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Research Article

A Framework for Coxeter Spectral Classification of Finite Posets and Their Mesh Geometries of Roots

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Following our paper [Linear Algebra Appl. 433(2010), 699–717], we present a framework and computational tools for the Coxeter spectral classification of finite posets $J \equiv (J, \preceq)$. One of the main motivations for the study is an application of matrix representations of posets in representation theory explained by Drozd [Funct. Anal. Appl. 8(1974), 219–225]. We are mainly interested in a Coxeter spectral classification of posets J such that the symmetric Gram matrix $G_J := (1/2)[C_J + C_J^{\mathrm{tr}}] \in \mathbb{M}_J(\mathbb{Q})$ is positive semidefinite, where $C_J \in \mathbb{M}_J(\mathbb{Z})$ is the incidence matrix of J. Following the idea of Drozd mentioned earlier, we associate to J its Coxeter matrix $\mathrm{Cox}_J := -C_J \cdot C_J^{-\mathrm{tr}}$, its Coxeter spectrum $\mathbf{specc}_J$, a Coxeter polynomial $\mathrm{cox}_J(t) \in \mathbb{Z}[t]$, and a Coxeter number $\mathbf{c}_J$. In case G_J is positive semi-definite, we also associate to J a reduced Coxeter number $\check{\mathbf{c}}_J$, and the defect homomorphism $\partial_J : \mathbb{Z}^J \to \mathbb{Z}$. In this case, the Coxeter spectrum $\mathbf{specc}_J$ is a subset of the unit circle and consists of roots of unity. In case G_J is positive semi-definite of corank one, we relate the Coxeter spectral properties of the posets J with the Coxeter spectral properties of a simply laced Euclidean diagram $DJ \in \{\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ associated with J. Our aim of the Coxeter spectral analysis of such posets J is to answer the question when the Coxeter type $\mathbf{Ctype}_J := (\mathbf{specc}_J, \mathbf{c}_J, \check{\mathbf{c}}_J)$ of J determines its incidence matrix C_J (and, hence, the poset J) uniquely, up to a $\mathbb{Z}$-congruency. In connection with this question, we also discuss the problem studied by Horn and Sergeichuk [Linear Algebra Appl. 389(2004), 347–353], if for any $\mathbb{Z}$-invertible matrix $A \in \mathbb{M}_n(\mathbb{Z})$, there is $B \in \mathbb{M}_n(\mathbb{Z})$ such that $A^{\mathrm{tr}} = B^{\mathrm{tr}} \cdot A \cdot B$ and $B^2 = E$ is the identity matrix.

1. Introduction

In the present paper, we continue our Coxeter spectral study of finite posets, started in [1], in a close connection with the Coxeter spectral technique introduced in [2–4] for acyclic edge-bipartite graphs or signed graphs in the sense of [5]. We also follow some of the techniques of representation theory, graph combinatorics, and the spectral graph theory; see [6–31].

Here, we use the terminology and notation introduced in [1, 4, 26–28]. We denote by $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ the set of nonnegative integers, the ring of integers, and the rational number field. Given $m \geq 1$, we view $\mathbb{Z}^m$ as a free abelian group and denote by $e_1, \ldots, e_m$ the standard $\mathbb{Z}$-basis of $\mathbb{Z}^m$. Given an index set J, we denote by $\mathbb{Z}^J$ the abelian group of all vectors $v = (v_j)_{j \in J}$, with integer coordinates $v_j \in \mathbb{Z}$, by $\mathbb{M}_J(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square J by J integral matrices, and by $E \in \mathbb{M}_J(\mathbb{Z})$ the identity matrix. In particular, $\mathbb{M}_m(\mathbb{Z})$, with $m \geq 1$, is the $\mathbb{Z}$-algebra of all square m by m matrices. The group

$$\mathrm{Gl}(m, \mathbb{Z}) := \{A \in \mathbb{M}_m(\mathbb{Z}), \det A \in \{-1, 1\}\} \subseteq \mathbb{M}_m(\mathbb{Z}) \quad (1)$$

is called the (integral) *general linear group*. We say that two square rational matrices $A, A' \in \mathbb{M}_m(\mathbb{Q})$ are $\mathbb{Z}$-*equivalent*, or $\mathbb{Z}$-*congruent*, (and denote $A \sim_{\mathbb{Z}} A'$) if there is a matrix $B \in \mathrm{Gl}(m, \mathbb{Z})$ such that $A' = B^{\mathrm{tr}} \cdot A \cdot B$. By a poset $J \equiv (J, \preceq)$ we mean a finite partially ordered set J with respect to a partial order relation $\preceq$. Following [26], a poset J is called a *one-peak poset* if J has a unique maximal element $*$. A finite poset J is uniquely determined by its *incidence matrix* $C_J \in \mathbb{M}_J(\mathbb{Z})$, that is, the square $J \times J$ matrix, as follows:

$$C_J = [c_{ij}]_{i, j \in J} \in \mathbb{M}_J(\mathbb{Z}), \quad \text{with } c_{ij} = \begin{cases} 1, & \text{for } i \preceq j, \\ 0, & \text{for } i \npreceq j. \end{cases} \quad (2)$$

Following an idea of Drozd [32] (developed in [27]), we have introduced in [1, 28] the *Tits matrix* $\widehat{C}_J \in \mathbb{M}_J(\mathbb{Z})$ of J to be the integral matrix

$$\widehat{C}_J = \left[\widehat{c}_{ij}\right]_{i,j\in J} \in \mathbb{M}_J(\mathbb{Z}),$$
$$\text{with } \widehat{c}_{ij} \begin{cases} 1, & i = j \text{ or } j \preceq i, i, j \notin \max J, \\ 0, & i, j \text{ incomparable, or } i \preceq j \text{ and } i, j \notin \max J, \\ -1, & \text{if } i \prec j \text{ and } j \in \max J, \end{cases} \tag{3}$$

where $\max J$ is the set of all maximal elements of J. Usually, we equip the elements of J with a numbering; that is, J is viewed as $J = \{a_1, \ldots, a_m\}$, $m = |J| \geq 1$. Throughout, we fix such a numbering and make the identifications $\mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z})$ and $\mathbb{Z}^m \equiv \mathbb{Z}^J$. The incidence matrix $C_J \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z})$ and the Tits matrix $\widehat{C}_J \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z})$ depend on the numbering of $a_1, \ldots, a_m$. Namely, if $J' = \{a'_1, \ldots, a'_m\}$ is obtained from $J = \{a_1, \ldots, a_m\}$ by a permutation $\sigma \in \mathbf{S}_m$ and $\widehat{\sigma} \in \mathrm{Gl}(m, \mathbb{Z})$ is the permutation matrix of σ, then

$$C_{J'} = \widehat{\sigma}^{-1} \cdot C_J \cdot \widehat{\sigma}, \qquad \widehat{C}_{J'} = \widehat{\sigma}^{-1} \cdot \widehat{C}_J \cdot \widehat{\sigma}. \tag{4}$$

Note that any poset J admits an *upper-triangular numbering* $J = \{a_1, \ldots, a_m\}$; that is, $a_i \preceq a_j$ implies $i \leq j$. In this case, $C_J \in \mathbb{M}_m(\mathbb{Z})$ is an upper-triangular matrix with 1 on the main diagonal, and, hence, $\det C_J = 1$, and $\det C_{J'} = 1$, for any numbering $J' = \{a'_1, \ldots, a'_m\}$.

Fix a numbering $a_1, \ldots, a_m$ of elements of J. Following [1, 28], by the *Euler matrix* of the poset J we mean the inverse

$$\overline{C}_J := C_J^{-1} \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z}) \tag{5}$$

of C_J. Following [3, 4], we call

$$\begin{gathered} Ad_J := C_J + C_J^{\mathrm{tr}} - 2 \cdot E, \\ P_J(t) = \det\left(t \cdot E - Ad_J\right) \in \mathbb{Z}[t] \end{gathered} \tag{6}$$

the *symmetric adjacency matrix* and the *characteristic polynomial* of the poset J. The set $\mathbf{spec}_J$ of all $m = |J|$ real roots of $P_J(t)$ is defined to be the (real) *spectrum* of the poset J.

We denote by $q_J, \widehat{q}_J, \overline{q}_J : \mathbb{Z}^J \equiv \mathbb{Z}^m \to \mathbb{Z}$ the *incidence* quadratic form, the *Tits* quadratic form, and the *Euler* quadratic form of J defined by the formulae

$$\begin{gathered} q_J(x) = x \cdot C_J \cdot x^{\mathrm{tr}} = \sum_{j\in J} x_j^2 + \sum_{i\prec j} x_i x_j, \\ \widehat{q}_J(x) = x \cdot \widehat{C}_J \cdot x^{\mathrm{tr}} = \sum_{j\in J} x_j^2 + \sum_{i\prec j\in \check{J}} x_i x_j - \sum_{p\in\max J} \sum_{i\prec p} x_i x_p, \\ \overline{q}_J(x) = x \cdot \overline{C}_J \cdot x^{\mathrm{tr}} = x \cdot C_J^{-1} \cdot x^{\mathrm{tr}}, \end{gathered} \tag{7}$$

respectively, where $\check{J} = J \setminus \max J$, $\max J$ is the set of all maximal elements in J, and $\widehat{C}_J \in \mathbb{M}_J(\mathbb{Z})$ is the Tits matrix of J; see (27) and [1, 28] for a definition. The matrices

$$\begin{gathered} G_J := \frac{1}{2}\left[C_J + C_J^{\mathrm{tr}}\right], \qquad \widehat{G}_J := \frac{1}{2}\left[\widehat{C}_J + \widehat{C}_J^{\mathrm{tr}}\right], \\ \overline{G}_J := \frac{1}{2}\left[\overline{C}_J + \overline{C}_J^{\mathrm{tr}}\right] \in \mathbb{M}_J(\mathbb{Q}), \end{gathered} \tag{8}$$

with rational coefficients, are called the *symmetric incidence Gram matrix*, the *symmetric Tits-Gram matrix*, and the *symmetric Euler-Gram matrix* of J. The matrices

$$\begin{gathered} \widehat{A}d_J := \widehat{C}_J + \widehat{C}_J^{\mathrm{tr}} - 2 \cdot E, \\ \overline{A}d_J := \overline{C}_J + \overline{C}_J^{\mathrm{tr}} - 2 \cdot E = C_J^{-1} + C_J^{-\mathrm{tr}} - 2 \cdot E, \end{gathered} \tag{9}$$

with integer coefficients, are called the *Tits adjacency matrix*, and the *Euler adjacency matrix* of J. The polynomials

$$\begin{gathered} P_J(t) := \det\left(t \cdot E - Ad_J\right) = \det\left(t \cdot E - \widehat{A}d_J\right), \\ \overline{P}_J(t) := \det\left(t \cdot E - \overline{A}d_J\right) \end{gathered} \tag{10}$$

are called the *characteristic polynomial* of J and the *Euler-characteristic polynomial* of J, respectively.

Example 1. (a) If I is the poset

$$I: \begin{tikzcd} & o_3 \arrow[dr] & \\ o_1 \arrow[ur] \arrow[r] & o_2 \arrow[r] & o_4 \end{tikzcd} \tag{11}$$

then $P_I(t) = \overline{P}_I(t) = t^4 - 5t^2 - 4t$; that is, the characteristic polynomial $P_I(t)$ of I coincides with the Euler-characteristic polynomial $\overline{P}_I(t)$ of I.

(b) If J is the poset

$$J: \begin{tikzcd} & & \overset{4}{\bullet} & & \\ \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} \arrow[u] \arrow[r] & \underset{5}{\bullet} \arrow[r] & \underset{6}{\bullet} \end{tikzcd}; \tag{12}$$

of the Dynkin type $\mathbb{E}_6$, then the characteristic polynomial $P_J(t)$ of J does not coincide with the Euler-characteristic polynomial $\overline{P}_J(t)$ of J, because

$$\begin{gathered} P_J(t) = t^6 - 13t^4 - 26t^3 - 15t^2 + 2t + 3, \\ \overline{P}_J(t) = t^6 - 5t^4 + 5t^2 - 1. \end{gathered} \tag{13}$$

Following [17, 33], we introduce the following definition.

Definition 2. (a) We define a poset J to be *positive* (resp., *nonnegative*) if the incidence form $q_J : \mathbb{Z}^J \to \mathbb{Z}$ of J is positive (resp., nonnegative); that is, $q_J(v) > 0$, for any nonzero $v \in \mathbb{Z}^J$ (resp., $q_J(v) \geq 0$, for any $v \in \mathbb{Z}^J$).

(b) We define a poset J to be *principal* if its incidence form $q_J : \mathbb{Z}^J \to \mathbb{Z}$ is principal in the sense of [34, Definition 2.1]; that is, q_J is nonnegative, not positive, and the kernel

$$\operatorname{Ker} q_J := \left\{ v \in \mathbb{Z}^J;\ q_J(v) = 0 \right\} \tag{14}$$

is an infinite cyclic subgroup of $\mathbb{Z}^J$.

Following the main idea of the Coxeter spectral analysis of acyclic edge-bipartite graphs (signed graphs) presented in [3, 4], we study finite posets J (with a fixed numbering $J = \{a_1, \ldots, a_m\}$) by means of the *Coxeter spectrum*

$$\mathbf{specc}_J \subseteq \mathbb{C} \tag{15}$$

of J, that is, the set $\mathbf{specc}_J$ of all $m = |J|$ eigenvalues of the *Coxeter matrix*

$$\operatorname{Cox}_J := -C_J \cdot C_J^{-\mathrm{tr}} \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z}) \tag{16}$$

of J, or equivalently, the set $\mathbf{specc}_J$ of all $m = |J|$ roots of the *Coxeter polynomial*

$$\begin{aligned} \operatorname{cox}_J(t) &:= \det\left(t \cdot E - \operatorname{Cox}_J\right) = \det\left(t \cdot E - \widehat{\operatorname{C}}\operatorname{ox}_J\right) \\ &= \det\left(t \cdot E - \overline{\operatorname{C}}\operatorname{ox}_J\right) \in \mathbb{Z}[t]; \end{aligned} \tag{17}$$

see (31) and [1]. It follows from (4) that the Coxeter spectrum $\mathbf{specc}_J$ of J and the spectrum $\mathbf{spec}_J$ of J do not depend on the numbering of the elements of the poset J.

A motivation. We recall from [26, 27] that the problems we study in the paper have a bimodule matrix problem interpretation and have essential applications in reducing some classes of partitioned matrices with coefficients in a field K to their canonical forms. For simplicity of its presentation, we illustrate it in case when $\widehat{q}_J(x)$ is the Tits quadratic form (7) of the poset $J = \{a_1, \ldots, a_n, *, +\}$, with an upper-triangular partial order $\preceq$ such that J has precisely two maximal elements $* := *_{n+1}$ and $+ := +_{n+2}$. In this case, we have

$$\widehat{q}_J(x) = \sum_{a_i \in J} x_i^2 + \sum_{a_i \prec a_j, i, j \le n} x_i x_j - \sum_{a_i \prec *} x_i x_* - \sum_{a_j \prec +} x_j x_+. \tag{18}$$

Fix a vector $v = (v_1, \ldots, v_n, v_*, v_+) \in \mathbb{N}^{n+2} \equiv \mathbb{N}^J$, and consider the K-vector space $\mathbf{Mat}_v^J$ of all partitioned matrices of the form (compare with [27])

$$A = \begin{array}{c} \begin{array}{|c|c|c|c|} \hline A_{1*} & A_{2*} & \cdots & A_{n*} \\ \hline A_{1+} & A_{2+} & \cdots & A_{n+} \\ \hline \end{array} \begin{array}{l} \}\, v_* \\ \}\, v_+ \end{array} \\ \begin{array}{cccc} \underbrace{\qquad}_{v_1} & \underbrace{\qquad}_{v_2} & \cdots & \underbrace{\qquad}_{v_n} \end{array} \end{array} \tag{19}$$

with coefficients in K, where $A_{i*} = 0$ if $a_i \npreceq *$ and $A_{j+} = 0$ if $a_j \npreceq +$. Consider the group $\mathbf{G}_v^J$ generated by the elementary transformations of the following three types:

(a) all simultaneous transformations on rows inside each horizontal block;

(b) all simultaneous transformations on columns inside each vertical block;

(c) all simultaneous transformations on columns from the ith column block to jth column block, if the relation $a_i \preceq a_j$ holds in the poset $J \setminus \{*, +\}$ (with natural zero-adjustments).

