Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 397-402

HEARING THE SHAPE OF A COMPACT RIEMANNIAN MANIFOLD WITH A FINITE NUMBER OF PIECEWISE IMPEDANCE BOUNDARY CONDITIONS

E.M.E. ZAYED

Mathematics Department
Faculty of Science
Zagazig University
Zagazig, EGYPT

(Received June 8, 1994 and in revised form March 21,1996)

ABSTRACT. The spectral function $\Theta(t) = \sum_{i=1}^{\infty} \exp(-t\lambda_j)$, where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of the negative Laplace-Beltrami operator $-\Delta$, is studied for a compact Riemannian manifold Ω of dimension "k" with a smooth boundary $\partial\Omega$, where a finite number of piecewise impedance boundary conditions $(\frac{\partial}{\partial n_i} + \gamma_i)u = 0$ on the parts $\partial\Omega_i (i = 1, ..., m)$ of the boundary $\partial\Omega$ can be considered, such that $\partial\Omega = \bigcup_{i=1}^{m} \partial\Omega_i$, and $\gamma_i (i = 1, ..., m)$ are assumed to be smooth functions which are not strictly positive

KEY WORDS AND PHRASES: Inverse problem, Laplace Beltrami operator, eigenvalues, spectral function, Riemannin manifold
1991 AMS SUBJECT CLASSIFICATION CODES: 35K, 35P, 58G32

1. INTRODUCTION

The underlying problem is to determine the geometry of a compact k-dimensional smooth Riemannian manifold Ω with metric tensor $g = (g_{\alpha\beta})$, from a complete knowledge of the eigenvalues for the negative Laplace-Beltrami operator $-\Delta = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_\alpha} \left[g^{\alpha\beta} \sqrt{\det g}\, \frac{\partial}{\partial x_\beta}\right]$ where $g^{-1} = (g^{\alpha\beta})$

Let Ω be a compact Riemannian manifold of dimension "k" with a smooth boundary $\partial\Omega$ Suppose that the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \to \infty \quad \text{as} \quad j \to \infty, \tag{1 1}$$

are known exactly for the eigenvalue equation

$$(\Delta + \lambda)u = 0 \quad \text{in} \quad \Omega, \tag{1 2}$$

together with the impedance boundary condition

$$\left(\frac{\partial}{\partial n} + \gamma\right)u = 0 \quad \text{on} \quad \partial\Omega, \tag{1 3}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$ and γ is a smooth function which is not strictly positive

Hsu [1] has investigated problem (1.2)-(1 3) and has determined the geometric quantities associated with the manifold Ω from the asymptotic expansion of the spectral function

$$\Theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j) \quad \text{as} \quad t \to 0. \tag{1 4}$$

Problem (1.2)-(1 3) has been investigated by many authors (see, for example, Mckean and Singer [3] and Hsu [1]) if $\gamma \equiv 0$ (Neumann problem) and have shown that

$$(4\pi t)^{k/2}\Theta(t) = a_0 + a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + 0(t^2) \quad \text{as} \quad t \to 0, \tag{1 5}$$

where

$$a_0 = |\Omega|,$$
$$a_1 = \frac{\sqrt{\pi}}{2}|\partial\Omega|,$$
$$a_2 = \frac{1}{6}\int_\Omega K(x)dx - \frac{1}{3}\int_{\partial\Omega} tr\, H(z)dz,$$

and

$$a_3 = \sqrt{\pi}\int_{\partial\Omega}\left\{\frac{1}{12}K^{\partial\Omega}(z) - \frac{37}{192}[tr\, H(z)]^2 + \frac{29}{96}tr\, H^2(z) + \frac{1}{8}Ric(n)(z)\right\}dz.$$

In these formulae, $|\Omega|$ is the (Riemannian) volume of Ω, $|\partial\Omega|$ is the (Riemannian) surface area of $\partial\Omega$, $K(x)$ is the scalar curvature of Ω at x, $H(z)$ is the second fundamental form of the boundary $\partial\Omega$, $K^{\partial\Omega}(z)$ is the scalar curvature of $\partial\Omega$ (equipped with the induced metric) at z, $Ric(n)z$ is the Ricci curvature of Ω at z in the normal direction "n" of the boundary $\partial\Omega$, and $tr\, H(z)$ is the mean curvature of $\partial\Omega$

