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NORMAL LATTICES AND COSEPARATION OF LATTICES

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ABSTRACT. Let X be an arbitrary non-empty set, and let $\mathcal{L}$ be a lattice of subsets of X such that $\emptyset$, $X \in \mathcal{L}$. We first summarize a number of known conditions which are equivalent to $\mathcal{L}$ being normal. We then develop new equivalent conditions in terms of set functions associated with $\mu \in I(\mathcal{L})$, the set of all non-trivial, zero-one valued, finitely additive measures on the algebra generated by $\mathcal{L}$. We finally generalize all the above to the situation where $\mathcal{L}_1$ and $\mathcal{L}_2$ are a pair of lattices of subsets of X with $\mathcal{L}_1 \subset \mathcal{L}_2$, and where we obtain equivalent conditions for $\mathcal{L}_1$ to coseparate $\mathcal{L}_2$.

KEY WORDS AND PHRASES: Normal lattice, coseparation of lattices, zero-one valued measures, associated outer measures.
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1. INTRODUCTION

Let X be an arbitrary non-empty set, and let $\mathcal{L}$ be a lattice of subsets of X such that $\emptyset$, $X \in \mathcal{L}$.

Various necessary and sufficient conditions for the lattice $\mathcal{L}$ to be normal are known (see [4,5,6]), and we summarize a number of these in section 2. We then give new necessary and sufficient conditions for the normality of $\mathcal{L}$ in section 3. These conditions are in terms of set functions associated with a $\mu \in I(\mathcal{L})$, where $I(\mathcal{L})$ is the set of non-trivial, zero-one valued, finitely additive measures on the algebra generated by $\mathcal{L}$.

Section 4 is devoted to the more general situation of a pair of lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ with $\mathcal{L}_1 \subset \mathcal{L}_2$ and for which $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. If $\mathcal{L}_1 = \mathcal{L}_2$, then $\mathcal{L}_1$ coseparates itself if and only if it is normal. We proceed, at first, to give necessary and sufficient conditions for $\mathcal{L}_1$ to coseparate $\mathcal{L}_2$ which extend known necessary and sufficient conditions for normal lattices which are summarized in section 2. Then we extend our new conditions for normality, to conditions both necessary and sufficient for $\mathcal{L}_1$ to coseparate $\mathcal{L}_2$ in terms of set functions associated with a $\mu \in I(\mathcal{L}_1)$.

We begin in section 2 with a brief summary of the notation and terminology used throughout the paper. Related matters can be found in [2,4,6]. We then turn our attention to normal lattices, and follow the program indicated above.

2. BACKGROUND AND NOTATION

Here we summarize briefly the notation and terminology that will be used throughout the paper Most of this is standard by now and follows that used in [1,3,4,7] for example. We will also assume for convenience that all lattices considered contain the $\emptyset$ and X.

X is an arbitrary non-empty set and $\mathcal{L}$ a lattice of subsets of X. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $I(\mathcal{L})$ those non-trivial, finitely additive, zero-one valued measures defined on $\mathcal{A}(\mathcal{L})$. $I_R(\mathcal{L})$

denotes those $\mu \in I(\mathcal{L})$, which are $\mathcal{L}$-regular, i.e

$$\mu(A) = \sup\{\mu(L) | L \subset A, L \in \mathcal{L}\}$$

where $A \in \mathcal{A}(\mathcal{L})$. We note, that $\mu \in I_R(\mathcal{L})$ if and only if $\mu \in I(\mathcal{L})$, and

$$\mu(L') = \sup\{\mu(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$$

where $L \in \mathcal{L}$. Here $L' = X - L$, and we denote by $\mathcal{L}' = \{L' | L \in \mathcal{L}\}$. $\mathcal{L}'$ is the complementary lattice to $\mathcal{L}$.

We note that there exists a one-to-one correspondence between all prime $\mathcal{L}$-filters and $I(\mathcal{L})$ given by associating with $\mu \in I(\mathcal{L})$ the prime $\mathcal{L}$-filter

$$\mathcal{F} = \{L \in \mathcal{L} | \mu(L) = 1\}. \tag{2.1}$$

Similarly, there exists a one-to-one correspondence between all $\mathcal{L}$-ultrafilters and $I_R(\mathcal{L})$ given by the same collection as in (2.1) only now $\mu \in I_R(\mathcal{L})$.

Finally, we note that if $\mathcal{H}$ is any collection of sets of $\mathcal{L}$ with the finite intersection property, i.e. the intersection of any finite number of sets of $\mathcal{H}$ is non-empty, then there exists a $\mu \in I_R(\mathcal{L})$ such that $\mu(A) = 1$ for all $A \in \mathcal{H}$.

$I_\sigma(\mathcal{L})$ denotes those $\mu \in I(\mathcal{L})$ such that μ is σ-smooth on $\mathcal{L}$, i.e. if $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$ then $\mu(L_n) \to 0$. There is a one-to-one correspondence between $I_\sigma(\mathcal{L})$ and all prime $\mathcal{L}$-filters with the countable intersection property.