It follows from [27, Section 2] (see also [16, 26, 32]) that the problem of finding canonical forms of matrices in $\mathbf{Mat}_v^J$, with respect to the elementary transformations from the set $\mathbf{G}_v^J$, is controlled by the Tits quadratic form $\widehat{q}_J$ in the following sense. For any $v \in \mathbb{N}^J$, there is only a finite number $\mathbf{G}_v^J$-canonical forms of matrices in $\mathbf{Mat}_v^J$ if and only if the form $\widehat{q}_J$ is weakly positive; that is, $\widehat{q}_J(v)$ is positive, for all nonzero vectors $v \in \mathbb{N}^J$. Moreover, there is one-to-one correspondence between the irreducible $\mathbf{G}_v^J$-canonical forms in $\mathbf{Mat}_v^J$ and the vectors $v \in \mathbb{N}^J$ satisfying the equation $\widehat{q}_J(v) = 1$. A solution of the problem is given in [27] and [1, Theorem 1.6]. A useful homological interpretation (in terms of the Euler characteristic) of the $\mathbb{Z}$-bilinear Tits form $\widehat{b}_J(x, y) = x \cdot \widehat{C}_J \cdot y^{\mathrm{tr}}$ (26) and $\mathbb{Z}$-bilinear Euler form $\overline{b}_J(x, y) = x \cdot \overline{C}_J \cdot y^{\mathrm{tr}}$ is given in [1, (1.3)]. The reader is referred to [6–8, 25] for a detailed study and a solution of other important matrix problems of high computational complexity that have many useful applications in representation theory; see [16, 26].

We show in Section 3 that the Coxeter spectral analysis of principal posets J essentially uses the Coxeter spectra of the simply laced Euclidean diagrams presented in Figure 1.

The nonsymmetric *Gram matrix* $\check{G}_\Delta$ of any graph $\Delta = (\Delta_0, \Delta_1) \in \{\widetilde{\mathbb{D}}_n, n \ge 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ of Figure 1, with the set of vertices $\Delta_0 = \{1, \ldots, n, n+1\}$ and the set of edges Δ_1, is defined to be the matrix

$$\check{G}_\Delta = \begin{bmatrix} 1 & d_{12}^\Delta & \ldots & d_{1n}^\Delta & d_{1n+1}^\Delta \\ 0 & 1 & \ldots & d_{2n}^\Delta & d_{2n+1}^\Delta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & d_{nn+1}^\Delta \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{M}_{n+1}(\mathbb{Z}), \tag{20}$$

where $d_{ij}^\Delta = -|\Delta_1(i, j)|$, if there is an edge $\bullet_i - \bullet_j$ and $i \le j$. We set $d_{ij}^\Delta = 0$, if $\Delta_1(i, j)$ is empty or $j < i$.

The Coxeter polynomial $\operatorname{cox}_\Delta(t) := \det(t \cdot E + \check{G}_\Delta \cdot \check{G}_\Delta^{-\mathrm{tr}})$ of any diagram $\Delta = (\Delta_0, \Delta_1) \in \{\widetilde{\mathbb{D}}_n, n \ge 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ does not depend on the numbering of the vertices in Δ_0 and is presented in (48). If $n \ge 1$ and $\Delta = \widetilde{\mathbb{A}}_n$, the Coxeter polynomial $\operatorname{cox}_\Delta(t) := \det(t \cdot E + \check{G}_\Delta \cdot \check{G}_\Delta^{-\mathrm{tr}})$ of Δ depends on the numbering of the vertices in Δ_0 and is one of

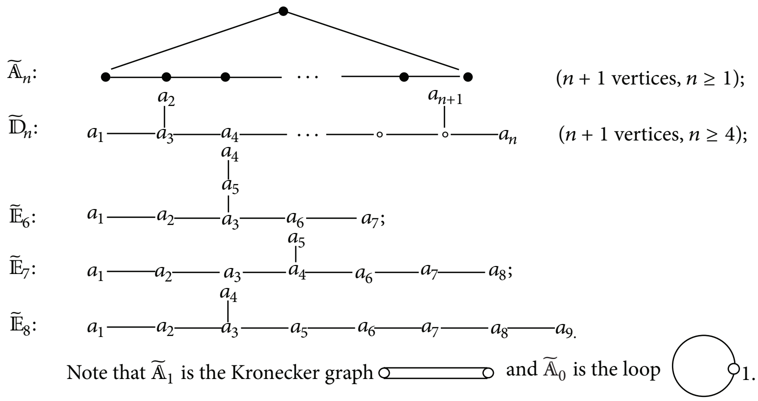


Figure 1: Simply laced Euclidean (extended Dynkin) diagrams.

the polynomials $F_{\Delta}^{(1)}(t), F_{\Delta}^{(2)}(t), \dots, F_{\Delta}^{(m_n)}(t)$ presented in [4], where

$$F_{\Delta}^{(j)}(t) = t^{n+1} - t^{n-j+1} - t^j + 1 = (t-1)^2 \cdot \mathfrak{v}_j(t) \cdot \mathfrak{v}_{n-j+1}(t),$$
$$m_n = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n+1}{2}, & \text{if } n+1 \text{ is even}, \end{cases} \tag{21}$$

for $j = 1, \dots, m_n$, and $\mathfrak{v}_m(t) = t^{m-1} + t^{m-2} + \cdots + t^2 + t + 1$. In particular, if $n+1$ is even and $j = m_n = (n+1)/2$, then $t^{n-j+1} = t^j$ and

$$F_{\Delta}^{(m_n)}(t) = F_{\Delta}^{((n+1)/2)}(t) = t^{n+1} - 2t^{(n+1)/2} + 1. \tag{22}$$

Following [4, 21], we associate (in Section 2) to any principal poset J a simply laced Euclidean diagram $DJ \in \{\widetilde{\mathbb{A}}_n, n \geq 1, \widetilde{\mathbb{D}}_n, n \geq 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ such that the incidence symmetric Gram matrix $G_J := (1/2)[C_J + C_J^{tr}] \in \mathbb{M}_J(\mathbb{Q})$ is $\mathbb{Z}$-congruent to the *symmetric Gram matrix*

$$G_{DJ} := \frac{1}{2}\left[\check{G}_{DJ} + \check{G}_{DJ}^{tr}\right] \in \mathbb{M}_{DJ}(\mathbb{Q}) = \mathbb{M}_J(\mathbb{Q}) \tag{23}$$

of DJ; that is, there is a $\mathbb{Z}$-invertible matrix B such that $G_{DJ} = B^{tr} \cdot G_J \cdot B$.

One of the aims of the Coxeter spectral analysis of nonnegative finite posets is to study the question when the Coxeter type

$$\mathbf{Ctype}_J := \left(\mathbf{specc}_J, \mathbf{c}_J, \check{\mathbf{c}}_J\right) \tag{24}$$

of a poset J determines the matrix C_J (and, hence, the poset J) uniquely, up to a $\mathbb{Z}$-congruency. Here, we set $\check{\mathbf{c}}_J = \mathbf{c}_J$, if J is positive. In other words, we claim that, for any pair J, I of nonnegative one-peak posets, $\mathbf{Ctype}_J = \mathbf{Ctype}_I$ if and only if the incidence matrices C_J and C_I are $\mathbb{Z}$-congruent. We also study the problem related with the results proved by Horn and Sergeichuk [35], if for any $\mathbb{Z}$-invertible matrix $A \in \mathbb{M}_n(\mathbb{Z})$, there exists $B \in \mathbb{M}_n(\mathbb{Z})$ such that $A^{tr} = B^{tr} \cdot A \cdot B$ and $B^2 = E$ is the identity matrix; see [17, 18].

The main results of the present paper on nonnegative posets J can be summarised as follows:

(1) canonical equivalences between the incidences, Tits, and Euler quadratic form (and corresponding Coxeter transformations and Coxeter spectra) of any poset J, established in Proposition 5;

(2) a characterization of principal posets given in Section 3. We show that a connected poset J is principal if and only if there exists a simply laced Euclidean diagram

$$DJ \in \left\{\widetilde{\mathbb{D}}_n, n \geq 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\right\} \tag{25}$$

such that the symmetric Gram matrix $G_J := (1/2)[C_J + C_J^{tr}] \in \mathbb{M}_J(\mathbb{Q})$ of J is $\mathbb{Z}$-congruent to the symmetric Gram matrix $G_{DJ} := (1/2)[\check{G}_{DJ} + \check{G}_{DJ}^{tr}] \in \mathbb{M}_{DJ}(\mathbb{Q})$ of DJ. Moreover, we show in Section 3 that, given a connected principal poset J, the Coxeter spectrum $\mathbf{specc}_J$ is a subset of a unit circle $\mathcal{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$, $1 \in \mathbf{specc}_J$, and any $z \in \mathbf{specc}_J$ is a root of unity;

(3) a Coxeter spectral classification result (Corollary 11) asserting that, given a pair I, J of one-peak principal posets with at most 13 elements, the following conditions are equivalent:

(3a) $DI = DJ$,

(3b) $\mathbf{specc}_I = \mathbf{specc}_J$,

(3c) $\check{\mathbf{c}}_I = \check{\mathbf{c}}_J$ and $|I| = |J|$,

(3d) the incidence matrix $C_J \in \mathbb{M}_J(\mathbb{Z})$ is $\mathbb{Z}$-congruent to the incidence matrix $C_I \in \mathbb{M}_I(\mathbb{Z})$; that is, there is a $\mathbb{Z}$-invertible matrix B such that $C_I = B^{tr} \cdot C_J \cdot B$.

In Section 3, we study principal posets by means of the defect and the reduced Coxeter number, and in Section 4, we present a framework for the study of nonnegative posets of corank $r \geq 2$ by means of their defect and the reduced Coxeter number. Examples are given in Sections 3–5.

The reader is referred to [1, 16, 17, 26] for a background of poset representation theory and elementary introduction to the poset matrix problems.

2. A Framework for the Coxeter Spectral Analysis of Finite Posets

The quadratic wanderings on finite posets J studied in [1] are playing a key role in the representation theory of posets, algebras, and coalgebras, as well as in the algebraic combinatorics of posets; see [6, 9–14, 16, 24–26, 28, 31, 32, 36–39]. Except for the incidence wandering and the Euler wanderings defined by the incidence matrix $C_J \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z})$ (2), with $\det C_J = 1$ and a fixed numbering $J = \{a_1, \ldots, a_m\}$, as well as the Euler matrix $\overline{C}_J := C_J^{-1}$, we study in [1, 26–28] the Tits wandering defined by the *Tits matrix* $\widehat{C}_J \in \mathbb{M}_m(\mathbb{Z}) \equiv \mathbb{M}_J(\mathbb{Z})$ of J (see [28, (3.6)]), that is, the Gram matrix of the *Tits* $\mathbb{Z}$*-bilinear form* $\widehat{b}_J : \mathbb{Z}^J \times \mathbb{Z}^J \to \mathbb{Z}$ given by

$$\widehat{b}_J(x, y) = \sum_{a_i \in J} x_i y_i + \sum_{a_j \prec a_i \in \check{J}} x_i y_j - \sum_{a_p \in \max J} \sum_{a_i \prec a_p} x_i y_p = x \cdot \widehat{C}_J \cdot y^{\mathrm{tr}}, \tag{26}$$

where $\max J$ is the set of all maximal elements in the poset J and $\check{J} := J \setminus \max J$. We call $\widehat{q}_J(x) := \widehat{b}_J(x, x) = x \cdot \widehat{C}_J \cdot x^{\mathrm{tr}}$ the *Tits quadratic form* of J.

A homological interpretation of the $\mathbb{Z}$-bilinear forms $\widehat{b}_J(x, y) = x \cdot \widehat{C}_J \cdot y^{\mathrm{tr}}$ and $\overline{b}_J(x, y) = x \cdot \overline{C}_J \cdot y^{\mathrm{tr}}$ is given in [1, (1.3)]. For a geometric interpretation of the Tits form $\widehat{q}_I$ of a one-peak poset I, the reader is referred to Drozd [32] and Simson [26].

Note that, given a one-peak poset I of the form $I = \{1, 2, \ldots, n, * = n+1\}$, with a unique maximal element $* = n+1$, we have

$$\widehat{C}_I = \left[\begin{array}{c|c} C_T^{\mathrm{tr}} & -u \\ \hline 0 & 1 \end{array}\right] \in \mathbb{M}_{n+1}(\mathbb{Z}), \quad \text{with } u = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{27}$$

where $C_T \in \mathbb{M}_T(\mathbb{Z}) = \mathbb{M}_n(\mathbb{Z})$ is the incidence matrix of the poset $T = I \setminus \{*\} = \{1, 2, \ldots, n\}$; see [26]. Note that $\widehat{q}_I(x) = x \cdot \widehat{C}_I \cdot x^{\mathrm{tr}}$.

Now, we show that, in the Coxeter spectral study of finite posets J, we can use the Coxeter spectral technique introduced in [2, 4], for the edge-bipartite graphs (signed graphs [5]), and developed in [2, 34, 40] for the matrix morsifications of unit quadratic forms.

Following [3, 4, 24], by an *edge-bipartite graph* (bigraph, in short), we mean a pair $\Delta = (\Delta_0, \Delta_1)$, where Δ_0 is a finite nonempty set of vertices and Δ_1 is a finite set of edges equipped with a bipartition $\Delta_1 = \Delta_1^- \cup \Delta_1^+$ such that the set $\Delta_1(i, j) = \Delta_1^-(i, j) \cup \Delta_1^+(i, j)$ of edges connecting the vertices i and j does not contain edges lying in $\Delta_1^-(i, j) \cap \Delta_1^+(i, j)$, for each pair of vertices $i, j \in \Delta_0$, and either $\Delta_1(i, j) = \Delta_1^-(i, j)$ or $\Delta_1(i, j) = \Delta_1^+(i, j)$. Note that the edge-bipartite graphs can be viewed as signed multigraphs satisfying a separation property; see [4, 5].

We visualize Δ as a graph in a Euclidean space $\mathbb{R}^m$, $m \geq 2$, with the vertices numbered by the integers $1, \ldots, n$; usually, we simply write $\Delta_0 = \{1, \ldots, n\}$. An edge in $\Delta_1^-(i, j)$ is visualised as a continuous one $\bullet_i \!—\! \bullet_j$, and an edge in $\Delta_1^+(i, j)$ is visualised as a dotted one $\bullet_i \text{- - -} \bullet_j$. A bigraph Δ is said to be *loop-free* if it has no loops.