The object of this paper is to discuss the following more general inverse problem Suppose that the eigenvalues (1 1) are known exactly for the eigenvalue equation (1.2) together with the following piecewise smooth impedance boundary conditions

$$\left(\frac{\partial}{\partial n_i} + \gamma_i\right)u = 0 \quad \text{on} \quad \partial\Omega_i (i = 1, ..., m), \tag{1 6}$$

where the boundary $\partial\Omega$ of Ω consists of a finite number of the parts $\partial\Omega_i$ $(i = 1, ..., m)$ such that $\partial\Omega = \bigcup_{i=1}^{m} \partial\Omega_i$, while $\frac{\partial}{\partial n_i}$ denote differentiations along the inward pointing normals to $\partial\Omega_i$ and γ_i are assumed to be smooth functions defined on $\partial\Omega_i$ which are not strictly positive.

The basic problem is that of determining the geometry of the manifold Ω as well as the impedance functions $\gamma_i (i = 1, ..., m)$ from the asymptotic expansion of the spectral function

$$\Theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j) \quad \text{as} \quad t \to 0. \tag{1 7}$$

Note that the main problem (1.2) and (1.6) has been discussed recently by Zayed and Younis [4] and Zayed [5-7] in the case where Ω is a general simply connected bounded domain in $R^k (k = 2 \text{ or } 3)$ with a smooth boundary $\partial\Omega$ and γ_i $(= 1, ..., m)$ are positive constants

2. STATEMENT OF RESULTS

THEOREM. Let $|\partial\Omega_i| (i = 1, ..., m)$ be the (Riemannian) surface areas of the parts $\partial\Omega_i (i = 1, ..., m)$ of the boundary $\partial\Omega$ respectively. Let $K^{\partial\Omega_i}(z)$, $(i = 1, ..., m)$ be the scalar curvatures of the parts $\partial\Omega_i$ $(i = 1, ..., m)$ of $\partial\Omega$ respectively. Let $Ric(n_i)(z)$ be the Ricci curvatures of $\partial\Omega_i$ at z in the normal directions n_i of the parts $\partial\Omega_i$ $(i = 1, ..., m)$ of $\partial\Omega$. Then the results of problem (1 2) and (1 6) can be written in the form

$$(4\pi t)^{k/2}\Theta(t) = a_0 + a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + 0(t^2) \quad \text{as} \quad t \to 0, \tag{2 1}$$

where

$$\begin{aligned} a_0 &= |\Omega| \\ a_1 &= \frac{1}{2}\sqrt{\pi}\sum_{i=1}^{m} |\partial\Omega_i|, \\ a_2 &= \frac{1}{6}\int_\Omega K(x)dx - \frac{1}{3}\sum_{i=1}^{m}\int_{\partial\Omega_i}[tr\, H(z) + 6\gamma_i(z)]dz, \end{aligned}$$

and

$$a_3 = \sqrt{\pi}\sum_{i=1}^{m}\int_{\partial\Omega_i}\left\{\frac{1}{12}K^{\partial\Omega_i}(z) - \frac{37}{192}[tr\, H(z)]^2 + \frac{29}{96}tr\, H^2(z) + \frac{1}{8}Ric(n_i)(z) + \frac{1}{2}\gamma_i(z)\, tr\, H(z) + \gamma_i^2(z)\right\}dz.$$

Note that the results of Neumann conditions on $\partial\Omega_i$ are obtained from (2 1) by setting $\gamma_i \equiv 0$ $(i = 1, ..., m)$ which are in agreement with the results (1 5) of Neumann conditions on $\partial\Omega$

REMARK 1. If Ω is a bounded domain in R^2 with a smooth boundary $\partial\Omega$, then $K^{\partial\Omega_i}(z) = 0$, $Ric(n_i)(z) = 0$, $tr\, H^2(z) = c^2(z)$, $tr\, H(z) = -c(z)$ where $c(z)$ is the curvature of $\partial\Omega$ at z and if γ_i are positive constants then, we get the result of Zayed [5] when $i = 1, 2$ and the result of Zayed and Younis [4] when $i = 1, ..., m$.