$I^\sigma(\mathcal{L})$ denotes these $\mu \in I(\mathcal{L})$ that are σ-smooth on $\mathcal{A}(\mathcal{L})$, or, equivalently, are countably additive.

$I_R^\sigma(\mathcal{L}) = I_\sigma(\mathcal{L}) \cap I_R(\mathcal{L})$, and it is easy to see that if $\mu \in I_R^\sigma(\mathcal{L})$ then $\mu \in I^\sigma(\mathcal{L})$.

If $\mu \in I(\mathcal{L})$, we denote by μ' the following set function defined on $\mathcal{P}(X)$: for $E \subset X$,

$$\mu'(E) = \inf\{\mu(L') | E \subset L', L \in \mathcal{L}\}.$$

μ' is a finitely subadditive outer measure.

If μ, ν are set functions defined on $\mathcal{L}$, we write $\mu \leq \nu(\mathcal{L})$ if $\mu(L) \leq \nu(L)$ for all $L \in \mathcal{L}$. It is now clear that

$$\mu \in I_R(\mathcal{L}) \quad \text{if and only if} \quad \mu = \mu'(\mathcal{L}). \tag{2.2}$$

A set function defined on $\mathcal{L}$ is called modular if $\nu(L_1 \cup L_2) + \nu(L_1 \cap L_2) = \nu(L_1) + \nu(L_2)$, for all $L_1, L_2 \in \mathcal{L}$. If $\nu(L_1 \cup L_2) + \nu(L_1 \cap L_2) \leq \nu(L_1) + \nu(L_2)$ for all $L_1, L_2 \in \mathcal{L}$, then ν is called submodular, and supermodular if the inequality is reversed.

We recall that $\mathcal{L}$ is countably compact (c.c.) if and only if $I(\mathcal{L}) = I_\sigma(\mathcal{L})$, or, equivalently, if and only if $I_R(\mathcal{L}) = I_R^\sigma(\mathcal{L})$.

$\mathcal{L}$ is countably paracompact (c.p.) if $A_n \downarrow \emptyset$, $A_n \in \mathcal{L}$ implies there exists $B_n \in \mathcal{L}$, $A_n \subset B_n' \downarrow \emptyset$. Clearly if $\mathcal{L}$ is c.p. then $I_\sigma(\mathcal{L}') \subset I_\sigma(\mathcal{L})$.

$\mathcal{L}$ is a normal lattice if for any $A_1, A_2 \in \mathcal{L}$ with $A_1 \cap A_2 = \emptyset$ there exist $B_1, B_2 \in \mathcal{L}$ with $A_1 \subset B_1'$, $A_2 \subset B_2'$ and $B_1' \cap B_2' = \emptyset$.

We summarize some equivalent characterizations of normality in the following theorem (see [4,5,6]).

THEOREM 2.1. $\mathcal{L}$ is normal (where $\emptyset, X \in \mathcal{L}$) is equivalent to any of the following:

1) For each $\mu \in I(\mathcal{L})$, there exists a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$.
2) For any $\mu \in I(\mathcal{L})$ and $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$ then $\mu \leq \nu = \nu' = \mu'(\mathcal{L})$.
3) If $\mu \leq \nu(\mathcal{L})$ where $\mu \in I(\mathcal{L})$, $\nu \in I_R(\mathcal{L})$ then $\nu(L') = \sup\{\mu(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$ where $L \in \mathcal{L}$.
4) If $L \subset L_1' \cup L_2'$ where $L, L_1, L_2 \in \mathcal{L}$, then $L = A \cup B$ where $A, B \in \mathcal{L}$ and $A \subset L_1'$, $B \subset L_2'$.
5) For any $\mu \in I(\mathcal{L})$, $\mathcal{F} = \{L \in \mathcal{L} | \mu'(L) = 1\}$ is an $\mathcal{L}$-ultrafilter.

Further characterizations of normality will be developed in section 3. We just note one consequence of normality. We denote by $I_w(\mathcal{L})$ those $\mu \in I(\mathcal{L})$ such that $\mu(L') = 1$ implies there exists an $\tilde{L} \in \mathcal{L}$,

$\tilde{L} \subset L'$, and $\mu'(\tilde{L}) = 1$ where $L \in \mathcal{L}$. If $\mathcal{L}$ is normal, then $I_w(\mathcal{L}) = I_R(\mathcal{L})$. The converse, however, is not true in general.

If we have a pair of lattices, $\mathcal{L}_1$, $\mathcal{L}_2$ of subsets of X with $\mathcal{L}_1 \subset \mathcal{L}_2$, and if $\mu \in I(\mathcal{L}_2)$ then its restriction to $\mathcal{A}(\mathcal{L}_1)$ will be denoted by $\mu|_{\mathcal{L}_1}$ or simply $\mu|$ if there is no ambiguity.