We view any finite graph $\Delta = (\Delta_0, \Delta_1)$ as an edge-bipartite one by setting $\Delta_1^-(i, j) = \Delta_1(i, j)$ and $\Delta_1^+(i, j) = \emptyset$, for each pair of vertices $i, j \in \Delta_0$.

To any loop-free edge-bipartite graph $\Delta = (\Delta_0, \Delta_1)$, with a fixed numbering $\Delta_0 = \{a_1, \ldots, a_m\}$ of its vertices, we associate the upper-triangular nonsymmetric *Gram matrix* $\check{G}_\Delta = E + [d_{ij}^\Delta] \in \mathbb{M}_m(\mathbb{Z})$ of the form (20), with $m := n + 1$, where $d_{ij}^\Delta = -|\Delta_1^-(i, j)|$, if there is an edge $\bullet_i \!—\! \bullet_j$ and $i \leq j$, $d_{ij}^\Delta = |\Delta_1^+(i, j)|$, if there is an edge $\bullet_i \text{- - -} \bullet_j$ and $i \leq j$. We set $d_{ij}^\Delta = 0$, if $\Delta_1(i, j)$ is empty or $j < i$. Since Δ is loop-free, we have $d_{11}^\Delta = \cdots = d_{mm}^\Delta = 0$ and the main diagonal of $\check{G}_\Delta$ consists of unities.

Following [4], we call $\Delta = (\Delta_0, \Delta_1)$ *positive* (resp., *nonnegative*), if the symmetric Gram matrix

$$G_\Delta := (1/2)\left(\check{G}_\Delta + \check{G}_\Delta^{\mathrm{tr}}\right) \tag{28}$$

of Δ is positive definite (resp., positive semidefinite).

Following [4], we associate to any loop-free edge-bipartite graph Δ, with $|\Delta_0| = n \geq 2$, the *Coxeter spectrum* $\mathbf{specc}_\Delta \subseteq \mathbb{C}$ defined to be the spectrum of the Coxeter (-Gram) matrix

$$\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-\mathrm{tr}} \in \mathbb{M}_n(\mathbb{Z}), \tag{29}$$

the Coxeter polynomial

$$\mathrm{cox}_\Delta(t) := \det\left(t \cdot E - \mathrm{Cox}_\Delta\right) \in \mathbb{Z}[t], \tag{30}$$

the Coxeter transformation $\Phi_\Delta : \mathbb{Z}^n \to \mathbb{Z}^n$, given by $x \mapsto \Phi_\Delta(x) := x \cdot \mathrm{Cox}_\Delta$, the Coxeter number $\mathbf{c}_\Delta$ (the order of Φ_Δ in the automorphism group of $\mathbb{Z}^n$, i.e., the minimal integer $r \geq 1$ such that $\Phi_\Delta^r = E$), the $\mathbb{Z}$-bilinear Gram form $b_\Delta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ of Δ given by $b_\Delta(x, y) := x \cdot \check{G}_\Delta \cdot y^{\mathrm{tr}}$, and the integral unit quadratic form

$$q_\Delta(x) := b_\Delta(x, x) = x_1^2 + \cdots + x_n^2 + \sum_{i<j} d_{ij}^\Delta x_i x_j = x \cdot G_\Delta \cdot x^{\mathrm{tr}}. \tag{31}$$

Conversely, following Ovsienko [24], to any integral unit form

$$q(x) = x_1^2 + \cdots + x_n^2 + \sum_{i<j} q_{ij} x_i x_j, \quad \text{with } q_{ij} \in \mathbb{Z}, \tag{32}$$

we associate the loop-free bigraph $\mathbf{bigr}(q)$ of q as follows (see also [34, 41]):

(a) the vertices of $\mathbf{bigr}(q)$ are the integers $1, \ldots, n$,

(b) two vertices $i \neq j$ are joined by $-q_{ij}$ continuous edges of the form $\bullet_i \!—\! \bullet_j$ if q_{ij} is negative, and by q_{ij} dotted edges of the form $\bullet_i \text{- - -} \bullet_j$, if q_{ij} is positive,

(c) there is no edge between i and j, if $q_{ij} = 0$, or $i = j$.

To any poset $J \equiv (J, \preceq)$, with a fixed numbering $J = \{a_1, \ldots, a_m\}$ of its points, we associate the following three edge-bipartite graphs:

$$\Delta_J := \mathbf{bigr}(q_J), \qquad \widehat{\Delta}_J := \mathbf{bigr}(\widehat{q}_J), \qquad \overline{\Delta}_J := \mathbf{bigr}(\overline{q}_J), \tag{33}$$

where $\mathbf{bigr}(q_J)$, $\mathbf{bigr}(\widehat{q}_J)$, and $\mathbf{bigr}(\overline{q}_J)$ are the bigraphs of the quadratic forms q_J, $\widehat{q}_J$, and $\overline{q}_J$, respectively; see (7). More precisely, the bigraphs (33) are defined as follows.

(i) The set of vertices of each of the bigraphs Δ_J, $\widehat{\Delta}_J$, and $\overline{\Delta}_J$ is the enumerated set $J = \{a_1, \ldots, a_m\}$.

(ii) There is an edge $a_i - - - a_j$ in Δ_J, if $a_i \prec a_j$ or $a_j \prec a_i$ holds in J.

(iii) There is an edge $a_i - - - a_j$ in $\widehat{\Delta}_J$, if a_i and a_j are not maximal in J and $a_i \prec a_j$ or $a_j \prec a_i$ holds in J. There is an edge $a_i — a_j$ in $\widehat{\Delta}_J$, if $a_i \prec a_j$ holds and a_j is maximal in J.

(iv) Let $\overline{C}_J = C_J^{-1} = [\overline{c}_{ij}] \in \mathbb{M}_m(\mathbb{Z})$ be the Euler matrix of J. There is an edge $a_i - - - a_j$ (resp., $a_i — a_j$) in $\overline{\Delta}_J$, if $\overline{c}_{ij} > 0$ or $\overline{c}_{ji} > 0$ (resp., $\overline{c}_{ij} < 0$ or $\overline{c}_{ji} < 0$).

We call Δ_J, $\widehat{\Delta}_J$, and $\overline{\Delta}_J$ the *incidence bigraph of* Δ, the *Tits bigraph of* Δ, and the *Euler bigraph of* Δ, respectively, (with respect to the numbering $J = \{a_1, \ldots, a_m\}$).

The following simple lemma is of importance.

Lemma 3. *Assume that J is a finite poset with a fixed numbering $J = \{a_1, \ldots, a_m\}$, and let Δ_J, $\widehat{\Delta}_J$, $\overline{\Delta}_J$ be the loop-free edge-bipartite graphs associated with J in* (33).

(a) *The symmetric Gram matrices G_J, $\widehat{G}_J$, $\overline{G}_J$ are $\mathbb{Z}$-congruent to the symmetric Gram matrices G_{Δ_J}, $G_{\widehat{\Delta}_J}$, $G_{\overline{\Delta}_J}$, respectively. The rank of each of the symmetric Gram matrices G_{Δ_J}, $G_{\widehat{\Delta}_J}$, $G_{\overline{\Delta}_J}$ does not depend of the numbering $J = \{a_1, \ldots, a_m\}$ and coincides with the common rank $\operatorname{rk} G_{\Delta_J} = \operatorname{rk} G_{\widehat{\Delta}_J} = \operatorname{rk} G_{\overline{\Delta}_J}$.*

(b) *$P_J(t) = P_{\Delta_J}(t) = P_{\widehat{\Delta}_J}(t)$.*

(c) *The poset J is positive (resp., nonnegative) if and only if the bigraph Δ_J (and $\widehat{\Delta}_J$, $\overline{\Delta}_J$) is positive (resp., nonnegative).*

(d) *The poset J is principal if and only if the bigraph Δ_J (and $\widehat{\Delta}_J$, $\overline{\Delta}_J$) is principal.*

Proof. For the proof of (a), we recall that the Gram matrices $G_J, \widehat{G}_J, \overline{G}_J, G_{\Delta_J}, G_{\widehat{\Delta}_J}, G_{\overline{\Delta}_J}$ are invariant, up to $\mathbb{Z}$-congruency, under permutations of the elements $\{a_1, \ldots, a_m\}$. Since J admits an upper-triangular numbering $J' = \{a_1', \ldots, a_m'\}$ and $G_{\Delta_{J'}} = G_{J'}$, then (a) follows. The proof of remaining statements is left to the reader. □

Following the terminology used in [2–4, 34], we introduce the following definition.

Definition 4. Let J be a finite poset, with a fixed numbering $J = \{a_1, \ldots, a_m\}$.

(a) We associate with J the following three Coxeter matrices:

(a1) the (incidence) Coxeter matrix $\mathrm{Cox}_J := -C_J \cdot C_J^{-\mathrm{tr}} \in \mathbb{M}_m(\mathbb{Z})$;

(a2) the Coxeter-Tits matrix $\widehat{\mathrm{Cox}}_J := -\widehat{C}_J \cdot \widehat{C}_J^{-\mathrm{tr}} \in \mathbb{M}_m(\mathbb{Z})$;

(a3) the Coxeter-Euler matrix $\overline{\mathrm{Cox}}_J := -C_J^{-1} \cdot C_J^{\mathrm{tr}} \in \mathbb{M}_m(\mathbb{Z})$.

Moreover, we define the following three *Coxeter transformations*:

(a4) the (incidence) Coxeter transformation $\Phi_m : \mathbb{Z}^m \to \mathbb{Z}^m$ of J;

(a5) the Coxeter-Tits transformation $\widehat{\Phi}_J : \mathbb{Z}^m \to \mathbb{Z}^m$ of J;

(a6) the Coxeter-Euler transformation $\overline{\Phi}_J : \mathbb{Z}^m \to \mathbb{Z}^m$ of J, by the following formulae:

$$\Phi_J(x) = x \cdot \mathrm{Cox}_J, \qquad \widehat{\Phi}_J(x) = x \cdot \widehat{\mathrm{Cox}}_J, \qquad \text{and } \overline{\Phi}_J(x) = x \cdot \overline{\mathrm{Cox}}_J. \tag{34}$$

(b) The integral polynomial

$$\begin{aligned} \mathrm{cox}_J(t) &= \det\left(t \cdot E - \mathrm{Cox}_J\right) = \det\left(t \cdot E - \widehat{\mathrm{Cox}}_J\right) \\ &= \det\left(t \cdot E - \overline{\mathrm{Cox}}_J\right) \in \mathbb{Z}[t] \end{aligned} \tag{35}$$

is called the *Coxeter polynomial of the poset J*.

(c) The *Coxeter spectrum* of J is the set $\mathbf{specc}_J \subseteq \mathbb{C}$ of all $m = |J|$ eigenvalues of the matrix Cox_J, or, equivalently, the set $\mathbf{specc}_J$ of all $m = |J|$ roots of the Coxeter polynomial $\mathrm{cox}_J(t)$.

(d) The order $\mathbf{c}_J := \operatorname{ord}(\Phi_J)$ of the Coxeter transformation $\Phi_J : \mathbb{Z}^J \to \mathbb{Z}^J$ is called the *Coxeter number* of the poset J. In other words, $\mathbf{c}_J$ is the minimal integer $r \geq 1$ such that $\Phi_J^r = id$. We set $\mathbf{c}_J = \infty$, if $\Phi_J^r \neq id$, for any $r \geq 1$.

(e) Assume that J is nonnegative. The *Coxeter type* of J is defined to be the pair $\mathbf{Ctype}_J := (\mathbf{specc}_J, \mathbf{c}_J)$ if J is positive, and the triple $\mathbf{Ctype}_J := (\mathbf{specc}_J, \mathbf{c}_J, \check{\mathbf{c}}_J)$ if J is not positive, where $\check{\mathbf{c}}_J$ is the reduced Coxeter number of J in the sense of Theorems 10 and 18.

The following proposition shows that equality (35) holds.

Proposition 5. *Let J be a finite poset, with a fixed numbering $J = \{a_1, \ldots, a_m\}$, let $q_J, \widehat{q}_J, \overline{q}_J : \mathbb{Z}^m \to \mathbb{Z}$ be the incidence, Tits, and Euler quadratic form of J, and let $\Phi_J, \widehat{\Phi}_J, \overline{\Phi}_J : \mathbb{Z}^m \to \mathbb{Z}^m$ be the corresponding Coxeter transformations.*

(a) *The following equalities hold* $\widehat{C}_J = B \cdot \overline{C}_J \cdot B^{tr}$ *and* $C_J^{tr} = B' \cdot \overline{C}_J \cdot B'^{tr}$, *and the following diagrams are commutative*

$$
\begin{tikzcd}
\mathbb{Z}^m \arrow[rr, "\widehat{q}_J"] \arrow[d, "h_B"', "\cong"] & & \mathbb{Z} \\
\mathbb{Z}^m \arrow[urr, "\overline{q}_J"] & & \mathbb{Z}^m \arrow[ll, "\cong", "h_{B'}"'] \arrow[u, "q_J"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z}^m \arrow[r, "h_B"] \arrow[d, "\widehat{\Phi}_J"', "\cong"] & \mathbb{Z}^m \arrow[d, "\overline{\Phi}_J"', "\cong"] & \mathbb{Z}^m \arrow[l, "h_{B'}"'] \arrow[d, "\Phi_J^{-1}"', "\cong"] \\
\mathbb{Z}^m \arrow[r, "h_B"] & \mathbb{Z}^m & \mathbb{Z}^m \arrow[l, "h_{B'}"']
\end{tikzcd}
\tag{36}
$$

where $B' = C_J^{tr}$, $B = \left[\begin{array}{c|c} C_{\check{J}} & 0 \\ \hline 0 & E \end{array}\right]$, $\check{J} = J \setminus \max J$, *and* $h_B, h_{B'} : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ *are the group isomorphisms defined by the formulae* $h_B(x) = x \cdot B$ *and* $h_{B'}(x) = x \cdot B'$, *for* $x \in \mathbb{Z}^m$.

(b) $\widehat{Cox}_J = B \cdot \overline{Cox}_J \cdot B^{-1}$, $Cox_J^{-1} = Cox_{J^{op}} = B' \cdot \overline{Cox}_J \cdot B'^{-1}$, *and* $\Phi_{J^{op}} = \Phi_J^{-1}$.

(c) *The Coxeter number* $\mathbf{c}_J = ord(\Phi_J)$ *of the poset* J *coincides with the Coxeter number of* J^{op}. *Moreover,* $\mathbf{c}_J = ord(\widehat{\Phi}_J) = ord(\overline{\Phi}_J)$ *and* $cox_{J^{op}}(t) = cox_J(t)$.

(d) *Assume that* J *is connected and nonnegative.*

(d1) *If the numbering* $J = \{a_1, \dots, a_m\}$ *is upper-triangular and* Δ_J *is the bigraph* (33) *associated to* J, *then* $Cox_{\Delta_J} = Cox_J$ *and* $cox_{\Delta_J}(t) = cox_J(t)$.

(d2) *The Coxeter type* $\mathbf{Ctype}_J := (\mathbf{specc}_J, \mathbf{c}_J, \check{\mathbf{c}}_J)$ *of* J *does not depend on the numbering* $J = \{a_1, \dots, a_m\}$.

(d3) *The Coxeter spectrum* $\mathbf{specc}_J$ *is a subset of a unit circle* $\mathcal{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$, *and any* $z \in \mathbf{specc}_J$ *is a root of unity.*

(d4) *The poset* J *is positive if and only if* $1 \notin \mathbf{specc}_J$.