REMARK 2. If Ω is a bounded domain in R^3 with a smooth surface $\partial\Omega$, then $K^{\partial\Omega_i}(z) = 2k_1k_2$, $Ric(n_i)(z) = 0$, $tr\, H^2(z) = k_1^2 + k_2^2$, $tr\, H(z) = -k_1 - k_2$, where k_1 and k_2 are the two principal curvatures of the boundary surface $\partial\Omega$ at z and if γ_i are positive constants, then we get the result of Zayed [6] when $i = 1, 2$ and also the result of Zayed [7] when $i = 1, .., m$

3. CONSTRUCTION OF RESULTS

Following the method of Kac [2] and Hsu [1], it is easily seen that $\Theta(t)$ associated with problem (1.2) and (1 6) is given by

$$\Theta(t) = \int_\Omega G(t, x, x)dx, \tag{3 1}$$

where the heat kernel $G(t, x, y)$ is defined on $(0, \infty) \times \overline{\Omega} \times \overline{\Omega}$, which satisfies the following

For fixed $x \in \overline{\Omega}$, it satisfies in t, y the heat equation

$$\left(\frac{\partial}{\partial t} - \triangle_y\right)G(t, x, y) = 0, \tag{3 2}$$

and the piecewise impedance boundary conditions

$$\left[\frac{\partial}{\partial n_{iy}} + \gamma_i(y)\right]G(t, x, y) = 0 \quad \text{on} \quad \partial\Omega_i (i = 1, ..., m), \tag{3 3}$$

and the initial condition

$$\lim_{t\to 0} G(t, x, y) = \delta(x - y), \tag{3 4}$$

where $\delta(x - y)$ is the Dirac delta function located at the source point $x = y$ Note that in (3 2)-(3 3) the subscript "y" means that the derivatives are taken in y-variables.

Thus by the superposition principle of the heat equation, we write

$$G(t, x, y) = G_N(t, x, y) + \chi(t, x, y) \tag{3 5}$$

where $G_N(t,x,y)$ is the Neumann heat kernel on Ω which satisfies the heat equation

$$\left(\frac{\partial}{\partial t} - \triangle_y\right) G_N(t,x,y) = 0, \tag{3 6}$$

and the piecewise Neumann boundary conditions

$$\frac{\partial}{\partial n_{iy}} G_N(t,x,y) = 0 \quad \text{on} \quad \partial\Omega_i (i = 1, ..., m), \tag{3 7}$$

and the initial condition

$$\lim_{t\to 0} G_N(t,x,y) = \delta(x-y), \tag{3 8}$$

while $\chi(t,x,y)$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t} - \triangle_y\right) \chi(t,x,y) = 0, \tag{3 9}$$

and the piecewise boundary conditions

$$\frac{\partial}{\partial n_{iy}} \chi(t,x,y) = -\gamma_i(y) G(t,x,y) \quad \text{on} \quad \partial\Omega_i (i = 1, ..., m), \tag{3 10}$$

and the initial condition

$$\lim_{t\to 0} \chi(t,x,y) = 0. \tag{3 11}$$

Now, the solution of problem (3.9), (3.10) and (3 11) is given by

$$\chi(t,x,y) = -\sum_{i=1}^{m} \int_0^t ds \int_{\partial\Omega_i} G_N(t-s,x,z)\gamma_i(z) G(s,z,y) dz. \tag{3 12}$$

From (3.5) and (3.12) we have the integral equation

$$G(t,x,y) = G_N(t,x,y) - \sum_{i=1}^{m} \int_0^t ds \int_{\partial\Omega_i} G_N(t-s,x,z)\gamma_i(z) G(s,z,y) dz. \tag{3 13}$$

