = \overline{V}V' \;\text{ in }\; \Phi' \;. \tag{2.9}$$

By the above condition we have that $\sqrt{g(L_2)}'\sqrt{g(L_2)}f \in \Phi$ if $f \in D_{V'} \subset \Phi$. However since it is assumed that $\overline{V}^{-1}$ exists, then equation 2.8 yields

$$\overline{V}^{-1}\left(\sqrt{g(L_2)}\right)'\sqrt{g(L_2)} = V' \;\text{ in }\; \Phi' \tag{2.10}$$

In order to proceed further we need to extend the operator V' to Φ'. For this observe that since $\overline{V} : \Phi \longrightarrow \Phi$ is a bounded operator, $\overline{V'} = V'$ is a bounded operator in Φ'. Hence V' is defined for all elements in Φ', and in particular for $y(\lambda)$, thus

$$V^{-1}\sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda) = V'y(\lambda).$$

We now need to compute $\sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda)$. Let $f \in D_{V'} \subset \Phi$ then

$$\begin{aligned} < f, \sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda) >_{\Phi\times\Phi'} &= < \sqrt{g(L_2)}f, \sqrt{g(L_2)}y(\lambda) >_{\Phi\times\Phi'} \\ &= < \sqrt{g(L_2)}'\sqrt{g(L_2)}f, y(\lambda) >_{\Phi\times\Phi'} \\ &= \overline{\sqrt{g(\lambda)}}\sqrt{g(\lambda)}\hat{f}^2(\lambda) \\ &= g(\lambda)\hat{f}^2(\lambda) \\ &= < f, g(\lambda)y(\lambda) > \end{aligned}$$

where we have used the fact that $\sqrt{g(L_2)}'\sqrt{g(L_2)}f = \overline{V}V'f \in \Phi$. Hence

$$\sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda) = g(\lambda)y(\lambda) \quad \text{in} \quad \Phi' \; d\Gamma_2 a.e.$$

where $g(\lambda) \equiv \frac{d\Gamma_1}{d\Gamma_2}(\lambda)$ is a real function. Hence we have

$$g(\lambda)\overline{V}^{-1}y(\lambda) = V'y(\lambda).$$

Since by definition we have $\overline{V}^{-1}y(\lambda) = \varphi(\lambda)$ we obtain

$$g(\lambda)\varphi(\lambda) - y(\lambda) = (V'-1)y(\lambda) \quad \text{in} \quad \Phi'.$$

We easily deduce the following result:

Corollary 1: Let conditions of Theorem 4 hold then

$$g(\lambda)\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0 \Leftrightarrow (V'-1)y(\lambda) \xrightarrow{\Phi'} 0$$

Corollary 2: Let conditions of Theorem 4 hold and $(V'-1)y(\lambda) \xrightarrow{\Phi'} 0 \quad \lambda \to \infty$ then

$$g(\lambda) \sim 1 \quad \text{as} \quad \lambda \to \infty \Leftrightarrow \varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

Proof: By hypothesis and Corollary 1 we have $\forall f \in \Phi$

$$g(\lambda)\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thus if $g(\lambda) \to 1$ then $\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0$ which means that $\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0$ as $\lambda \to \infty$. Conversely $\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0$ together with $y(\lambda) - g(\lambda)\varphi(\lambda) \xrightarrow{\Phi'} 0$ implies that

$$g(\lambda)\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0$$

i.e. $g(\lambda) \to 1$ as $\lambda \to \infty$.

Corollary 2 suggests to write $V = 1 + K$. In this case Theorem 2 would read

$$g(\lambda)\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0 \Leftrightarrow K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

The question we would like to answer now is under what condition would

$$K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

First we need to observe that the above convergence holds in Φ'. Indeed by construction the function $y(\lambda)$ is in Φ' and so the operator K' originally was defined in Φ must be extended to Φ'. This is easily achieved if the operator K, i.e. $\overline{V}$, is bounded in $\Phi \longrightarrow \Phi$.

Theorem 5: Let

- $V : \Phi \longrightarrow \Phi$ be a bounded operator.
- $K \equiv \overline{V} - 1$, be such that $\Phi \xrightarrow{L_2 K} H$ is densely defined in Φ

then

$$K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

Proof: Recall that for each λ, there exists a bounded sequence $\varphi_{n,\lambda} \in D_{L_2}$ such that

$$\varphi_{n,\lambda} \in D_{L_2}, \quad \| \varphi_{n,\lambda} \| = 1, \text{ and } \quad \| L_2\varphi_{n,\lambda} - \lambda\varphi_{n,\lambda} \| \longrightarrow 0$$