Proof. The first equality $\widehat{C}_J = B \cdot \overline{C}_J \cdot B^{\mathrm{tr}}$ is obvious, and the second one $C_J^{\mathrm{tr}} = B' \cdot \overline{C}_J \cdot B'^{\mathrm{tr}}$ follows by a direct calculation. Hence, (b) follows and, consequently, the diagrams (36) are commutative; see [1, Proposition 3.13]. Hence, the statement (c) follows from the commutativity of the diagrams (36).

(d1) We recall from Section 1 that, given two numberings $J = \{a_1, \dots, a_m\}$ and $J' = \{a_1', \dots, a_m'\}$ of elements in J, we have $C_{J'} = \widehat{\sigma}^{-1} \cdot C_J \cdot \widehat{\sigma}$, where $\widehat{\sigma} \in \mathrm{Gl}\,(m, \mathbb{Z})$ is the permutation matrix of a permutation $\sigma \in \mathbf{S}_m$. Hence, (d1) easily follows.

(d2) It is sufficient to note that the incidence matrix C_J is upper triangular. Hence, $C_J = \check{G}_{\Delta_J}$ and $\mathrm{Cox}_{\Delta_J} = \mathrm{Cox}_J$.

To prove (d3) and (d4), we recall from [2] and [3, Proposition 2.6] that the Coxeter spectrum $\mathbf{specc}_A$ of any matrix morsification $A \in \mathbf{Mor}_\Delta$ of a nonnegative bigraph Δ is a subset of the unit circle $\mathcal{S}^1$ and any $z \in \mathbf{specc}_A$ is a root of unity (see also [41, 42]). Moreover, Δ is positive iff $1 \notin \mathbf{specc}_A$. Assume that J is connected and nonnegative. Then, the bigraph Δ_J (33) associated to J is nonnegative, $A := \check{G}_{\Delta_J} = \nabla(C_J)$ is a morsification of Δ_J, and $\mathbf{specc}_J = \mathbf{specc}_A$, because the incidence matrix C_J is quasitriangular and [4, Proposition 2.2] applies. This completes the proof. □

Corollary 6. *For any poset* J, *equality* (35) *holds.*

Proof. Apply Proposition 5(b). □

The following example shows that the correspondence $J \mapsto \Delta_J$ defined in (33) does not preserve the Coxeter types of J and Δ_J. In particular, it shows that the equality $\mathrm{cox}_J(t) = \mathrm{cox}_{\Delta_J}(t)$ does not hold in general and the Coxeter polynomial $\mathrm{cox}_{\Delta_J}(t)$ depends on the numbering of J, whereas the Coxeter polynomial $\mathrm{cox}_J(t)$ does not depend on the numbering of J.

Example 7. Consider the poset J such that its Hasse quiver has the form

$$
\begin{array}{ll}
\mathscr{H}_J: \begin{tikzcd}[row sep=small] & 3 \\ 1 \arrow[r] & 2 \arrow[u] \arrow[d] \\ & 4 \end{tikzcd} & \mathrm{cox}_J(t) = t^4 + t^3 + t + 1 \\
\Delta_J: \begin{tikzcd}[row sep=small] & 3 \\ 1 \arrow[r, dash, dotted] \arrow[ur, dash, dotted] \arrow[dr, dash, dotted] & 2 \arrow[u, dash, dotted] \arrow[d, dash, dotted] \\ & 4 \end{tikzcd} & \mathrm{cox}_{\Delta_J}(t) = t^4 + t^3 + t + 1
\end{array}
\tag{37}
$$

By a permutation of the elements in J, we get

$$
\begin{array}{ll}
\mathscr{H}_{J'}: \begin{tikzcd}[row sep=small] & 2 \\ 3 \arrow[r] & 1 \arrow[u] \arrow[d] \\ & 4 \end{tikzcd} & \mathrm{cox}_{J'}(t) = t^4 + t^3 + t + 1 \\
\Delta_{J'}: \begin{tikzcd}[row sep=small] & 2 \\ 3 \arrow[r, dash, dotted] \arrow[ur, dash, dotted] \arrow[dr, dash, dotted] & 1 \arrow[u, dash, dotted] \arrow[d, dash, dotted] \\ & 4 \end{tikzcd} & \mathrm{cox}_{\Delta_{J'}}(t) = t^4 + 2t^2 + 1
\end{array}
\tag{38}
$$

Note that the first numbering is upper-triangular, whereas the second one is not upper-triangular.

3. Principal Posets

We recall that a poset J is *principal* if its incidence unit form q_J is principal in the sense of [34, Definition 2.1]; that is, $q_J : \mathbb{Z}^J \rightarrow \mathbb{Z}$ is nonnegative and not positive, and the kernel $\mathrm{Ker}\, q_J := \{v \in \mathbb{Z}^J; q_J(v) = 0\}$ is an infinite cyclic subgroup of $\mathbb{Z}^J$.

We start with the following useful observation.

Lemma 8. *Assume that J is a connected principal poset.*

(a) *The Coxeter number $\mathbf{c}_J$ of J is infinite.*

(b) *The Coxeter spectrum $\mathbf{specc}_J$ is a subset of a unit circle $\mathcal{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$, $1 \in \mathbf{specc}_J$, and any $z \in \mathbf{specc}_J$ is a root of unity.*

(c) *If $\operatorname{Ker} q_J = \mathbb{Z} \cdot \mathbf{h}$, then $\operatorname{Ker} \widehat{q}_J = \mathbb{Z} \cdot \widehat{\mathbf{h}}$ and $\operatorname{Ker} \overline{q}_J = \mathbb{Z} \cdot \overline{\mathbf{h}}$, where*

(i) $\overline{\mathbf{h}} = \mathbf{h} \cdot B'$, $\overline{\mathbf{h}} = \widehat{\mathbf{h}} \cdot B$, $\widehat{\mathbf{h}} = \mathbf{h} \cdot B' \cdot B^{-1}$,

(ii) $B' = C_J^{tr}$, $B = \left[\begin{array}{c|c} C_{\check{J}} & 0 \\ \hline 0 & E \end{array}\right]$, *and* $\check{J} = J \setminus \max J$

are as in Proposition 5.

Proof. (a) By Proposition 5 (d2), $\mathbf{c}_J$ is independent of the numbering of J. Then, without loss of generality, we may suppose that the numbering of J is upper-triangular. Then, by Lemma 3(d) and Proposition 5(d1), the Coxeter number $\mathbf{c}_J$ coincides with the Coxeter number of the principal edge-bipartite graph Δ_J associated with J in (33). Then, (a) is a consequence of [3, Proposition 3.12].

The statements (b) and (c) follow by applying Proposition 5 and the commutative diagram (36). □

Proposition 9. *Let J be a connected poset, $m = |J| \geq 2$, and let $G_J, \widehat{G}_J, \overline{G}_J, \in \mathbb{M}_J(\mathbb{Q})$ be the symmetric incidence Gram matrix of J, the symmetric Tits-Gram matrix of J, and the symmetric Euler-Gram matrix of J, respectively. The following five conditions are equivalent.*

(a) *The poset J is principal.*

(b) *The Gram matrix G_J is positive indefinite of rank $m-1$.*

(c) *The Tits quadratic form $\widehat{q}_J$ of J is nonnegative and $\operatorname{Ker} \widehat{q}_J = \mathbb{Z} \cdot \widehat{\mathbf{h}}$, for some nonzero vector $\widehat{\mathbf{h}} \in \mathbb{Z}^J$.*

(d) *The Euler quadratic form $\overline{q}_J$ of J is nonnegative and $\operatorname{Ker} \overline{q}_J = \mathbb{Z} \cdot \overline{\mathbf{h}}$, for some nonzero vector $\overline{\mathbf{h}} \in \mathbb{Z}^J$.*

(e) *If G is any of the symmetric Gram matrices $G_J, \widehat{G}_J, \overline{G}_J, \in \mathbb{M}_J(\mathbb{Q})$ of J, then there exists a simply laced Euclidean diagram $DJ \in \{\widetilde{\mathbb{A}}_s, s \geq 3, \widetilde{\mathbb{D}}_n, n \geq 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ (uniquely determined by J) such that the matrix G is $\mathbb{Z}$-congruent to the symmetric Gram matrix $G_{DJ} \in \mathbb{M}_{DJ}(\mathbb{Q})$ of the Euclidean diagram DJ; that is, there is a $\mathbb{Z}$-invertible matrix $B \in Gl(m, \mathbb{Z})$ such that $G_{DJ} = B^{tr} \cdot G \cdot B$.*

Proof. (a)⇔(b) If $m = |J|$ and

$$\begin{gathered} Dq_J : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m, \\ v \longmapsto Dq_J(v) = \left(\frac{\partial q_J}{\partial x_1}(v), \ldots, \frac{\partial q_J}{\partial x_m}(v)\right), \end{gathered} \tag{39}$$

is the gradient group homomorphism of q_J, then $\operatorname{Ker} q_J = \operatorname{Ker}[Dq_J : \mathbb{Z}^m \to \mathbb{Z}^m]$ and the subgroup $\operatorname{Ker} q_J$ of $\mathbb{Z}^m$ is of rank $m - \operatorname{rk} G_J$ and consists of all integral solutions of the system $2 \cdot G_J \cdot x^{\mathrm{tr}} = 0$ of linear equations with integral coefficients; see [34, Proposition 2.8]. Then, (a)⇔(b) follows.

The equivalences (a)⇔(c)⇔(d) follow from Proposition 5 (a) and the commutativity of the diagram (36).

(e)⇒(a) Assume that there exist a simply laced Euclidean diagram $DJ \in \{\widetilde{\mathbb{A}}_s, s \geq 3, \widetilde{\mathbb{D}}_n, n \geq 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and a $\mathbb{Z}$-invertible matrix $B \in \mathrm{Gl}(m, \mathbb{Z})$ such that $G_{DJ} = B^{\mathrm{tr}} \cdot G \cdot B$. It follows that the quadratic form $q_{DJ}(x) = x \cdot G_{DJ} \cdot x^{\mathrm{tr}}$ is $\mathbb{Z}$-congruent to q_J and $q_J = q_{DJ} \circ h_B$. Then, (a) is a consequence of [36, Lemma VII.4.2].

(a)⇒(e) Let $\overline{\Delta}_J$ be the Euler edge-bipartite graph defined in (33) of J. By (a) and Lemma 3 (d), $\overline{\Delta}_J$ is principal and the inflation algorithm defined in [4, 21] applies to Δ_J. Consequently, there exists a simply laced Euclidean diagram $DJ \in \{\widetilde{\mathbb{A}}_s, s \geq 3, \widetilde{\mathbb{D}}_n, n \geq 4, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and a $\mathbb{Z}$-invertible matrix $B \in \mathrm{Gl}(n, \mathbb{Z})$ defining the congruence $\overline{\Delta}_J \approx_{\mathbb{Z}} DJ$; that is, the equality $G_{DJ} = B^{\mathrm{tr}} \cdot G_{\overline{\Delta}_J} \cdot B = B^{\mathrm{tr}} \cdot G_J \cdot B$ holds. Then, in view of Proposition 5, the implication (a)⇒(e) follows from Lemma 3 (d); see also Section 6. □

Theorem 10. *Let J be a finite principal poset, with a numbering $\{a_1, \ldots, a_m\}$ of elements of J. Fix a nonzero vector $\mathbf{h}_J \in \mathbb{Z}^J \equiv \mathbb{Z}^m$ such that $\operatorname{Ker} q_J = \mathbb{Z} \cdot \mathbf{h}_J$.*

(a) *There exist a minimal integer $\check{\mathbf{c}}_J \geq 2$ (called the reduced Coxeter number of J) and a group homomorphism $\partial_J : \mathbb{Z}^J \to \mathbb{Z}$ (called the incidence defect of J) such that*

$$\begin{gathered} \Phi_J^{\check{\mathbf{c}}_J}(v) = v + \partial_J(v) \cdot \mathbf{h}_J, \\ \partial_J(\Phi_J(v)) = \partial_J(v), \quad \forall v \in \mathbb{Z}^J, \\ \partial_J(\mathbf{h}_J) = 0. \end{gathered} \tag{40}$$

(b) *Assume that $\check{\mathbf{c}}_J \geq 1$ and $\partial_J : \mathbb{Z}^J \to \mathbb{Z}$ are as in (a), and one sets $\overline{\mathbf{h}}_J = \mathbf{h}_J \cdot B'$, $\widehat{\mathbf{h}}_J = \mathbf{h}_J \cdot B' \cdot B^{-1}$, where $B', B \in \mathbb{M}_J(\mathbb{Z})$ are as in Proposition 5.*

(b1) *There exists a group homomorphism $\overline{\partial}_J : \mathbb{Z}^J \to \mathbb{Z}$ (called the Euler defect of J) such that*

$$\begin{gathered} \overline{\Phi}_J^{\check{\mathbf{c}}_J}(v) = v + \overline{\partial}_J(v) \cdot \overline{\mathbf{h}}_J, \quad \forall v \in \mathbb{Z}^J, \\ \overline{\partial}_J \circ \Phi_J = \overline{\partial}_J, \\ \overline{\partial}_J \circ h_{B'} = \partial_J, \quad \overline{\partial}_J(\overline{\mathbf{h}}_J) = 0. \end{gathered} \tag{41}$$

(b2) *There exists a group homomorphism $\widehat{\partial}_J : \mathbb{Z}^J \to \mathbb{Z}$ (called the Tits defect of J) such that*

$$\begin{gathered} \widehat{\Phi}_J^{\check{\mathbf{c}}_J}(v) = v + \widehat{\partial}_J(v) \cdot \widehat{\mathbf{h}}_J, \quad \forall v \in \mathbb{Z}^J, \\ \widehat{\partial}_J = \overline{\partial}_J \circ h_B = \partial_J \circ h_{B'}^{-1} \circ h_B, \\ \widehat{\partial}_J = \widehat{\partial}_J \circ \Phi_J, \qquad \widehat{\partial}_J(\widehat{\mathbf{h}}_J) = 0. \end{gathered} \tag{42}$$

(c) *The Coxeter number* $\mathbf{c}_J$ *of* J *is infinite, and the incidence defect* $\partial_J : \mathbb{Z}^J \to \mathbb{Z}$ *is nonzero.*

(d) *Given* $v \in \mathbb{Z}^J$, *the order* $\mathbf{s}_v := |\mathcal{O}(v)|$ *of the* Φ_J*-orbit* $\mathcal{O}(v)$ *is finite if and only if* $\partial_J(v) = 0$. *If* $\mathbf{s}_v = |\mathcal{O}(v)|$ *is finite, then* $\mathbf{s}_v$ *divides* $\check{\mathbf{c}}_J$ *and there is a unique integer* m_v *such that*

$$\begin{aligned} m_v \cdot \mathbf{h} &= v + \Phi_J(v) + \Phi_J^2(v) + \cdots + \Phi_J^{\mathbf{s}_v - 1}(v) \\ &= v + \Phi_J^{-1}(v) + \Phi_J^{-2}(v) + \cdots + \Phi_J^{-\mathbf{s}_v + 1}(v). \end{aligned} \tag{43}$$

Proof. We recall from the proof of Proposition 9 that

$$\mathbb{Z} \cdot \mathbf{h}_J = \operatorname{Ker} q_J = \operatorname{Ker}\left[Dq_J : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m\right], \tag{44}$$

where $m = |J| \geq 2$ and $Dq_J : \mathbb{Z}^m \to \mathbb{Z}^m$, $v \mapsto Dq_J(v) = ((\partial q_J/\partial x_1)(v), \ldots, (\partial q_J/\partial x_m)(v))$, is the gradient group homomorphism. It follows that $\mathbb{Z}^m/\mathbb{Z} \cdot \mathbf{h}_J \cong \operatorname{Im} Dq_J \cong \mathbb{Z}^{m-1}$. Denote by $\phi : \mathbb{Z}^m \to \mathbb{Z}^{m-1}$ the composite quotient epimorphism. Then, the form q_J induces the form $\widetilde{q}_J : \mathbb{Z}^{m-1} \to \mathbb{Z}$ such that $\widetilde{q}_J(\phi(x)) = q_J(x)$, for all $x \in \mathbb{Z}^m$. Moreover, the Coxeter transformation $\Phi_J : \mathbb{Z}^m \to \mathbb{Z}^m$ induces a group automorphism $\widetilde{\Phi}_J : \mathbb{Z}^{m-1} \to \mathbb{Z}^{m-1}$ such that

$$\widetilde{\Phi}_J \circ \phi = \phi \circ \Phi_J, \qquad \widetilde{q}_J\left(\widetilde{\Phi}_J(y)\right) = \widetilde{q}_J(y), \quad \forall y \in \mathbb{Z}^{m-1}. \tag{45}$$

It follows that $\widetilde{q}_J$ is positive definite and there exists a minimal integer $\check{\mathbf{c}}_J \geq 1$ such that $\widetilde{\Phi}_J^{\check{\mathbf{c}}_J}$ is the identity map on $\mathbb{Z}^{m-1}$. Hence, (a) follows, because the equalities $\partial_J(\mathbf{h}_J) = 0$ and $\partial_J(\Phi_J(v)) = \partial_J(v)$, for all $v \in \mathbb{Z}^J$, are almost obvious; see [34, Theorem 4.7].