On applying the iteration method (see [5]) to the integral equation (3 13) we obtain the series

$$G(t,x,y) = \sum_{r=0}^{\infty} (-1)^r F_r(t,x,y), \tag{3.14}$$

where

$$F_0(t,x,y) = G_N(t,x,y),$$

and

$$F_r(t,x,y) = \sum_{i=1}^{m} \int_0^t ds \int_{\partial\Omega_i} G_N(t-s,x,z)\gamma_i(z) F_{r-1}(s,z,y) dz, \quad (r = 1, 2, ...). \tag{3.15}$$

From (3.1), (3 5), (3.14) and with the help of the following well known estimate (see [1], [3])

$$(4\pi t)^{k/2} \sum_{r=3}^{\infty} \int_\Omega |F_r(t,x,x)| dx = 0(t^2) \quad \text{as} \quad t \to 0, \tag{3 16}$$

we deduce as $t \to 0$ that

$$\Theta(t) = \Theta_N(t) - \int_\Omega F_1(t,x,x) dx + \int_\Omega F_2(t,x,x) dx + 0\left(t^{(4-k)/2}\right), \tag{3 17}$$

where $\Theta_N(t) = \int_\Omega G_N(t,x,x)dx$, which has the same asymptotic expansion (2 1) with $\gamma_i \equiv 0$

The problem now is to study the integrals of $F_r(t,x,x)$, $(r = 1,2)$ over the manifold Ω

LEMMA 1. We have as $t \to 0$,

$$(4\pi t)^{k/2}\int_\Omega F_1(t,x,x)dx = 2t\sum_{i=1}^{m}\int_{\partial\Omega_i}\gamma_i(z)dz - \frac{1}{2}\sqrt{\pi}\, t^{3/2}\sum_{i=1}^{m}\int_{\partial\Omega_i}\gamma_i(z)tr\, H(z)dz + 0(t^2). \tag{3 18}$$

PROOF. The definition of $F_1(t,x,x)$ and the Chapman-Kolmogorov equation of the heat kernel imply

$$\int_\Omega F_1(t,x,x)dx = t\sum_{i=1}^{m}\int_{\partial\Omega_i}G_N(t,z,z)\gamma_i(z)dz. \tag{3 19}$$

Let us now introduce the following well known estimate of the Neumann heat kernel (see [1])

$$(4\pi t)^{k/2}G_N(t,z,z) = 2\left[1 - \frac{1}{4}\sqrt{\pi t}\, tr\, H(z)\right] + 0(t) \quad \text{as} \quad t \to 0, \tag{3 20}$$

which is valid uniformly in $z \in \partial\Omega_i$ $(i = 1, ..., m)$.

On inserting (3.20) into (3.19) we arrive at the proof of Lemma 1

LEMMA 2. We have as $t \to 0$

$$(4\pi t)^{k/2}\int_\Omega F_2(t,x,x)dx = \sqrt{\pi}\, t^{3/2}\sum_{i=1}^{m}\int_{\partial\Omega_i}\gamma_i^2(z)dz + 0(t^2). \tag{3 21}$$

PROOF. From the definition of $F_2(t,x,x)$ and with the help of the expression of $F_1(t,x,x)$ we deduce that

$$\int_\Omega F_2(t,x,x)dx = \sum_{i=1}^{m}\int_0^t (t-u)du\int_{\partial\Omega_i}\gamma_i(z)dz\int_{\partial\Omega_i}G_N(t-u,z,y)\gamma_i(y)G_N(u,y,z)dy. \tag{3 22}$$