In general if we have a pair of lattices $\mathcal{L}_1$, $\mathcal{L}_2$ we define:

$\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ if for all $A \in \mathcal{L}_1$, $B \in \mathcal{L}_2$ such that $A \cap B = \emptyset$, there exists an $A_1 \in \mathcal{L}_1$ such that $B \subset A_1$ and $A \cap A_1 = \emptyset$.

$\mathcal{L}_1$ separates $\mathcal{L}_2$ if for all B_1, $B_2 \in \mathcal{L}_2$ such that $B_1 \cap B_2 = \emptyset$, there exists A_1, $A_2 \in \mathcal{L}_1$ such that $B_1 \subset A_1$, $B_2 \subset A_2$ and $A_1 \cap A_2 = \emptyset$.

$\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if for all B_1, $B_2 \in \mathcal{L}_2$ such that $B_1 \cap B_2 = \emptyset$, there exists A_1, $A_2 \in \mathcal{L}_1$ such that $B_1 \subset A_1'$, $B_2 \subset A_2'$ and $A_1' \cap A_2' = \emptyset$.

$\mathcal{L}_2$ coallocates $\mathcal{L}_1$ if $A \subset B_1' \cup B_2'$, where $A \in \mathcal{L}_1$, B_1, $B_2 \in \mathcal{L}_2$, then there exists A_1, $A_2 \in \mathcal{L}_1$ such that $A = A_1 \cup A_2$, $A_1 \subset B_1'$ and $A_2 \subset B_2'$.

Under our assumption that all lattices involved contain $\emptyset$, X and assuming that $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\mathcal{L}_1$coseparates $\mathcal{L}_2$ if and only if $\mathcal{L}_2$ coallocates $\mathcal{L}_1$.

Finally, if ν is a finite outer measure (either finitely subadditive or countably subadditive) defined on $P(X)$, we denote by $\mathcal{S}_\nu$ the ν-measurable sets, so

$$\mathcal{S}_\nu = \{E \subset X | \nu(G) = \nu(G \cap E) + \nu(G \cap E') \text{ for all } G \subset X\}.$$

ν is regular if for any $A \subset X$, there exists an $E \in \mathcal{S}_\nu$ with $A \subset E$ and $\nu(A) = \nu(E)$. If ν is a regular outer measure which is finite then

$$\mathcal{S}_\nu = \{E \subset X | \nu(X) = \nu(E) + \nu(E')\}.$$

If ν is any outer measure that just assumes the values zero and one, then ν is clearly regular.

3. NORMAL LATTICES

In this section we wish to get characterizations of normal lattices in terms of certain set functions associated with a $\mu \in I(\mathcal{L})$. In the presence of normality, these set functions have been investigated [2,4,5,6]. We will summarize briefly these results, but we wish to go beyond this, and show that properties of these set functions can be utilized to give necessary and sufficient conditions for a lattice to be normal.

DEFINITION 3.1. Let $\mu \in I(\mathcal{L})$ and let $E \subset X$.

a) $\mu_i(E) = \sup\{\mu(L) | L \subset E, L \in \mathcal{L}\}$.

b) $\overline{\mu}(E) = \inf\{\mu_i(L') | E \subset L', L \in \mathcal{L}\}$.

It follows readily from the definition that, for $\mu \in I(\mathcal{L})$,

$$\mu_i = \mu \leq \overline{\mu} \leq \mu'(\mathcal{L}), \quad \text{and} \tag{3.1}$$

$$\overline{\mu} = \mu_i \leq \mu = \mu'(\mathcal{L}').$$

Now, we have

THEOREM 3.1. If $\mathcal{L}$ is a normal lattice and if $\mu \in I(\mathcal{L})$, then

a) μ_i is finitely additive and finitely subadditive on $\mathcal{L}'$.

b) $\overline{\mu}$ is a finitely subadditive outer measure.

PROOF. a) Let A, $B \in \mathcal{L}$, and suppose $\mu_i(A' \cup B') = 1$. Then there exists $L \in \mathcal{L}$ such that $L \subset A' \cup B'$ and $\mu(L) = 1$. From Theorem 2.1 part 4 of section 2, it follows that there exists A_1, $B_1 \in \mathcal{L}$ with $L = A_1 \cup B_1$, $A_1 \subset A'$, $B_1 \subset B'$. Thus $\mu(L) \leq \mu(A_1) + \mu(B_1)$, consequently, $\mu(A_1) = 1$ or $\mu(B_1) = 1$. By (3.1) this implies that $\mu_i(A_1) = 1$ or $\mu_i(B_1) = 1$ so $\mu_i(A') = 1$ or $\mu_i(B') = 1$ since μ_i is monotone. It is now clear that $\mu_i(A' \cup B') \leq \mu_i(A') + \mu_i(B')$ for any A, $B \in \mathcal{L}$. Again, if $\mu_i(A' \cup B') = 1$ and if $A' \cap B' = \emptyset$, A, $B \in \mathcal{L}$, then using the previous notation