In view of Proposition 5, the statements (b)–(d) are a consequence of (a) and Lemma 8(a). The reader is referred to [34, Theorem 4.7, Corollary 4.15] for more details. □

Corollary 11. *(a) If* J *is a principal connected poset with at most* 13 *elements, then its Coxeter spectrum* $\mathbf{specc}_J$ *is a subset of a unit circle* $\mathcal{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$, $1 \in \mathbf{specc}_J$, *and any* $z \in \mathbf{specc}_J$ *is an mth root of unity, where* $m \leq \check{\mathbf{c}}_J$ *and* $\check{\mathbf{c}}_J$ *is the reduced Coxeter number of* J.

(b) If I *and* J *are one-peak principal posets with at most* 13 *elements and* DI, DJ *are the associated Euclidean diagrams, then the following conditions are equivalent:*

(b1) $DI = DJ$,

(b2) $\mathbf{specc}_I = \mathbf{specc}_J$,

(b3) $\check{\mathbf{c}}_I = \check{\mathbf{c}}_J$ *and* $|I| = |J|$,

(b4) *the incidence matrix* $C_J \in \mathbb{M}_J(\mathbb{Z})$ *is* $\mathbb{Z}$*-congruent to the incidence matrix* $C_I \in \mathbb{M}_I(\mathbb{Z})$*; that is, there is a* $\mathbb{Z}$*-invertible matrix* B *such that* $C_I = B^{tr} \cdot C_J \cdot B$.

Proof. (a) By Lemma 8, $\mathbf{specc}_J \subseteq \mathcal{S}^1$ and $1 \in \mathbf{specc}_J$. Assume that DJ is the associated Euclidean diagram of J, as in Proposition 9. By a computer search (using the results of [43] and the inflation algorithm given in [4, 21]), we show that

$$\operatorname{cox}_J(t) = \operatorname{cox}_{DJ}(t), \quad \check{\mathbf{c}}_J = \check{\mathbf{c}}_{DJ}, \tag{46}$$

for any poset J, with at most 13 elements. Hence, in view of [4, Proposition 2.17], we have

$$\operatorname{cox}_J(t) = \operatorname{cox}_{DJ}(t) = F_{DJ}(t), \tag{47}$$

where

$$\begin{aligned} &F_{DJ}(t) = \\ &t^{n+1} + t^n - t^{n-1} - t^{n-2} - t^3 - t^2 + t + 1 \\ &\quad = (t-1)^2 \mathfrak{v}_2^2(t)\, \mathfrak{v}_{n-2}(t), \quad \text{for } DJ = \widetilde{\mathbb{D}}_n, \\ &t^7 + t^6 - 2t^4 - 2t^3 + t + 1 \\ &\quad = (t-1)^2 \mathfrak{v}_2(t)\, \mathfrak{v}_3^2(t), \quad \text{for } DJ = \widetilde{\mathbb{E}}_6, \\ &t^8 + t^7 - t^5 - 2t^4 - t^3 + t + 1 \\ &\quad = (t-1)^2 \mathfrak{v}_2(t)\, \mathfrak{v}_3(t)\, \mathfrak{v}_4(t), \quad \text{for } DJ = \widetilde{\mathbb{E}}_7, \\ &t^9 + t^8 - t^6 - t^5 - t^4 - t^3 + t + 1 \\ &\quad = (t-1)^2 \mathfrak{v}_2(t)\, \mathfrak{v}_3(t)\, \mathfrak{v}_5(t), \quad \text{for } DJ = \widetilde{\mathbb{E}}_8, \end{aligned} \tag{48}$$

where $\mathfrak{v}_m(t) = t^{m-1} + t^{m-2} + t^{m-3} + \cdots + t^2 + t + 1$. For $DJ \in \{\widetilde{\mathbb{D}}_4, \widetilde{\mathbb{D}}_5\}$, we have

$$F_{DJ}(t) = \begin{cases} t^5 + t^4 - 2t^3 - 2t^2 + t + 1, & \text{for } DJ = \widetilde{\mathbb{D}}_4, \\ t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1, & \text{for } DJ = \widetilde{\mathbb{D}}_5. \end{cases} \tag{49}$$

Then, (a) follows by applying [38, Lemma XIII.1.3]. Hence, we also easily conclude that the statements (b1)–(b3) are equivalent.

To finish the proof of (b), we note that the equality $C_I = B^{\mathrm{tr}} \cdot C_J \cdot B$ in (b4) implies that the matrices Cox_I and Cox_J are conjugate, and, hence, we get $\mathbf{specc}_I = \mathbf{specc}_J$; that is, the implication (b4)⇒(b2) holds. To prove the inverse implication (b2)⇒(b4), we apply the technique used in [18, Section 6]. On this way, given a principal poset J, with at most 13 elements and the associated Euclidean diagram DJ, we construct (by a computer search) a $\mathbb{Z}$-invertible matrix B_J such that $\check{G}_{DJ} = B_J^{\mathrm{tr}} \cdot C_J \cdot B_J$ (compare with [17, 18, 33, 43]). Hence, (b4) follows, and the proof is complete. □

If J is a principal poset, then the sets

$$\begin{aligned} \mathcal{R}_{q_J} &= \left\{v \in \mathbb{Z}^m;\ q_J(v) = 1\right\}, \\ \mathcal{R}_{\widehat{q}_J} &= \left\{v \in \mathbb{Z}^m;\ \widehat{q}_J(v) = 1\right\}, \\ \mathcal{R}_{\overline{q}_J} &= \left\{v \in \mathbb{Z}^m;\ \overline{q}_J(v) = 1\right\} \end{aligned} \tag{50}$$

of roots of the unit forms $q_J, \widehat{q}_J$, and $\overline{q}_J$ have the disjoint union decompositions

$$\begin{aligned}
\mathscr{R}_{q_J} &= \partial_J^- \mathscr{R}_{q_J} \cup \partial_J^+ \mathscr{R}_{q_J} \cup \partial_J^0 \mathscr{R}_{q_J}, \\
\mathscr{R}_{\widehat{q}_J} &= \widehat{\partial}_J^- \mathscr{R}_{\widehat{q}_J} \cup \widehat{\partial}_J^+ \mathscr{R}_{\widehat{q}_J} \cup \widehat{\partial}_J^0 \mathscr{R}_{\widehat{q}_J}, \\
\mathscr{R}_{\overline{q}_J} &= \overline{\partial}_J^- \mathscr{R}_{\overline{q}_J} \cup \overline{\partial}_J^+ \mathscr{R}_{\overline{q}_J} \cup \overline{\partial}_J^0 \mathscr{R}_{\overline{q}_J},
\end{aligned} \tag{51}$$

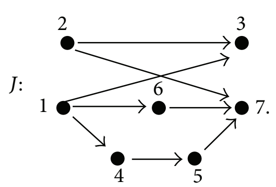
where

$$\begin{aligned}
\partial_J^- \mathscr{R}_{q_J} &= \left\{ v \in \mathscr{R}_{q_J};\ \partial_J(v) < 0 \right\}, \\
\partial_J^+ \mathscr{R}_{q_J} &= \left\{ v \in \mathscr{R}_{q_J};\ \partial_J(v) > 0 \right\}, \\
\partial_J^0 \mathscr{R}_{q_J} &= \left\{ v \in \mathscr{R}_{q_J};\ \partial_J(v) = 0 \right\}; \\
\widehat{\partial}_J^- \mathscr{R}_{\widehat{q}_J} &= \left\{ v \in \mathscr{R}_{\widehat{q}_J};\ \widehat{\partial}_J(v) < 0 \right\}, \\
\widehat{\partial}_J^+ \mathscr{R}_{\widehat{q}_J} &= \left\{ v \in \mathscr{R}_{\widehat{q}_J};\ \widehat{\partial}_J(v) > 0 \right\}, \\
\widehat{\partial}_J^0 \mathscr{R}_{\widehat{q}_J} &= \left\{ v \in \mathscr{R}_{\widehat{q}_J};\ \widehat{\partial}_J(v) = 0 \right\}; \\
\overline{\partial}_J^- \mathscr{R}_{\overline{q}_J} &= \left\{ v \in \mathscr{R}_{\overline{q}_J};\ \overline{\partial}_J(v) < 0 \right\}, \\
\overline{\partial}_J^+ \mathscr{R}_{\overline{q}_J} &= \left\{ v \in \mathscr{R}_{\overline{q}_J};\ \overline{\partial}_J(v) > 0 \right\}, \\
\overline{\partial}_J^0 \mathscr{R}_{\overline{q}_J} &= \left\{ v \in \mathscr{R}_{\overline{q}_J};\ \overline{\partial}_J(v) = 0 \right\}.
\end{aligned} \tag{52}$$

Note that the group isomorphism $\mathbb{Z}^J \to \mathbb{Z}^J$, $v \mapsto \widehat{v} := -v$, restricts to the bijections

$$\begin{gathered}
\partial_J^- \mathscr{R}_{q_J} \xrightarrow[1-1]{\simeq} \partial_J^+ \mathscr{R}_{q_J}, \qquad \widehat{\partial}_J^- \mathscr{R}_{\widehat{q}_J} \xrightarrow[1-1]{\simeq} \widehat{\partial}_J^+ \mathscr{R}_{\widehat{q}_J}, \\
\overline{\partial}_J^- \mathscr{R}_{\overline{q}_J} \xrightarrow[1-1]{\simeq} \overline{\partial}_J^+ \mathscr{R}_{\overline{q}_J}.
\end{gathered} \tag{53}$$

Example 12. We compute the reduced Coxeter number, the Coxeter polynomial, and the Euler defect of the following principal two-peak poset



(54)

Note that J is principal, because

$$\begin{aligned}
\overline{q}_J(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \\
&\quad + x_7^2 - (x_1 + x_2) x_3 - (x_1 + x_5) x_4 \\
&\quad - x_1 x_6 + (x_1 - x_2 - x_5 - x_6) x_7 \\
&= \left(x_1 - \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2}x_6 + \frac{1}{2}x_7 \right)^2 \\
&\quad + \left(x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_7 \right)^2 \\
&\quad + \frac{5}{12}\left(x_3 - \frac{2}{5}x_5 - \frac{4}{5}x_6 + \frac{1}{5}x_7 \right)^2 \\
&\quad + \frac{3}{4}\left(-\frac{1}{3}x_3 + x_4 - \frac{2}{3}x_5 - \frac{1}{3}x_6 + \frac{1}{3}x_7 \right)^2 \\
&\quad + \frac{3}{5}\left(x_5 - \frac{1}{2}x_6 - \frac{1}{2}x_7 \right)^2 + \frac{1}{4}(x_6 - x_7)^2.
\end{aligned} \tag{55}$$

It follows that $\overline{q}_J$ is nonnegative and $\operatorname{Ker} \overline{q}_J = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h} = (1,1,1,1,1,1,1)$; $\overline{q}_J$ is critical in the sense of Ovsienko [24]; see also [38, 44]. Note that the Euler matrix $\overline{C}_J = C_J^{-1}$ of J and the inverse of the Coxeter-Euler matrix $\overline{\mathrm{Cox}}_J := -C_J^{-1} \cdot C_J^{\mathrm{tr}}$ have the forms

$$\begin{gathered}
\overline{C}_J = C_J^{-1} = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
\overline{\mathrm{Cox}}_J^{-1} = \begin{bmatrix} -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.
\end{gathered} \tag{56}$$

Moreover, we have $G_{\widetilde{\mathbb{E}}_6} = B^{\mathrm{tr}} \cdot \overline{G}_J \cdot B$, and the matrix $A := B^{\mathrm{tr}} \cdot \overline{C}_J \cdot B$ is a morsification of the Euclidean diagram $\widetilde{\mathbb{E}}_6$ (see [34, 40]), where

$$\begin{gathered}
B = \begin{bmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
A = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.
\end{gathered} \tag{57}$$

Hence, in view of [2, Proposition 2.8], we get the following:

(i) the Euclidean type DJ of J is the diagram $\widetilde{\mathbb{E}}_6$, and the Coxeter polynomial of the poset J has the form

$$\operatorname{cox}_J(t) = t^7 - t^5 - t^2 + 1 = \operatorname{cox}_A(t); \tag{58}$$

that is, $\operatorname{cox}_J(t)$ is the Coxeter polynomial $F_\Delta^{(2)}(t)$ (21), of the Euclidean diagram $\Delta = \widetilde{\mathbb{A}}_6$ (with a particular numbering of vertices), and is the Coxeter polynomial $\operatorname{cox}_A(t)$ of the morsification $A \in \mathbb{M}_7(\mathbb{Z})$ of the diagram $\widetilde{\mathbb{E}}_6$,

(ii) the Coxeter number $\mathbf{c}_J$ is infinite and the reduced Coxeter number $\check{\mathbf{c}}_J$ equals 10,

(iii) the Euler defect has the form $\overline{\partial}_J(x) = 3(x_1 + x_2 - x_3 - x_7)$,

(iv) the $\overline{\Phi}_J$-orbit of any vector of defect zero in $\overline{\partial}_J^0 \mathcal{R}_{\widehat{q}_J}$ is of length 2 or of length 5. It is shown in [1, Remark 4.5] and [34, Example 5.14] that they lie on one sand-glass tube $\mathcal{T}_{1,2}$ of rank 2 and on six sand-glass tubes of rank five.

4. Nonnegative Posets of Positive Corank

In the study of nonnegative posets, the following extensions of [34, Definition 2.2] are of importance.

Definition 13. Assume that $m \geq 2$, $r \geq 0$, and $q : \mathbb{Z}^m \to \mathbb{Z}$ is a unit quadratic form.