The last condition can be written as

$$\lambda\varphi_{n,\lambda} = L_2\varphi_{n,\lambda} + \epsilon(n,\lambda)$$

where $\epsilon(n,\lambda) \to 0$ in H as $n \to \infty$. This allows us to obtain the following limit

$$\begin{aligned} < f, K'y(\lambda) >_{\Phi\times\Phi'} &= < Kf, y(\lambda) >_{\Phi\times\Phi'} \\ &= \lim_{n\to\infty} < Kf, \varphi_{n,\lambda} >_{\Phi\times\Phi'} \\ &= \frac{1}{\lambda}\lim_{n\to\infty} (\lambda\varphi_{n,\lambda}, Kf) \\ &= \frac{1}{\lambda}\lim_{n\to\infty} (L_2\varphi_{n,\lambda} + \epsilon(n,\lambda), Kf) \\ &= \frac{1}{\lambda}\lim_{n\to\infty} (L_2\varphi_{n,\lambda}, Kf) + \frac{1}{\lambda}\lim_{n\to\infty} (\epsilon(n,\lambda), Kf) \\ &= \frac{1}{\lambda}\lim_{n\to\infty} (\varphi_{n,\lambda}, L_2Kf) + \frac{1}{\lambda}\lim_{n\to\infty} (\epsilon(n,\lambda), Kf) \\ &\leq \frac{1}{\lambda} \| \varphi_{n,\lambda} \| \| L_2Kf \| + \frac{1}{\lambda}\lim_{n\to\infty} \|(\epsilon(n,\lambda)\| \|Kf\| \end{aligned}$$

So as $\lambda \to \infty$ we shall obtain $< f, K'y(\lambda) >_{\Phi\times\Phi'} \longrightarrow 0$. This last limit means that

$$K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

Recall that in order for the conclusion to hold we need L_2K to be at least densely defined in Φ.

Remark: The condition $V : \Phi \longrightarrow \Phi$ bounded can be replaced by densely defined. This forces us to use Baire's Theorem to obtain the density of $\Phi \cap D_V \cap D_{L_2K}$ in Φ.

Theorem 6: Let the conditions of Theorem 2 hold, and

- $V : \Phi \longrightarrow \Phi$ be a bounded operator
- $(g(L_2) - 1)^{-1}K$ be a bounded operator in Φ

then

$$(g(\lambda) - 1)y(\lambda) \xrightarrow{\Phi'} 0 \quad \Rightarrow \quad K'y(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Proof:

$$\begin{aligned} < f, K'y(\lambda) >_{\Phi\times\Phi'} &= < Kf, y(\lambda) >_{\Phi\times\Phi'} \\ &= \widehat{Kf}^2(\lambda) \\ &= (g(\lambda) - 1)(g(\lambda) - 1)^{-1}\widehat{Kf}^2(\lambda) \\ &= (g(\lambda) - 1)\{\widehat{(g(L_2) - 1)^{-1}Kf}^2\} \\ &= (g(\lambda) - 1) < (g(L_2) - 1)^{-1}Kf), y(\lambda) >_{\Phi\times\Phi'} \\ &= < (g(L_2) - 1)^{-1}Kf), (g(\lambda) - 1)y(\lambda) >_{\Phi\times\Phi'} \end{aligned}$$

Since the $[g(\lambda) - 1]y(\lambda) \xrightarrow{\Phi'} 0$ we obtain $< f, K'y(\lambda) >_{\Phi\times\Phi'} \to 0 \quad \forall f \in \Phi$ and so $K'y(\lambda) \to 0$ as $\lambda \to \infty$.

Corollary 3: Assume that conditions of Theorem 4, hold and

- $y(\lambda)$ are bounded functionals for large λ
- $(g(L_2) - 1)^{-1}K$ be a bounded operator in Φ

then

$$g(\lambda) - 1 \stackrel{\lambda\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad y(\lambda) - \varphi(\lambda) \stackrel{\Phi'}{\longrightarrow} 0 \quad \text{as} \quad \lambda \to \infty$$

Proof: It suffices to see that $(g(\lambda) - 1)y(\lambda) \stackrel{\Phi'}{\longrightarrow} 0$, and since Theorem 6, is applicable

$$K'y(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

From Theorem 4, we deduce that

$$g(\lambda)\varphi(\lambda) - y(\lambda) \stackrel{\Phi'}{\longrightarrow} 0.$$

It remains to see that since $g(\lambda) \asymp 1$ as $\lambda \to \infty \Rightarrow \varphi(\lambda) \stackrel{\Phi'}{\longrightarrow} y(\lambda)$ as $\lambda \to \infty$.

3 EXAMPLES

Below we shall consider two simple examples to illustrate the above results.