$A_1 \cap B_1 = \emptyset$, and $1 = \mu(L) = \mu(A_1 \cup B_1) = \mu(A_1) + \mu(B_1)$. Say, $\mu(A_1) = 1$ and $\mu(B_1) = 0$, and clearly $\mu_i(A') = 1$ and $\mu_i(B') = 0$ from which the additivity of μ_i on $\mathcal{L}'$ follows.

b) Clearly $\overline{\mu}(\emptyset) = 0$, $\overline{\mu}$ is monotone, and all we need prove is the finite subadditivity of $\overline{\mu}$. Let E_1 and E_2 be arbitrary subsets with $\overline{\mu}(E_1 \cup E_2) = 1$. Then $\mu_i(L') = 1$ for all $L \supset E_1 \cup E_2$, $L \in \mathcal{L}$. If both $\overline{\mu}(E_1) = 0$ and $\overline{\mu}(E_2) = 0$, then there exists $L_1', L_2' \in \mathcal{L}'$ with $L_1' \supset E_1$, $L_2' \supset E_2$ and $\mu_i(L_1') = \mu_i(L_2') = 0$. Then $L_1' \cup L_2' \supset E_1 \cup E_2$ and $\mu_i(L_1' \cup L_2') = 0$ by part a), which is a contradiction, and completes the proof.

Since $\overline{\mu}$ is a finitely subadditive outer measure, we denote by $\mathcal{S}_{\overline{\mu}}$, the $\overline{\mu}$-measurable sets, i.e.

$$\mathcal{S}_{\overline{\mu}} = \{E \subset X | \overline{\mu}(G) = \overline{\mu}(G \cap E) + \overline{\mu}(G \cap E') \text{ for all } G \subset X\}.$$

Clearly $\mathcal{S}_{\overline{\mu}}$ is an algebra, and since μ and, therefore, $\overline{\mu}$ just assume the values 0 and 1, we have

$$\mathcal{S}_{\overline{\mu}} = \{E \subset X | \overline{\mu}(X) = \overline{\mu}(E) + \overline{\mu}(E')\}.$$

Now we show

THEOREM 3.2. If $\mathcal{L}$ is normal and if $\mu \in I(\mathcal{L})$, then $\mathcal{A}(\mathcal{L}) \subset \mathcal{S}_{\overline{\mu}}$.

PROOF. We need only show that if $L \in \mathcal{L}$, then $\overline{\mu}(X) \geq \overline{\mu}(L) + \overline{\mu}(L')$. Suppose $\overline{\mu}(L') = 1$. By (3.1), this implies $\mu_i(L') = 1$. Hence, there exists $\tilde{L} \in \mathcal{L}$, $\tilde{L} \subset L'$ and $\mu(\tilde{L}) = 1$. Since $\tilde{L} \cap L = \emptyset$, and since $\mathcal{L}$ is normal, there exists $B_1', B_2' \in \mathcal{L}'$ with $L \subset B_1'$, $\tilde{L} \subset B_2'$ and $B_1' \cap B_2' = \emptyset$. Clearly, $\mu(B_2') = 1$ and $\mu(B_1') = 0$. Thus $\overline{\mu}(B_1') = 0$ by (3.1), and $\overline{\mu}(L) = 0$. Thus $\overline{\mu}(L')$ and $\overline{\mu}(L)$ can't both be one, which completes the proof.

Finally we have

THEOREM 3.3. If $\mathcal{L}$ is normal and if $\mu \in I(\mathcal{L})$ then

a) $\overline{\mu}|_{\mathcal{A}(\mathcal{L})} \in I_R(\mathcal{L})$, and

b) $\overline{\mu} = \mu'(\mathcal{L})$.

PROOF. a) Since $\overline{\mu}$ is a finitely additive measure on $\mathcal{S}_{\overline{\mu}}$, it follows that $\overline{\mu}|_{\mathcal{A}(\mathcal{L})} \in I(\mathcal{L})$.

Also, for $L \in \mathcal{L}$,

$$\overline{\mu}'(L) = \inf\{\overline{\mu}(A') | L \subset A', A \in \mathcal{L}\},$$

but $\overline{\mu} = \mu_i(\mathcal{L}')$. Thus $\overline{\mu}' \underset{\mathcal{L}}{=} \overline{\mu}$, therefore, $\overline{\mu}|_{\mathcal{A}(\mathcal{L})} \in I_R(\mathcal{L})$ (see section 2).

b) Since $\mu \leq \overline{\mu}(\mathcal{L})$, by a) and by normality (see section 2), $\mu \leq \overline{\mu} = \overline{\mu}' = \mu'(\mathcal{L})$.

Let ν be a set function defined on all subsets of X. Recall ν is submodular if and only if

$$\nu(E_1 \cup E_2) + \nu(E_1 \cap E_2) \leq \nu(E_1) + \nu(E_2)$$

for all $E_1, E_2 \subset X$.

It is easy to see that the following holds.

LEMMA 3.4. If ν is a monotone set function defined on all sets $E \subset X$ that assumes only the values 0 and 1, then ν is finitely subadditive if and only if ν is submodular.