(a) The form q is defined to be *nonnegative of corank* $r \geq 0$, if q is nonnegative and the $\mathbb{Q}$-rank $\operatorname{rk}_{\mathbb{Q}} G_q$ of the rational Gram matrix $G_q \in \mathbb{M}_m(\mathbb{Q})$ equals $\operatorname{rk}_{\mathbb{Q}} G_q = m - r$.

(b) The form q is defined to be *nonnegative critical of corank* $r \geq 1$, if q is nonnegative of corank $r \geq 1$ and each of the nonnegative quadratic forms $q^{(1)}, \ldots, q^{(m)} : \mathbb{Z}^{m-1} \to \mathbb{Z}$ is of corank at most $r - 1 \geq 0$, where

$$\begin{aligned} q^{(j)}&\left(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m\right) \\ &= q\left(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_m\right). \end{aligned} \tag{59}$$

Lemma 14. *Assume that $m \geq 2$, $r \geq 0$, and $q : \mathbb{Z}^m \to \mathbb{Z}$ is an integral quadratic form.*

(a) *q is nonnegative of corank $r \geq 0$ if and only if q is nonnegative and the subgroup* $\operatorname{Ker} q$ *of the abelian group $\mathbb{Z}^m$ is free of rank r.*

(b) *q is nonnegative of corank $r = 0$ if and only if q is positive, and q is nonnegative of corank one if and only if q is principal.*

(c) *q is nonnegative critical of corank $r \geq 1$ if and only if q is nonnegative and, for any $j \in \{1, \ldots, m\}$, the abelian subgroup $\mathbb{Z}^{m,j} \cap$* $\operatorname{Ker} q$ *of $\mathbb{Z}^m$ is free of rank at most $r - 1$, where*

$$\begin{aligned} \mathbb{Z}^{m,j} &:= \mathbb{Z}^{m-j-1} \times \{0\} \times \mathbb{Z}^{j-1} \\ &= \left\{v = (v_1, \ldots, v_m) \in \mathbb{Z}^m;\ v_j = 0\right\} \subseteq \mathbb{Z}^m \end{aligned} \tag{60}$$

is viewed as a subgroup of $\mathbb{Z}^m$.

(d) *q is nonnegative critical of corank $r = 1$ if and only if q is P-critical in the sense of [34, Definition 2.2] and [44].*

Proof. The proof of (a) follows by applying the arguments used in the proof of the equivalence (a)⇔(b) in Proposition 9. The statements (b) and (c) follow from (a).

(c) First, we note that the quadratic forms $q^{(1)}, \ldots, q^{(m)} : \mathbb{Z}^{m-1} \to \mathbb{Z}$ are nonnegative, if q is nonnegative. Then, (c) is a consequence of the group isomorphism

$$\begin{aligned} \operatorname{Ker} q^{(j)} &\xrightarrow{\simeq} \mathbb{Z}^{m,j} \cap \operatorname{Ker} q, \\ w \longmapsto \widehat{w}^{(j)} &:= \left(w_1, \ldots, w_{j-1}, 0, w_{j+1}, \ldots, w_m\right). \end{aligned} \tag{61}$$

Since (d) is a consequence of (c), the proof is complete. □

Definition 15. Assume that J is a connected poset and $q_J, \widehat{q}_J : \mathbb{Z}^J \to \mathbb{Z}$ are its incidence and Tits quadratic forms (6), respectively.

(a) J is defined to be *nonnegative of corank* $r \geq 0$ if its incidence quadratic form $q_J : \mathbb{Z}^J \to \mathbb{Z}$ (resp., one of the forms $\widehat{q}_J$ and $\overline{q}_J$) is nonnegative and the free abelian subgroup $\operatorname{Ker} q_J$ of $\mathbb{Z}^J$ is of $\mathbb{Z}$-rank r (resp., $\operatorname{Ker} \widehat{q}_J \cong \operatorname{Ker} \overline{q}_J \cong \operatorname{Ker} q_J$ is of $\mathbb{Z}$-rank r); see (36).

(b) J is defined to be *nonprincipal critical* if the incidence quadratic form $q_J : \mathbb{Z}^J \to \mathbb{Z}$ is nonnegative and not positive, J is not principal, and the quadratic form $q_{J'} : \mathbb{Z}^{J'} \to \mathbb{Z}$ is principal or positive, for every proper subposet J' of J.

(c) A one-peak poset I, with $\max I = \{*\}$, is defined to be *nonprincipal Tits-critical* if the Tits quadratic form $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ is nonnegative and not positive, I is not principal, and the Tits quadratic form $\widehat{q}_{I'} : \mathbb{Z}^{I'} \to \mathbb{Z}$ is principal or positive, for every proper subposet I' of

I containing the peak $*$. We call a nonprincipal Tits-critical poset I *exceptional*, if the subposet $T = I \setminus \{*\}$ is nonprincipal Tits-critical; see [33, 34].

(d) A poset J is defined to be *P-hypercritical* if J is not nonnegative and each of its proper subposet is nonnegative; see [34, Definition 2.2].

Remark 16. Assume that T is a poset and $T^* = T \cup \{*\}$ is its one-peak enlargement.

(a) If T^* is P-hypercritical, then T is NP-critical in the sense of [14], but not conversely.

(b) By [43], many of the NP-critical posets T listed in [14, Table 2] are of corank at most two.

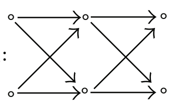
(c) A Coxeter spectral classification of one-peak positive (resp., almost Tits P-critical) posets is given in [17, 18] (resp., in [33]).

We frequently use the following important characterisation.

Theorem 17. *Assume that J is a connected poset and $q_J, \widehat{q}_J : \mathbb{Z}^J \to \mathbb{Z}$ are the incidence and the Tits quadratic forms of J (7), respectively.*

(a) *If J is nonnegative of corank two, then J contains at least 6 elements, and $|J| = 6$ if and only if J is the garland*

$\mathcal{G}_3$:



with $cox_{\mathcal{G}_3}(t) = t^6 + 2t^5 - t^4 - 4t^3 - t^2 + 2t + 1$ (62)

and $\operatorname{Ker} q_{\mathcal{G}_3} := \mathbb{Z} \cdot \mathbf{h}^{(1)} \oplus \mathbb{Z} \cdot \mathbf{h}^{(2)}$, *where* $\mathbf{h}^{(1)} = (1, 1, -1, -1, 0, 0)$ *and* $\mathbf{h}^{(2)} = (1, 1, 0, 0, -1, -1)$. *The garland $\mathcal{G}_3$ is nonprincipal critical.*

(b) *The following four conditions are equivalent.*

(b1) *The poset J is nonprincipal critical.*

(b2) $|J| \geq 6$ *and the form* $q_J : \mathbb{Z}^J \to \mathbb{Z}$ *is nonnegative critical of corank two.*

(b3) $|J| \geq 6$ *and* $q_J : \mathbb{Z}^J \to \mathbb{Z}$ *is nonnegative, the group* $\operatorname{Ker} q_J$ *is of $\mathbb{Z}$-rank two, and for any $j \in J$, the subposet $J^{(j)} := J \setminus \{j\}$ of J is principal or positive.*

(b4) $|J| \geq 6$ *and* $q_J : \mathbb{Z}^J \to \mathbb{Z}$ *is nonnegative, and the group* $\operatorname{Ker} q_J$ *has a $\mathbb{Z}$-basis $\mathbf{h}, \mathbf{h}'$ such that there is no $j \in J$, with $h_j = h'_j = 0$.*

(c) *Let I be a one-peak poset I, with* $\max I = \{*\}$. *The following three conditions are equivalent.*

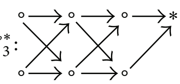
(c1) *I is nonprincipal Tits-critical.*

(c2) $|I| \geq 7$, *the Tits form* $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ *is nonnegative, the group* $\operatorname{Ker} \widehat{q}_I$ *is of $\mathbb{Z}$-rank two, and for any $j \in I \setminus \{*\}$, the one-peak subposet $I^{(j)} := I \setminus \{j\}$ of I is principal or positive.*

(c3) $|I| \geq 7$ *and* $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ *is nonnegative, and the group* $\operatorname{Ker} \widehat{q}_I$ *has a $\mathbb{Z}$-basis $\mathbf{h}, \mathbf{h}'$ such that there is no $j \in I \setminus \{*\}$, with $h_j = h'_j = 0$.*

(d) *A nonprincipal Tits-critical one-peak poset I, with* $\max I = \{*\}$ *and* $|I| = 7$, *is exceptional if and only if I is the one-peak garland*

$\mathcal{G}_3^*$:



with $cox_{\mathcal{G}_3^*}(t) = t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1$ (63)

and $\operatorname{Ker} \widehat{q}_{\mathcal{G}_3^*} := \mathbb{Z} \cdot \widehat{\mathbf{h}}^{(1)} \oplus \mathbb{Z} \cdot \widehat{\mathbf{h}}^{(2)}$, *where* $\widehat{\mathbf{h}}^{(1)} = (1, 1, -1, -1, 0, 0, 0)$, $\widehat{\mathbf{h}}^{(2)} = (1, 1, 0, 0, -1, -1, 0)$.

Proof. (a) It is easy to check that any poset J with at most 5 elements is either positive or principal. Moreover, if J is nonnegative of corank two and $|J| = 6$, then J is the garland $\mathcal{G}_3$. Since

$$q_{\mathcal{G}_3}(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + (x_1 + x_2)(x_3 + x_4 + x_5 + x_6) + (x_3 + x_4)(x_5 + x_6), \tag{64}$$

the Lagrange's algorithm yields

$$q_{\mathcal{G}_3}(x) = \left(x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6\right)^2 + \left(x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6\right)^2 + \frac{1}{2}(x_3 - x_4)^2 + \frac{1}{2}(x_5 - x_6)^2. \tag{65}$$

It follows that $q_{\mathcal{G}_3} : \mathbb{Z}^6 \to \mathbb{Z}$ is nonnegative and its kernel is a rank-two free abelian group of the form shown in (a). Hence, (a) follows.

(b) We show by a computer search that there is no nonprincipal critical poset J such that $|J| \leq 5$. Then, in view of Lemma 14, the equivalences (b1)⇔(b2)⇔(b3)⇔(b4) easily follow.

(c) We show by a computer search that there is no one-peak nonprincipal Tits-critical poset I such that $|I| \leq 6$. Then, in view of Lemma 14, the equivalences (c1)⇔(c2)⇔(c3) easily follow.

(d) Note that

$$\begin{aligned}\widehat{q}_{\mathscr{G}_3^*}(x) = {} & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 \\ & + (x_1 + x_2)(x_3 + x_4 + x_5 + x_6) \\ & + (x_3 + x_4)(x_5 + x_6) \\ & - (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)x_7,\end{aligned} \tag{66}$$

and the Lagrange's algorithm yields

$$\begin{aligned}\widehat{q}_{\mathscr{G}_3^*}(x) = {} & \left(x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6 - \frac{1}{2}x_7\right)^2 \\ & + \left(x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6 - \frac{1}{2}x_7\right)^2 \\ & + \frac{1}{2}(x_3 - x_4)^2 + \frac{1}{2}(x_5 - x_6)^2 + \frac{1}{2}x_7^2.\end{aligned} \tag{67}$$

It follows that $\widehat{q}_{\mathscr{G}_3^*} : \mathbb{Z}^7 \to \mathbb{Z}$ is nonnegative and its kernel is a rank-two free abelian group of the form shown in (d). Hence, the one-peak garland $\mathscr{G}_3^*$ is nonprincipal Tits-critical and exceptional. On the other hand, one shows by a computer search that $\mathscr{G}_3^*$ is the only one-peak poset that is nonprincipal Tits-critical and exceptional. This finishes the proof. □

Following [34, Section 4], we will study nonnegative posets J of corank $r \geq 2$ by means of the spectrum $\mathbf{specc}_J$, the reduced Coxeter number $\check{\mathbf{c}}_J$, and the rank $r \geq 2$ defects

$$\begin{gathered}\partial_J = \left(\partial_J^{(1)}, \ldots, \partial_J^{(r)}\right), \qquad \overline{\partial}_J = \left(\overline{\partial}_J^{(1)}, \ldots, \overline{\partial}_J^{(r)}\right), \\ \widehat{\partial}_J = \left(\widehat{\partial}_J^{(1)}, \ldots, \widehat{\partial}_J^{(r)}\right) : \mathbb{Z}^J \longrightarrow \mathbb{Z}^r\end{gathered} \tag{68}$$

defined in the following extension of Theorem 10.

Theorem 18. *Let J be a finite nonnegative poset of corank $r \geq 2$, and let $m = |J| \geq 2$. One fixes nonzero vectors $\mathbf{h}_J^{(1)}, \ldots, \mathbf{h}_J^{(r)} \in \mathbb{Z}^J$ such that $\operatorname{Ker} q_J = \mathbb{Z} \cdot \mathbf{h}_J^{(1)} \oplus \cdots \oplus \mathbb{Z} \cdot \mathbf{h}_J^{(r)} \cong \mathbb{Z}^r$, and one sets $\mathbf{h}_J = (\mathbf{h}_J^{(1)}, \ldots, \mathbf{h}_J^{(r)})$.*

(a) *There exist a minimal integer $\check{\mathbf{c}}_J \geq 1$ (called the reduced Coxeter number of J) and a group homomorphism $\partial_J = (\partial_J^{(1)}, \ldots, \partial_J^{(r)}) : \mathbb{Z}^J \to \mathbb{Z}^r \cong \operatorname{Ker} q_J$ (called the incidence defect of J) such that*

$$\begin{aligned}\Phi_J^{\check{\mathbf{c}}_J}(v) & = v + \partial_J(v) \bullet \mathbf{h}_J \\ & = v + \partial_J^{(1)}(v) \cdot \mathbf{h}_J^{(1)} + \cdots + \partial_J^{(r)}(v) \cdot \mathbf{h}_J^{(r)}, \\ & \partial_J(\Phi_J(v)) = \partial_J(v), \quad \forall v \in \mathbb{Z}^J,\end{aligned} \tag{69}$$

and $\partial_J(\mathbf{h}) = 0$, for all $\mathbf{h} \in \operatorname{Ker} q_J$, where one sets

$$\begin{gathered}\partial_J(v) = \left(\partial_J^{(1)}(v), \ldots, \partial_J^{(r)}(v)\right), \\ \partial_J(v) \bullet \mathbf{h}_J := \partial_J^{(1)}(v) \cdot \mathbf{h}_J^{(1)} + \cdots + \partial_J^{(r)}(v) \cdot \mathbf{h}_J^{(r)}.\end{gathered} \tag{70}$$

(b) *Assume that $\check{\mathbf{c}}_J \geq 1$ and $\partial_J : \mathbb{Z}^J \to \mathbb{Z}^r$ are as in (a), and one sets*

$$\begin{gathered}\overline{\mathbf{h}}_J^{(1)} = \mathbf{h}_J^{(1)} \cdot B', \ldots, \overline{\mathbf{h}}_J^{(r)} = \mathbf{h}_J^{(r)} \cdot B', \\ \widehat{\mathbf{h}}_J^{(1)} = \mathbf{h}_J^{(1)} \cdot B' \cdot B^{-1}, \ldots, \widehat{\mathbf{h}}_J^{(r)} = \mathbf{h}_J^{(r)} \cdot B' \cdot B^{-1},\end{gathered} \tag{71}$$

where $B', B \in \mathbb{M}_J(\mathbb{Z})$ are as in Proposition 5.