We replace $\gamma_i(y)$ in the above integral by $\gamma_i(z) + 0(|z-y|)$ and split the integral into two integrals accordingly Using the following estimate for the Neumann heat kernel There exist positive constants t_0, c_1 such that for all $t < t_0$, $(x,y) \in \overline{\Omega}\times\overline{\Omega}$,

$$G_N(t,x,y) \le c_1 t^{-k/2}\exp\left\{-\frac{|x-y|^2}{c_1 t}\right\}, \tag{3.23}$$

we deduce that

$$\int_{\partial\Omega_i}|z-y|G_N(t-u,y,z)G_N(u,z,y)dy \le c_1[u(t-u)]^{-k/2}\int_{R^{k-1}}|y|\exp\left\{-\frac{c_2|y|^2 t}{u(t-u)}\right\}dy. \tag{3 24}$$

Since the integral in the right-hand side of (3.24) is bounded by $c_3 t^{-k/2}$ where c_2 and c_3 are positive constants, we deduce as $t \to 0$ that

$$\int_\Omega F_2(t,x,x)dx = \sum_{i=1}^{m}\int_{\partial\Omega_i}g(t,z)\gamma_i(z)dz + 0\left(t^{(4-k)/2}\right), \tag{3 25}$$

where

$$g(t,z) = \int_0^t (t-u)du\int_{\partial\Omega_i}G_N(t-u,y,z)G_N(u,z,y)dy. \tag{3.26}$$

The right-hand side of (3.26) can be computed by taking the first term in the series expansion of the Neumann heat kernels (see [1])

$$G_N(t-u,y,z) = 2q(t-u,y,z) \quad \text{and} \quad G_N(u,z,y) = 2q(u,z,y),$$

where

$$q(t,y,z) = (4\pi t)^{-k/2}\exp\left\{ -\frac{|y-z|^2}{4t} \right\}. \tag{3 27}$$

The explicit computation can be carried out with the help of a suitably chosen local coordinate system and the localization principle (see [1]). We leave the details of this computation to the interested reader and we content ourselves with the statement that the leading term of $g(t,z)$ is equal to the same integral in the Euclidean space, i.e ,

$$(4\pi t)^{k/2}g(t,z) = 4\int_0^t \left[\frac{t}{4\pi u(t-u)}\right]^{k/2}(t-u)du\int_{R^{k-1}}\exp\left\{ -\frac{|z-y|^2}{4(t-u)} - \frac{|y-z|^2}{4u}\right\}dy + 0(t^2). \tag{3 28}$$

After some reduction, we deduce that

$$\begin{aligned}(4\pi t)^{k/2}g(t,z) &= 2\left(\frac{t}{\pi}\right)^{1/2}\int_0^t \left(\frac{t-u}{u}\right)^{1/2}du + 0(t^2) \\ &= \sqrt{\pi}\, t^{3/2} + 0(t^2).\end{aligned} \tag{3.29}$$

On inserting (3.29) into (3.25) we arrive at the proof of Lemma 2.

Finally, our result (2.1) follows immediately from (3.17), (3.18), (3 21) and the expansion of $\Theta_N(t)$ for the Neumann conditions on $\partial\Omega_i$ $(i = 1, ..., m)$.

REFERENCES

[1] HSU, P., On the θ-function of a compact Riemannian manifold with boundary, *C.R. Acad. Sci.*, Paris, Ser. I, **309** (1989), 507-510.

[2] KAC, M., Can one hear the shape of a drum? *Amer. Math. Monthly*, **73**, Part II (1966), 1-23.

[3] McKEAN, H.P., Jr. and SINGER, I.M., Curvature and the eigenvalues of the Laplacian, *J. Diff. Geom.*, **1** (1967), 43-69

[4] ZAYED, E.M.E. and YOUNIS, A.I., An inverse problem for a general convex domain with impedance boundary conditions, *Quart. Applied Math.* **48** (1) (1990), 181-188

[5] ZAYED, E.M.E., Hearing the shape of a general convex domain, *J. Math. Anal. Appl.*, **142** (1989), 170-187

[6] ZAYED, E.M.E., Hearing the shape of a general convex domain An extension to higher dimensions, *Portugaliae Math.*, **48** (1991), 259-280

[7] ZAYED, E.M.E., Heat equation for a general convex domain in R^3 with a finite number of piecewise impedance boundary conditions, *J. Applicable Anal.*, **42** (1991), 209-220.