Now we establish:

THEOREM 3.5. If $\mu \in I(\mathcal{L})$, then $\mathcal{L}$ is normal if and only if $\overline{\mu}$ is submodular (or equivalently if and only if $\overline{\mu}$ is a finitely subadditive outer measure).

PROOF. If $\mathcal{L}$ is normal, then $\overline{\mu}$ is a finitely subadditive outer measure by Theorem 3.1 b), and, therefore, submodular by the Lemma.

Conversely, suppose $\overline{\mu}$ is submodular. If $\mathcal{L}$ is not normal, then there exists $A_1, A_2 \in \mathcal{L}$ such that $A_1 \cap A_2 = \emptyset$, but for all $B_1', B_2' \in \mathcal{L}'$ with $A_1 \subset B_1'$, $A_2 \subset B_2'$, we have $B_1' \cap B_2' \neq \emptyset$.

This implies that the set

$$\mathcal{B} = \{B' \in \mathcal{L}' | B' \supset A_1 \text{ or } B' \supset A_2\}$$

has the finite intersection property. Consequently there exists a $\mu \in I_R(\mathcal{L})$ such that $\mu(B') = 1$ for all $B' \in \mathcal{B}$. Hence, $\mu(B) = 0$ for all $B' \in \mathcal{B}$, which implies $\mu_*(A_1') = 0$ and $\mu_*(A_2') = 0$. Thus, $\overline{\mu}(A_1') = \overline{\mu}(A_2') = 0$ by (3.1). But $1 = \overline{\mu}(X) = \overline{\mu}(A_1' \cup A_2') = 0$ which is a contradiction since $\overline{\mu}$ is submodular.

As our characterization theorem, we have

THEOREM 3.6. If $\mu \in I(\mathcal{L})$, then $\mathcal{L}$ is normal if and only if $\overline{\mu} = \mu'(\mathcal{L})$.

PROOF. If $\mathcal{L}$ is normal then $\overline{\mu} = \mu'(\mathcal{L})$ by Theorem 3.3 part b).

Conversely, suppose $\overline{\mu} = \mu'(\mathcal{L})$ and $\mathcal{L}$ is not normal. Then using the same notation and construction as in Theorem 3.5, we have $\mu_*(A_2') = 0$, while $A_1 \subset B'$, $B \in \mathcal{L}$ implies $\mu(B') = 1$, but $A_1 \subset A_2'$. Thus $\overline{\mu}(A_1) = 0$, while clearly $\mu'(A_1) = 1$. This contradiction proves the theorem.

4. COSEPARATION OF LATTICES

In the present section, we will extend the results of sections 2 and 3 on normal lattices to a pair of lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1 \subset \mathcal{L}_2$, and where $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. Clearly, if $\mathcal{L}_1 = \mathcal{L}_2$ then $\mathcal{L}_1$ coseparates itself if and only if it is normal. The work done here also extends that done in [2,5,6].

Our first result directly generalizes Theorem 2.1 part 1).

THEOREM 4.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Then $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if and only if for any $\mu \in I(\mathcal{L}_1)$ and any $\nu_1, \nu_2 \in I_R(\mathcal{L}_2)$ such that $\mu \leq \nu_1(\mathcal{L}_1)$ and $\mu \leq \nu_2(\mathcal{L}_1)$ then $\nu_1 = \nu_2$.

PROOF. 1) We assume that $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. If $\nu_1 \neq \nu_2$ then there exists $B_1, B_2 \in \mathcal{L}_2$ such that $\nu_1(B_1) = 1$, $\nu_2(B_2) = 1$, and $B_1 \cap B_2 = \emptyset$. Hence, there exists $A_1, A_2 \in \mathcal{L}_1$ with $B_1 \subset A_1'$, $B_2 \subset A_2'$ and $A_1' \cap A_2' = \emptyset$. Consequently, $A_1 \cup A_2 = X$ so $\mu(A_1) = 1$ or $\mu(A_2) = 1$. But $A_1 \subset B_1'$, and $\nu_1(B_1') = 0$, so $\nu_1(A_1) = 0$; hence $\mu(A_1) = 0$. Similarly $A_2 \subset B_2'$, and $\nu_2(B_2') = 0$ so $\mu(A_2) = 0$, from this contradiction, we conclude that $\nu_1 = \nu_2$.