(b1) *There exists a group homomorphism $\overline{\partial}_J = (\overline{\partial}_J^{(1)}, \ldots, \overline{\partial}_J^{(r)}) : \mathbb{Z}^J \to \mathbb{Z}^r \cong \operatorname{Ker} \overline{q}_J$ (called the Euler defect of J) such that*

$$\begin{aligned}\overline{\Phi}_J^{\check{\mathbf{c}}_J}(v) & = v + \overline{\partial}_J(v) \bullet \overline{\mathbf{h}}_J \\ & = v + \overline{\partial}_J^{(1)}(v) \cdot \overline{\mathbf{h}}_J^{(1)} + \cdots + \overline{\partial}_J^{(r)}(v) \cdot \overline{\mathbf{h}}_J^{(r)}, \\ & \forall v \in \mathbb{Z}^J,\end{aligned} \tag{72}$$

$\overline{\partial}_J \circ \overline{\Phi}_J = \overline{\partial}_J$, $\overline{\partial}_J = \overline{\partial}_J \circ h_{B'}$, and $\overline{\partial}_J(\mathbf{h}) = 0$, for all $\mathbf{h} \in \operatorname{Ker} \overline{q}_J$, where one sets

$$\begin{gathered}\overline{\mathbf{h}}_J = \left(\overline{\mathbf{h}}_J^{(1)}, \ldots, \overline{\mathbf{h}}_J^{(r)}\right), \\ \overline{\partial}_J(v) \bullet \overline{\mathbf{h}}_J := \overline{\partial}_J^{(1)}(v) \cdot \overline{\mathbf{h}}_J^{(1)} + \cdots + \overline{\partial}_J^{(r)}(v) \cdot \overline{\mathbf{h}}_J^{(r)}.\end{gathered} \tag{73}$$

(b2) *There exists a group homomorphism $\widehat{\partial}_J = (\widehat{\partial}_J^{(1)}, \ldots, \widehat{\partial}_J^{(r)}) : \mathbb{Z}^J \to \mathbb{Z}^r \cong \operatorname{Ker} \widehat{q}_J$ (called the Tits defect of J) such that*

$$\begin{aligned}\widehat{\Phi}_J^{\check{\mathbf{c}}_J}(v) & = v + \widehat{\partial}_J(v) \bullet \widehat{\mathbf{h}}_J \\ & = v + \overline{\partial}_J^{(1)}(v) \cdot \widehat{\mathbf{h}}_J^{(1)} + \cdots + \widehat{\partial}_J^{(r)}(v) \cdot \widehat{\mathbf{h}}_J^{(r)}, \\ & \forall v \in \mathbb{Z}^J,\end{aligned} \tag{74}$$

$\widehat{\partial}_J \circ \widehat{\Phi}_J = \widehat{\partial}_J$, $\widehat{\partial}_J = \overline{\partial}_J \circ h_B = \partial_J \circ h_{B'}^{-1} \circ h_B$, and $\widehat{\partial}_J(\mathbf{h}) = 0$, for all $\mathbf{h} \in \operatorname{Ker} \widehat{q}_J$, where one sets

$$\begin{gathered}\widehat{\mathbf{h}}_J = \left(\widehat{\mathbf{h}}_J^{(1)}, \ldots, \widehat{\mathbf{h}}_J^{(r)}\right), \\ \widehat{\partial}_J(v) \bullet \widehat{\mathbf{h}}_J := \widehat{\partial}_J^{(1)}(v) \cdot \widehat{\mathbf{h}}_J^{(1)} + \cdots + \widehat{\partial}_J^{(r)}(v) \cdot \widehat{\mathbf{h}}_J^{(r)}.\end{gathered} \tag{75}$$

(c) *The Coxeter number $\mathbf{c}_J$ of J is finite if and only if the incidence defect $\partial_J : \mathbb{Z}^J \to \mathbb{Z}^r$ is zero. In this case, $\check{\mathbf{c}}_J = \mathbf{c}_J$.*

(d) *Given $v \in \mathbb{Z}^m \equiv \mathbb{Z}^J$, the order $\mathbf{s}_v := |\mathscr{O}(v)|$ of the Φ_J-orbit $\mathscr{O}(v)$ is finite if and only if $\partial_J(v) = 0$. If $\mathbf{s}_v = |\mathscr{O}(v)|$ is finite, then $\mathbf{s}_v$ divides $\check{\mathbf{c}}_J$ and there is a unique integer m_v such that*

$$\begin{aligned}m_v \cdot \mathbf{h} & = v + \Phi_J(v) + \Phi_J^2(v) + \cdots + \Phi_J^{\mathbf{s}_v - 1}(v) \\ & = v + \Phi_J^{-1}(v) + \Phi_J^{-2}(v) + \cdots + \Phi_J^{-\mathbf{s}_v + 1}(v).\end{aligned} \tag{76}$$

(e) *The statement (d) holds with Φ_J and $\widehat{\Phi}_J$ (resp., $\overline{\Phi}_J$) interchanged.*

Proof. For simplicity of presentation, we assume that $r = 2$. We recall from the proof of Proposition 9 that $\mathbb{Z} \cdot \mathbf{h}_J^{(1)} \oplus \mathbb{Z} \cdot \mathbf{h}_J^{(2)} = \operatorname{Ker} q_J = \operatorname{Ker}[Dq_J : \mathbb{Z}^m \to \mathbb{Z}^m]$, where $m = |J| \geq 2$ and

$$Dq_J : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m,$$
$$v \longmapsto Dq_J(v) = \left(\frac{\partial q_J}{\partial x_1}(v), \ldots, \frac{\partial q_J}{\partial x_m}(v)\right), \tag{77}$$

is the gradient group homomorphism. It follows that

$$\frac{\mathbb{Z}^m}{\mathbb{Z} \cdot \mathbf{h}_J^{(1)} \oplus \mathbb{Z} \cdot \mathbf{h}_J^{(2)}} = \frac{\mathbb{Z}^m}{\operatorname{Ker} q_J} \cong \operatorname{Im} Dq_J \cong \mathbb{Z}^{m-2}. \tag{78}$$

Denote by $\phi : \mathbb{Z}^m \to \mathbb{Z}^{m-2}$ the composite quotient epimorphism. Then, the form q_J induces the form $\widetilde{q}_J : \mathbb{Z}^{m-2} \to \mathbb{Z}$ such that $\widetilde{q}_J(\phi(x)) = q_J(x)$, for all $x \in \mathbb{Z}^m$. Moreover, the Coxeter transformation $\Phi_J : \mathbb{Z}^m \to \mathbb{Z}^m$ induces a group automorphism $\widetilde{\Phi}_J : \mathbb{Z}^{m-2} \to \mathbb{Z}^{m-2}$ such that

$$\widetilde{\Phi}_J \circ \phi = \phi \circ \Phi_J, \qquad \widetilde{q}_J\left(\widetilde{\Phi}_J(y)\right) = \widetilde{q}_J(y), \tag{79}$$

for all $y \in \mathbb{Z}^{m-2}$. It follows that $\widetilde{q}_J$ is positive definite, and there exists a minimal integer $\check{\mathbf{c}}_J \geq 1$ such that $\widetilde{\Phi}_J^{\check{\mathbf{c}}_J}$ is the identity map on $\mathbb{Z}^{m-2}$. Hence, given $v \in \mathbb{Z}^m$, the element $\Phi_J^{\check{\mathbf{c}}_J}(v) - v$ lies in the kernel of q_J; that is, it has the form

$$\Phi_J^{\check{\mathbf{c}}_J}(v) - v = \partial_J^{(1)}(v) \cdot \mathbf{h}_J^{(1)} + \partial_J^{(2)}(v) \cdot \mathbf{h}_J^{(2)}, \tag{80}$$

where $\partial_J^{(1)}(v), \partial_J^{(2)}(v)$ are integers uniquely determined by v. Since Φ_J is a group homomorphism, then

$$\partial_J^{(1)}\left(v + v'\right) = \partial_J^{(1)}(v) + \partial_J^{(1)}\left(v'\right),$$
$$\partial_J^{(2)}\left(v + v'\right) = \partial_J^{(2)}(v) + \partial_J^{(2)}\left(v'\right); \tag{81}$$

that is, we have defined a pair of group homomorphisms $\partial_J^{(1)}, \partial_J^{(2)} : \mathbb{Z}^J \to \mathbb{Z}$; hence, $\partial_J = (\partial_J^{(1)}, \partial_J^{(2)}) : \mathbb{Z}^J \to \mathbb{Z}^2$ is a group homomorphism. It is easy to see that ∂_J has the properties required in (a), and (a) follows.

In view of Proposition 5, the statements (b)–(e) are a consequence of (a). The reader is referred to [34, Theorem 4.17] for more details and a generalization. □

Corollary 19. *Assume that J is a finite nonnegative poset of corank $r \geq 2$.*

(a) *The Coxeter number $\mathbf{c}_J$ of J is infinite if and only if the defect homomorphism $\partial_J : \mathbb{Z}^J \to \mathbb{Z}^r$ is nonzero, or, equivalently, if and only if the Φ_J-orbit $\mathcal{O}(e_j)$ of some basis vector $e_j \in \mathbb{Z}^J$ is infinite.*

(b) *The Coxeter transformation Φ_J is weakly periodic in the sense of Sato [42]; that is, $\Phi_J^s - id$ is nilpotent, for some $s \geq 1$.*

Proof. The statement (a) follows immediately from Theorem 18. To prove (b), we check that $(\Phi_J^{\check{\mathbf{c}}_J} - id)^2 = 0$. □

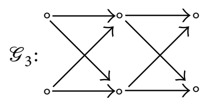
Remark 20. (a) It was shown in [34, Example 5.18] that, for the one-peak garland $I = \mathcal{G}_3^*$ of Theorem 17(d), we have

(i) $\partial_I = \widehat{\partial}_I = \overline{\partial}_I = 0$ and $\mathbf{c}_I = \check{\mathbf{c}}_I = 4$,

(ii) the set $\mathcal{R}_{\widehat{q}_I}$ of Tits roots of I lies on 22 sand-glass tubes; six of them are of rank two, and each of the remaining fourteen tubes is of rank four; see [34, pp. 459–461] for details.

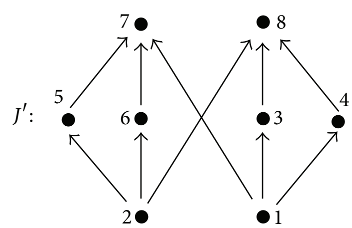
(b) By Lemma 8(a), the Coxeter number $\mathbf{c}_J$ is infinite, for every principal poset J.

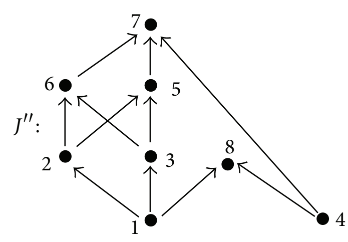
(c) By Theorem 17, there is no nonnegative connected poset J of corank 2, with $|J| \leq 5$. Moreover, a minimal such a poset is the garland



(82)

(d) We show in [43] that most of the nonnegative connected posets J of corank 2, with at most 15 elements, are of zero defect. We also show there that a smallest nonnegative connected poset J with nonzero defect has 8 elements and is one of the following two posets:



(83)


It is easy to check that

(i) $\check{\mathbf{c}}_{J'} = \check{\mathbf{c}}_{J''} = 2$,

(ii) $\operatorname{cox}_{J'}(t) = \operatorname{cox}_{J''}(t) = t^8 - 4t^6 + 6t^4 - 4t^2 + 1$,

(iii) the coordinates of the Tits defect $\widehat{\partial}_{J'} = (\widehat{\partial}_{J'}^{(1)}, \widehat{\partial}_{J'}^{(2)}) : \mathbb{Z}^8 \to \mathbb{Z}^2$ of J', with respect to the $\mathbb{Z}$-basis

$$\widehat{\mathbf{h}}_{J'}^{(1)} = (2, 0, -1, -1, 1, 1, 2, 0),$$
$$\widehat{\mathbf{h}}_{J'}^{(2)} = (0, 2, 1, 1, -1, -1, 0, 2) \tag{84}$$

of Ker $\widehat{q}_{J'}$, are given by the formulae

$$\begin{aligned}\widehat{\partial}_{J'}^{(1)}(x) &= x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6 - x_7,\\ \widehat{\partial}_{J'}^{(2)}(x) &= x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2}x_6 - x_8,\end{aligned} \tag{85}$$

(iv) the coordinates of the Tits defect $\widehat{\partial}_{J''} = (\widehat{\partial}_{J''}^{(1)}, \widehat{\partial}_{J''}^{(2)}) : \mathbb{Z}^8 \to \mathbb{Z}^2$ of J'', with respect to the $\mathbb{Z}$-basis

$$\begin{aligned}\widehat{\mathbf{h}}_{J''}^{(1)} &= (0, -1, -1, 0, 1, 1, 0, 0),\\ \widehat{\mathbf{h}}_{J''}^{(2)} &= (1, 0, 0, 1, 0, 0, 1, 1)\end{aligned} \tag{86}$$

of Ker $\widehat{q}_{J''}$, are given by

$$\begin{aligned}\widehat{\partial}_{J'}^{(1)}(x) &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - x_7 - x_8,\\ \widehat{\partial}_{J''}^{(2)}(x) &= 2x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 - 2x_7 - 2x_8.\end{aligned} \tag{87}$$

5. An Example

In this section, we illustrate the results of Section 3 by an example of a principal one-peak poset I of the Euclidean type $DI = \widetilde{\mathbb{D}}_4$. We give a description of the set $\mathcal{R}_{\widehat{q}_I}$ of roots of $\widehat{q}_I$ and the mesh translation quiver $\Gamma(\mathcal{R}_{\widehat{q}_I}, \widehat{\Phi}_I)$ together with the decomposition (see (51))

$$\begin{aligned}\Gamma\left(\mathcal{R}_{\widehat{q}_I}, \widehat{\Phi}_I\right) = {} & \Gamma\left(\widehat{\partial}_I^{-}\mathcal{R}_{\widehat{q}_I}, \widehat{\Phi}_I\right) \cup \Gamma\left(\widehat{\partial}_I^{+}\mathcal{R}_{\widehat{q}_I}, \widehat{\Phi}_I\right)\\ & \cup \Gamma\left(\widehat{\partial}_I^{0}\mathcal{R}_{\widehat{q}_I}, \widehat{\Phi}_I\right).\end{aligned} \tag{88}$$

Let I be the one-peak garland

$$I = \mathcal{G}_2^*: \quad \begin{tikzcd} 1 \arrow[r] \arrow[dr] & 3 \arrow[dr] & \\ 2 \arrow[r] \arrow[ur] & 4 \arrow[r] & 5. \end{tikzcd} \tag{89}$$

The incidence matrix C_I, the Tits matrix $\widehat{C}_I$, and the Coxeter-Tits matrix $\widehat{\mathrm{Cox}}_I = -\widehat{C}_I \cdot \widehat{C}_I^{-\mathrm{tr}}$ of I are the following:

$$C_I = \begin{bmatrix} 1 & 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\widehat{C}_I = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & -1\\ 1 & 1 & 1 & 0 & -1\\ 1 & 1 & 0 & 1 & -1\\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right], \tag{90}$$

$$\widehat{\mathrm{Cox}}_I = \begin{bmatrix} 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\\ -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