Let L_1 and L_2 be two self-adjoint differential operators in $L^2[0,\infty)$ defined by

$$\begin{cases} L_1 f \equiv -f''(x) + q(x)f(x) \\ nf(0) - f'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} L_2 f \equiv -f''(x) \\ nf(0) - f'(0) = 0. \end{cases}$$

where $|n| < \infty$. Let the eigenfunctionals associated with L_1 and L_2 be defined by

$$\begin{cases} L_1\varphi(x,\lambda) \equiv \lambda\varphi(x,\lambda) \\ \varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = n \end{cases} \quad \text{and} \quad \begin{cases} L_2 y(x,\lambda) = \lambda y(x,\lambda) \\ y(0,\lambda) = 1, \;\; y'(0,\lambda) = n \end{cases}$$

where $y(x,\lambda) = \cos(\sqrt{\lambda}x) + n\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$. It is clear that

$$\varphi(x,\lambda) = y(x,\lambda) + \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} q(t)\varphi(t,\lambda)dt.$$

By the Riemman-Lebesgue theorem we have

$$\varphi(x,\lambda) - y(x,\lambda) \longrightarrow 0 \;\; as \;\; \lambda \longrightarrow \infty.$$

It is also known that the following representation holds

$$\varphi(x,\lambda) = y(x,\lambda) + \int_0^x K(x,t)y(t,\lambda)dt.$$

Then formally

$$(V' - 1)y(x,\lambda) = \int_x^\infty K(t,x)y(t,\lambda)dt$$

Therefore if $(V' - 1)y(x,\lambda) \stackrel{\Phi'}{\longrightarrow} 0$ then

$$\frac{d\Gamma_1(\lambda)}{d\lambda} \asymp \frac{1}{\pi}\frac{\sqrt{\lambda}}{\lambda + n^2} \quad \text{as} \quad \lambda \to \infty.$$

Remark: It is known that if $q'(x) \in L^{1,loc}[0,\infty)$ then for each fixed x $K_{tt}(x,t) \in L^{1,loc}[0,\infty)$ and hence L_2K is densely defined. Therefore Theorem 5 is applicable.

The next example deals with the generalized Sturm Liouville operator. Let

$$\begin{cases} L_1 f \equiv -\frac{1}{w(x)}f''(x) + q(x)f(x) \\ f'(0) = 0. \end{cases} \quad \text{and} \quad \begin{cases} L_2 f \equiv \frac{-1}{x^\alpha}f'(x) \\ f'(0) = 0. \end{cases}$$

where $w(x) \asymp x^\alpha$ as $x \to 0$ and $\alpha > 0$. In this case the operator L_2 corresponds to a string whose length and mass are infinite, and is known to be self-adjoint in the space $L^2_{x^\alpha dx}$, see [5, p. 151] and [9].

We shall see that the behaviour of $w(x) \to 0$ dictates the behaviour of the spectral function at infinity. Although this result is known, see [6], we shall provide a different treatment as it is stated in [7]. For simplicity let the eigenfunctionals associated with $L_1$and L_2be defined by

$$\begin{cases} L_1\varphi(x,\lambda) \equiv \lambda\varphi(x,\lambda) \\ \varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = 0 \end{cases} \quad \text{and} \quad \begin{cases} L_2 y(x,\lambda) = \lambda y(x,\lambda) \\ y(0,\lambda) = 1,\ y'(0,\lambda) = 0. \end{cases}$$

It is clear that

$$\varphi(x,\lambda) = y(x,\lambda) + \int_0^x R(x,t,\lambda)q(t)\varphi(t,\lambda)dt.$$

where $R(x,t,\lambda)$ is the Greens' function and it is shown, by the semi-classical approximation, see [8], that $R(x,t,\lambda) \longrightarrow 0$ as $\lambda \to \infty$. Therefore we have that $\varphi(x,\lambda) - y(x,\lambda) \longrightarrow 0$ as $\lambda \to \infty$. The solution $y(x,\lambda)$ are known explicitly,

$$y(x,\lambda) = \sqrt{x}AJ_{-\nu}((\frac{2\sqrt{\lambda}}{\alpha+2})x^{\frac{\alpha+2}{2}}).$$

where $\nu = \frac{1}{\alpha+2}$ and $A = \{\frac{2\sqrt{\lambda}}{\alpha+2}\}^{\frac{1}{\alpha+2}}\frac{1}{\Gamma(1-\nu)}$.

Therefore provided $(V'-1)y(x,\lambda) \xrightarrow{\Phi'} 0$,we shall have

$$\Gamma_1(\lambda) \asymp \Gamma_2(\lambda) \quad \text{as} \quad \lambda \to \infty.$$

where, see [3], $\Gamma_2(\lambda) = c\lambda^{\frac{\alpha+1}{\alpha+2}}$ for $\lambda > 0$.

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