2) Conversely, assuming the condition of the theorem holds, if $\mathcal{L}_1$ does not coseparate $\mathcal{L}_2$ then there exists $B_1, B_2 \in \mathcal{L}_2$ such that

$$\mathcal{H} = \{A' \in \mathcal{L}_1' | B_1 \subset A' \text{ or } B_2 \subset A'\}$$

has the finite intersection property. Therefore, there exists a $\mu \in I(\mathcal{L}_1)$ such that $\mu(A') = 1$ for all $A' \in \mathcal{H}$. Now let $L_1 \in \mathcal{L}_1$ with $\mu(L_1) = 1$. Then $\mu(L_1') = 0$ so $L_1' \notin \mathcal{H}$. Hence, $L_1 \cap B_1 \neq \emptyset$ and $L_1 \cap B_2 \neq \emptyset$ for all $L_1 \in \mathcal{L}_1$ with $\mu(L_1) = 1$. Thus if we let

$$\mathcal{H}_1 = \{L_2 \in \mathcal{L}_2 | L_2 \supset L_1 \cap B_1, L_1 \in \mathcal{L}_1 \text{ with } \mu(L_1) = 1\},$$

and

$$\mathcal{H}_2 = \{L_2 \in \mathcal{L}_2 | L_2 \supset L_1 \cap B_2, L_1 \in \mathcal{L}_1 \text{ with } \mu(L_1) = 1\}$$

then $\mathcal{H}_1$ and $\mathcal{H}_2$ are $\mathcal{L}_2$-filters Consequently, there exists $\nu_1, \nu_2 \in I_R(\mathcal{L}_2)$ such that $\nu_1(L_2) = 1$ for all $L_2 \in \mathcal{H}_1$ and $\nu_2(L_2) = 1$ for all $L_2 \in \mathcal{H}_2$. $\nu_1 \neq \nu_2$ since $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$, and $B_1 \cap B_2 = \emptyset$. Also $\mu \leq \nu_1(\mathcal{L}_1)$ since, if $A \in \mathcal{L}_1$ and $\mu(A) = 1$ then $A \in \mathcal{H}$, clearly, so $\nu_1(A) = 1$; similarly $\mu \leq \nu_2(\mathcal{L}_1)$ which completes the proof.

The next theorem generalizes Theorem 2.1 part 2).

THEOREM 4.2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Then $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if and only if for any $\mu \in I(\mathcal{L}_1)$ and $\nu \in I_R(\mathcal{L}_2)$ such that $\mu \leq \nu(\mathcal{L}_1)$, we have $\nu = \mu'(\mathcal{L}_2)$.

PROOF. 1) Suppose $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. Clearly $\nu' \leq \mu'$. Suppose $\nu(L_2) = 0$ where $L_2 \in \mathcal{L}_2$. Then $\nu(L_2') = 1$; hence $L_2' \supset \tilde{L}_2 \in \mathcal{L}_2$ and $\nu(\tilde{L}_2) = 1$. Now $L_2 \cap \tilde{L}_2 = \emptyset$ so there exists $L_1, \tilde{L}_1 \in \mathcal{L}_1$, $\tilde{L}_2 \subset \tilde{L}_1'$, $L_2 \subset L_1'$ and $L_1' \cap \tilde{L}_1' = \emptyset$, so $\nu(\tilde{L}_1') = 1$, and, therefore, $\mu(\tilde{L}_1') = 1$. Consequently, $\mu(L_1') = 0$, so $\mu'(L_2) = 0$. This implies that

$$\nu = \nu' = \mu'(\mathcal{L}_2).$$

Conversely, if this condition holds, and if $\mu \in I(\mathcal{L}_1)$, and $\nu_1, \nu_2 \in I_R(\mathcal{L}_2)$ with $\mu \leq \nu_1(\mathcal{L}_1)$, $\mu \leq \nu_2(\mathcal{L}_1)$ then $\nu_1 = \nu_1' = \mu'(\mathcal{L}_2)$ and $\nu_2 = \nu_2' = \mu'(\mathcal{L}_2)$, so $\nu_1 = \nu_2$, and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ by Theorem 4.1.

Note: Clearly Theorem 4.2 is equivalent to the following: Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be lattices of subsets of X, and let $\mu \in I(\mathcal{L}_1)$ and $\nu \in I_R(\mathcal{L}_2)$ be arbitrary with $\mu \leq \nu(\mathcal{L}_1)$. Then $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if and only if $\nu(B') = 1$, $B \in \mathcal{L}_2$, there exists an $A \in \mathcal{L}_1$ with $A \subset B'$, and $\mu(A) = 1$. Clearly this result extends Theorem 2.1 part 3.

We now extend the comment following Theorem 2.1.

Theorem 4.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Also let $\mathcal{L}_1$ coseparate $\mathcal{L}_2$. Then, if for $\mu \in I(\mathcal{L}_1)$ and $\nu \in I(\mathcal{L}_2)$ with $\mu \leq \nu(\mathcal{L}_1)$ and with $\nu(B') = \sup\{\mu'(A) | A \subset B', A \in \mathcal{L}_1\}$, $B \in \mathcal{L}_2$, we have $\nu \in I_R(\mathcal{L}_2)$.

PROOF. Suppose $\nu(B') = 1$, where $B \in \mathcal{L}_2$. Then there exists $A \in \mathcal{L}$, $A \subset B'$ and $\mu'(A) = 1$. Since $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ there exists $A_1, A_2 \in \mathcal{L}_1$ with $A \subset A_1'$, $B \subset A_2'$, and $A_1' \cap A_2' = \emptyset$, so $A \subset A_1' \subset A_2 \subset B'$. Thus $\mu'(A_1') = 1 = \mu(A')$, so $\mu(A_2) = 1$. Hence $\nu(A_2) = 1$, and $A_2 \in \mathcal{L}_1 \subset \mathcal{L}_2$ so $\nu \in I_R(\mathcal{L}_2)$.

We recall:

DEFINITION 4.1. The lattice $\mathcal{L}$ is almost countably compact if $I_R(\mathcal{L}') \subset I_\sigma(\mathcal{L})$.

Clearly if $\mathcal{L}$ is countably compact then $\mathcal{L}$ is almost countably compact. While, if $\mathcal{L}$ is normal, countably paracompact and almost countably compact, then $\mathcal{L}$ is countably compact.

THEOREM 4.4. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. If $\mu \in I_\sigma(\mathcal{L}_1)$ and $\nu \in I_R(\mathcal{L}_2)$ such that $\mu \leq \nu(\mathcal{L}_1)$ then $\nu \in I_\sigma(\mathcal{L}_2')$.

PROOF. If $\nu \notin I_\sigma(\mathcal{L}_2')$, then there exists a sequence $\{B_n'\}$, $B_n \in \mathcal{L}_2$ such that $B_n' \downarrow \emptyset$ and $\nu(B_n') = 1$ for all n. By the note after Theorem 4.2 there exists $A_n \in \mathcal{L}_1$ with $A_n \subset B_n'$ and $\mu(A_n) = 1$ for all n. Clearly, we may assume $A_n \downarrow$, so $A_n \downarrow \emptyset$, which implies $\mu \notin I_\sigma(\mathcal{L}_1)$, a contradiction.

THEOREM 4.5. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. If $\nu \in I_R(\mathcal{L}_2)$ and if $\mathcal{L}_1$ is almost countably compact then $\nu \in I_\sigma(\mathcal{L}_2')$.

PROOF. Let $\lambda = \nu|_{\mathcal{A}(\mathcal{L}_1)}$. $\lambda \in I_R(\mathcal{L}_1)$, and since $\mathcal{L}_1$ is almost countably compact, there exists a $\mu \in I_\sigma(\mathcal{L})$ such that

$$\mu \leq \lambda(\mathcal{L}_1).$$

Now, by Theorem 4.4, it follows that $\nu \in I_\sigma(\mathcal{L}_2')$, which completes the proof.

REMARK. Under the assumption of Theorem 4.5, if in addition, $I_\sigma(\mathcal{L}_2') \subset I_\sigma(\mathcal{L}_2)$, then $\nu \in I_R^\sigma(\mathcal{L}_2)$, in which case $\mathcal{L}_2$ is countably compact.

We now wish to extend the results of Theorems 3.1-3.3, to the situation of a pair of lattices $\mathcal{L}_1$, $\mathcal{L}_2$ with $\mathcal{L}_1 \subset \mathcal{L}_2$. We define for $\mu \in I(\mathcal{L})$ and any $E \subset X$.

a) $\mu_i(E) = \sup\{\mu(L_1) | L_1 \subset E, L_1 \in \mathcal{L}_1\}$.

b) $\overline{\mu}_i(E) = \inf\{\mu_i(L_2') | E \subset L_2', L_2 \in \mathcal{L}_2\}$.

Arguing analogously to the proofs in Theorems 3.1-3.3, we obtain readily,

THEOREM 4.6. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and let $\mathcal{L}_1$ coseparate $\mathcal{L}_2$. Then, for $\mu \in I(\mathcal{L}_1)$,

a) μ_i is finitely additive and finitely subadditive on $\mathcal{L}_2'$.

b) $\overline{\mu}$ is a finitely subadditive outer measure.

c) $\mathcal{A}(\mathcal{L}_1) \subset \mathcal{S}_{\overline{\mu}}$.

d) $\nu = \overline{\mu}|_{\mathcal{A}(\mathcal{L}_2)} \in I_R(\mathcal{L}_2)$, and $\mu \leq \nu(\mathcal{L}_1)$.

Furthermore, adhering to the above notation, we have

THEOREM 4.7. $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if and only if

$$\mathcal{G}_{\mu'} = \{L_2 \in \mathcal{L}_2 | \mu'(L_2) = 1\}$$

is a prime $\mathcal{L}_2$-filter for any $\mu \in I(\mathcal{L}_1)$.

PROOF. 1) If $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ then $\nu = \nu' = \mu'(\mathcal{L}_2)$ since $\nu \in I_R(\mathcal{L}_2)$ and using Theorem 4.2. From this it follows immediately that μ' is modular on $\mathcal{L}_2$ and $\mathcal{G}_{\mu'}$ is a prime $\mathcal{L}_2$-filter.

2) Suppose for any $\mu \in I(\mathcal{L}_1)$, $\mathcal{G}_{\mu'}$ is a prime $\mathcal{L}_2$-filter. If $\mathcal{L}_1$ does not coseparate $\mathcal{L}_2$, then, by Theorem 4.1, there exists $\mu \in I(\mathcal{L}_1)$, $\nu_1, \nu_2 \in I_R(\mathcal{L}_2)$ with $\nu_1 \neq \nu_2$, and such that $\mu \leq \nu_1(\mathcal{L}_1)$ and $\mu \leq \nu_2(\mathcal{L}_1)$. Hence

$$\nu_1 = \nu_1' \leq \mu'(\mathcal{L}_2),$$

and

$$\nu_2 = \nu_2' \leq \mu'(\mathcal{L}_2).$$

But $\mathcal{G}_{\mu'}$ is a prime $\mathcal{L}_2$-filter, while ν_1 and ν_2 determine $\mathcal{L}_2$ ultrafilters. Thus, we must have $\nu_1 = \nu_2$, a contradiction. Hence, $\mathcal{L}_1$ coseparates $\mathcal{L}_2$.

Next, we extend Theorem 3.5.

THEOREM 4.8. Continuing with the notation prior to and in Theorem 4.6, we have that $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if and only if $\overline{\mu}$ is submodular or, equivalently, a finitely subadditive outer measure for any $\mu \in I(\mathcal{L}_1)$.

PROOF. 1) If $\overline{\mu}$ is submodular, then, in particular, $\overline{\mu}$ is submodular on $\mathcal{L}_2'$. However, $\overline{\mu} = \mu_i(\mathcal{L}_2')$, so μ_i is submodular on $\mathcal{L}_2'$. Hence μ' is supermodular on $\mathcal{L}_2$. Thus if $B_1, B_2 \in \mathcal{L}_2$, and, if $\mu'(B_1) = 1 = \mu'(B_2)$, then $\mu'(B_1 \cap B_2) = 1$, and clearly $\mathcal{G}_{\mu'}$ is a prime $\mathcal{L}_2$-filter, so $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ by Theorem 4.7.

2) Conversely suppose $\mathcal{L}_1$ coseparates $\mathcal{L}_2$, then by Theorem 4.6 b) $\overline{\mu}$ is a finitely subadditive outer measure for any $\mu \in I(\mathcal{L}_1)$.

We note if $\mathcal{L}_1$ coseparates $\mathcal{L}_2$, then $\nu = \nu' = \mu'(\mathcal{L}_2)$ since $\nu \in I_R(\mathcal{L}_2)$ and by Theorem 4.2, however, $\nu = \overline{\mu}(\mathcal{L}_2)$, so $\overline{\mu} = \mu'(\mathcal{L}_2)$.

Suppose conversely for any $\mu \in I(\mathcal{L}_1)$, $\overline{\mu} = \mu'(\mathcal{L}_2)$. We note for any $E \subset X$, $\mu'(E) + \mu_i(E) = \mu(X) = 1$.

If $\mathcal{L}_1$ does not coseparate $\mathcal{L}_2$, then there exists $B_1, B_2 \in \mathcal{L}_2$ such that $B_1 \cap B_2 = \emptyset$, and

$$\mathcal{H} = \{A' \in \mathcal{L}_1' | A' \supset B_1, \text{ or } A' \supset B_2\}$$

has the finite intersection property. Consequently there exists a $\mu \in (\mathcal{L}_1)$ such that for any $A \in \mathcal{L}_1$ and $A' \supset B_1$ or $A' \supset B_2$ then $\mu(A') = 1$. Hence, $\mu'(B_1) = \mu'(B_2) = 1$. Thus $\mu_i(B_2') = 0$. But, $B_1 \subset B_2'$, so $\overline{\mu}(B_1) = 0$, a contradiction. Therefore, $\mathcal{L}_1$ coseparates $\mathcal{L}_2$.

Summarizing, we have extended Theorem 3.6 to:

THEOREM 4.9. Using the above notation $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if and only if $\overline{\mu} = \mu'(\mathcal{L}_2)$ for any $\mu \in I(\mathcal{L}_1)$.

REFERENCES

[1] BACHMAN, G. and SULTAN, A., Regular lattice measures: mappings and spaces, *Pacific J. Math.* **67**, 2 (1976), 291-321.

[2] CHAN, J.K., Measures on coallocation and normal lattices, *Internat. J. Math. & Math. Sci.* **15**, 4 (1992), 701-718.

[3] FROLIK, Z., Prime filters with the c.i.p., *Comm. Math. Univ. Caroline* **13**, 3 (1972), 553-573.

[4] SZETO, M., On separation of lattices, *Internat. J. Math. & Math. Sci.* **14**, 2 (1991), 325-338.

[5] SZETO, M., On normal lattices and separation properties of lattices, *Journal of the Indian Mathematical Society* **58**, 1 (1992), 51-64.

[6] VLAD, C., On normal lattices and semi-separation of lattices, *Journal of the Indian Mathematical Society* **56** (1991), 259-273.

[7] WALLMAN, H., Lattices and topological spaces, *Ann. of Math.* **39** (1938), 112-126.