The Coxeter polynomial $\mathrm{cox}_I(t)$, the Tits quadratic form $\widehat{q}_I : \mathbb{Z}^5 \to \mathbb{Z}$, and the Coxeter-Tits transformation $\widehat{\Phi}_I : \mathbb{Z}^5 \to \mathbb{Z}^5$ of I are

$$\begin{aligned}\mathrm{cox}_I(t) &= t^5 + t^4 - 2t^3 - 2t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_4}(t),\\ \widehat{q}_I(x) &= x \cdot \widehat{C}_I \cdot x^{\mathrm{tr}} = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\\ &\quad + (x_1 + x_2)(x_3 + x_4) - (x_1 + x_2 + x_3 + x_4)x_5,\\ \widehat{\Phi}_I(x) &= x \cdot \widehat{\mathrm{Cox}}_I = (x_2 - x_5, x_1 - x_5, x_4 + x_5,\\ &\quad x_3 + x_5, x_1 + x_2 + x_3 + x_4 - x_5),\end{aligned} \tag{91}$$

for $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^I \equiv \mathbb{Z}^5$. Note that the $\widehat{\Phi}_I$-orbit $\mathcal{O}(e_5)$ of e_5 consists of two vectors e_5 and $\widehat{\Phi}_I(e_5) = (-1, -1, 1, 1, -1) = \widehat{\Phi}_I^{-1}(e_5)$. Since

$$\begin{aligned}\widehat{q}_I(x) = {} & \left(x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_5\right)^2\\ & + \left(x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_5\right)^2\\ & + \frac{1}{2}(x_3 - x_4)^2 + \frac{1}{2}x_5^2,\end{aligned} \tag{92}$$

then the form $\widehat{q}_I : \mathbb{Z}^5 \to \mathbb{Z}$ is positive semidefinite, is not positive definite,

$$\mathrm{Ker}\,\widehat{q}_I = \mathbb{Z} \cdot \widehat{\mathbf{h}}_I, \quad \text{where } \widehat{\mathbf{h}}_I = (1, 1, -1, -1, 0), \tag{93}$$

and $\widehat{\Phi}_I(\widehat{\mathbf{h}}_I) = \widehat{\mathbf{h}}_I$. This means that $\widehat{q}_I$ is principal, but not P-critical; see [44]. One easily shows that the reduced Coxeter number of I equals $\check{\mathbf{c}}_I = 2$ and the Tits defect $\widehat{\partial}_I : \mathbb{Z}^5 \to \mathbb{Z}$ of I is given by $\widehat{\partial}_I(x) = -(x_1 + x_2 + x_3 + x_4)$, because $\widehat{\Phi}_I \neq id$ and $\widehat{\Phi}_I^2(v) = v + \widehat{\partial}_I(v) \cdot \widehat{\mathbf{h}}_I$, for any $v \in \mathbb{Z}^5$. The set $\mathcal{R}_{\widehat{q}_I}$ of roots of $\widehat{q}_I$ has the disjoint union decomposition (see (51))

$$\mathcal{R}_{\widehat{q}_I} = \widehat{\partial}_I^{-}\mathcal{R}_{\widehat{q}_I} \cup \widehat{\partial}_I^{+}\mathcal{R}_{\widehat{q}_I} \cup \widehat{\partial}_I^{0}\mathcal{R}_{\widehat{q}_I}, \tag{94}$$

and $\widehat{\partial}_I^{-}\mathcal{R}_{\widehat{q}_I}$, $\widehat{\partial}_I^{+}\mathcal{R}_{\widehat{q}_I}$, $\widehat{\partial}_I^{0}\mathcal{R}_{\widehat{q}_I}$ are $\widehat{\Phi}_I$-invariant subsets of $\mathcal{R}_{\widehat{q}_I}$. Obviously, the $\widehat{\Phi}_I$-orbit $\mathcal{O}(v)$ of any $v \in \widehat{\partial}_I^{0}\mathcal{R}_{\widehat{q}_I}$ is of length two, whereas the $\widehat{\Phi}_I$-orbit $\mathcal{O}(w)$ of any vector $w \in \widehat{\partial}_I^{-}\mathcal{R}_{\widehat{q}_I} \cup \widehat{\partial}_I^{+}\mathcal{R}_{\widehat{q}_I}$ is infinite. By (92), a vector $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{Z}^5$ is a root of $\widehat{q}_I : \mathbb{Z}^5 \to \mathbb{Z}$ if and only if $(2v_1 + v_3 + v_4 - v_5)^2 + (2v_2 + v_3 + v_4 - v_5)^2 + 2(v_3 - v_4)^2 + 2v_5^2 = 4$. Hence, looking at all possible decompositions $4 = a_1^2 + a_2^2 + 2a_3^2 + 2a_4^2$, with $a_1, a_2, a_3, , a_4, \in \mathbb{Z}$, we show that $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{Z}^5$ is a root of $\widehat{q}_I : \mathbb{Z}^5 \to \mathbb{Z}$ if and only if v or $\widetilde{v} := -v$ is one of the vectors listed in Table 1 or in Table 2.

(1) *The $\widehat{\Phi}_I$-orbits in $\widehat{\mathcal{P}}_I := \widehat{\partial}_I^{-}\mathcal{R}_{\widehat{q}_I}$.* Since $\widehat{\partial}_I(u) < 0$, if $u \in \{e_1, e_2, e_3, e_4\}$ or u is the vector $\mathbf{p}_{12} = (1, 1, 0, 0, 1)$, then the $\widehat{\Phi}_I$-orbits of the vectors $e_1, e_2, e_3, e_4, \mathbf{p}_{12}$ lie in $\widehat{\mathcal{P}}_I := \widehat{\partial}_I^{-}\mathcal{R}_{\widehat{q}_I}$, because $\widehat{\mathcal{P}}_I$ is a $\widehat{\Phi}_I$-invariant subset of $\mathcal{R}_{\widehat{q}_I}$. It is easy to see that the $\widehat{\Phi}_I$-orbits consist of the vectors listed in Table 1.

Table 1

j	$\widehat{\Phi}_I^j(e_1)$	$\widehat{\Phi}_I^j(e_2)$	$\widehat{\Phi}_I^j(\mathbf{p}_{12})$	$\widehat{\Phi}_I^j(e_3)$	$\widehat{\Phi}_I^j(e_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$j=7$	$(-3,-2,3,3,1)$	$(-2,-3,3,3,1)$	$(-6,-6,7,7,1)$	$(-3,-3,3,4,1)$	$(-3,-3,4,3,1)$
$j=6$	$(-2,-3,3,3,0)$	$(-3,-2,3,3,0)$	$(-5,-5,6,6,1)$	$(-3,-3,4,3,0)$	$(-3,-3,3,4,0)$
$j=5$	$(-2,-1,2,2,1)$	$(-1,-2,2,2,1)$	$(-4,-4,5,5,1)$	$(-2,-2,2,3,1)$	$(-2,-2,3,2,1)$
$j=4$	$(-1,-2,2,2,0)$	$(-2,-1,2,2,0)$	$(-3,-3,4,4,1)$	$(-2,-2,3,2,0)$	$(-2,-2,2,3,0)$
$j=3$	$(-1,0,1,1,1)$	$(0,-1,1,1,1)$	$(-2,-2,3,3,1)$	$(-1,-1,1,2,1)$	$(-1,-1,2,1,1)$
$j=2$	$(0,-1,1,1,0)$	$(-1,0,1,1,0)$	$(-1,-1,2,2,1)$	$(-1,-1,2,1,0)$	$(-1,-1,1,2,0)$
$j=1$	$(0,1,0,1,0)$	$(1,0,0,0,1)$	$(0,0,1,1,1)$	$(0,0,0,1,1)$	$(0,0,1,0,1)$
$j=0$	$(1,0,0,0,0)$	$(0,1,0,0,0)$	$(1,1,0,0,1)$	$(0,0,1,0,0)$	$(0,0,0,1,0)$
$j=-1$	$(1,2,-1,-1,1)$	$(2,1,-1,-1,1)$	$(2,2,-1,-1,1)$	$(1,1,-1,0,1)$	$(1,1,0,-1,1)$
$j=-2$	$(2,1,-1,-1,0)$	$(1,2,-1,-1,0)$	$(3,3,-2,-2,1)$	$(1,1,0,-1,0)$	$(1,1,-1,0,0)$
$j=-3$	$(2,3,-2,-2,1)$	$(3,2,-2,-2,1)$	$(4,4,-3,-3,1)$	$(2,2,-2,-1,1)$	$(2,2,-1,-2,1)$
$j=-4$	$(3,2,-2,-2,0)$	$(2,3,-2,-2,0)$	$(5,5,-4,-4,1)$	$(2,2,-1,-2,0)$	$(2,2,-2,-1,0)$
$j=-5$	$(3,4,-3,-3,1)$	$(4,3,-3,-3,1)$	$(6,6,-5,-5,1)$	$(3,3,-3,-2,1)$	$(3,3,-2,-3,1)$
$j=-6$	$(4,3,-3,-3,0)$	$(3,4,-3,-3,0)$	$(7,7,-6,-6,1)$	$(3,3,-2,-3,0)$	$(3,3,-3,-2,0)$
$j=-7$	$(4,5,-4,-4,1)$	$(5,4,-4,-4,1)$	$(8,8,-7,-7,1)$	$(4,4,-4,-3,1)$	$(4,4,-3,-4,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2

j	$u^{(j)}$	$u_+^{(j)}$	$w^{(j)}$	$w_+^{(j)}$	$v^{(j)}$	$v_+^{(j)}$
$j=0$	$-e_5=(0,0,0,0,-1)$	$(1,1,-1,-1,1)$	$(1,0,0,-1,0)$	$(0,1,-1,0,0)$	$(0,1,0,-1,0)$	$(1,0,-1,0,0,)$
$j=1$	$(2,2,-2,-2,1)$	$(1,1,-1,-1,-1)$	$(1,2,-2,-1,0)$	$(2,1,-1,-2,0)$	$(2,1,-2,-1,0)$	$(1,2,-1,-2,0)$
$j=2$	$(2,2,-2,-2,-1)$	$(3,3,-3,-3,1)$	$(3,2,-2,-3,0)$	$(2,3,-3,-2,0)$	$(2,3,-2,-3,0)$	$(3,2,-3,-2,0)$
$j=3$	$(4,4,-4,-4,1)$	$(3,3,-3,-3,-1)$	$(3,4,-4,-3,0)$	$(4,3,-3,-4,0)$	$(4,3,-4,-3,0)$	$(3,4,-3,-4,0)$
$j=4$	$(4,4,-4,-4,-1)$	$(5,5,-5,-5,1)$	$(5,4,-4,-5,0)$	$(4,5,-5,-4,0)$	$(4,5,-4,-5,0)$	$(5,4,-5,-4,0)$
$j=5$	$(6,6,-6,-6,1)$	$(5,5,-5,-5,-1)$	$(5,6,-6,-5,0)$	$(6,5,-5,-6,0)$	$(6,5,-6,-5,0)$	$(5,6,-5,-6,0)$
$j=6$	$(6,6,-6,-6,-1)$	$(7,7,-7,-7,1)$	$(7,6,-6,-7,0)$	$(6,7,-7,-6,0)$	$(6,7,-6,-7,0)$	$(7,6,-7,-6,0)$
$j=7$	$(8,8,-8,-8,1)$	$(7,7,-7,-7,-1)$	$(7,8,-8,-7,0)$	$(8,7,-7,-8,0)$	$(8,7,-8,-7,0)$	$(7,8,-7,-8,0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Throughout this section, we freely use the $\widehat{\Phi}_I$-mesh terminology and notation introduced in [2, 34, 40].

(2) *$\widehat{\Phi}_I$-mesh quiver* $\Gamma(\widehat{\mathscr{P}}_I, \widehat{\Phi}_I) = \Gamma(\widehat{\partial}_I^- \mathscr{R}_{\widehat{q}_I}, \widehat{\Phi}_I)$. It follows from our earlier remarks that the set $\widehat{\mathscr{P}}_I := \widehat{\partial}_I^- \mathscr{R}_{\widehat{q}_I}$ of the negative defect roots of $\widehat{q}_I$ splits into the five $\widehat{\Phi}_I$-orbits $\mathcal{O}(e_1)$, $\mathcal{O}(e_2)$, $\mathcal{O}(e_3)$, $\mathcal{O}(e_4)$, $\mathcal{O}(\mathbf{p}_{12})$. By applying the mesh toroidal algorithm defined in [2, 34], one constructs the following infinite $\widehat{\Phi}_I$-mesh translation quiver of the negative defect roots of $\widehat{q}_I$; see Figure 2, where we set $\widehat{a} := -a$ for any positive integer $a \geq 1$.

(3) *$\widehat{\Phi}_I$-mesh quiver* $\Gamma(\widehat{\mathscr{Q}}_I, \widehat{\Phi}_I) = \Gamma(\widehat{\partial}_I^+ \mathscr{R}_{\widehat{q}_I}, \widehat{\Phi}_I)$. Since the group isomorphism $\mathbb{Z}^I \to \mathbb{Z}^I$, $v \mapsto -v$, carries roots to roots, $\widehat{\Phi}_I$-meshes to $\widehat{\Phi}_I$-meshes, and $\widehat{\Phi}_I$-orbits to $\widehat{\Phi}_I$-orbits, then it defines the bijections $\widehat{\partial}_I^- \mathscr{R}_{\widehat{q}_I} \to \widehat{\partial}_I^+ \mathscr{R}_{\widehat{q}_I}$ and $\widehat{\partial}_I^0 \mathscr{R}_{\widehat{q}_I} \to \widehat{\partial}_I^0 \mathscr{R}_{\widehat{q}_I}$, because $\widehat{\partial}_I(-v) = -\widehat{\partial}_I(v)$. It follows that the set $\widehat{\mathscr{Q}}_I := \widehat{\partial}_I^+ \mathscr{R}_{\widehat{q}_I}$ of the positive defect roots of $\widehat{q}_I$ splits into the five $\widehat{\Phi}_I$-orbits $\mathcal{O}(\widehat{e}_1)$, $\mathcal{O}(\widehat{e}_2)$, $\mathcal{O}(\widehat{e}_3)$, $\mathcal{O}(\widehat{e}_4)$, $\mathcal{O}(\widehat{\mathbf{p}}_{12})$, and one constructs the infinite $\widehat{\Phi}_I$-mesh translation quiver

$$\Gamma\left(\widehat{\mathscr{Q}}_I, \widehat{\Phi}_I\right) = \Gamma\left(\widehat{\partial}_I^+ \mathscr{R}_{\widehat{q}_I}, \widehat{\Phi}_I\right) \tag{95}$$

of the positive defect roots of $\widehat{q}_I$ by interchanging any vector v in $\Gamma(\widehat{\mathscr{P}}_I, \widehat{\Phi}_I) = \Gamma(\widehat{\partial}_I^- \mathscr{R}_{\widehat{q}_I}, \widehat{\Phi}_I)$ with its negative $\widehat{v} := -v$.

(4) $\widehat{\Phi}_I$*-mesh quiver* $\Gamma(\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}, \widehat{\Phi}_I)$. By the equality $\widehat{\Phi}_I^2(v) = v - \widehat{\partial}_I(v)\cdot\widehat{\mathbf{h}}_I$, the $\widehat{\Phi}_I$-orbit of any $v \in \widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}$ consists of two vectors v and $\widehat{\Phi}_I(v)$. Now, we show that the $\widehat{\Phi}_I$-orbits in $\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}$ form a $\widehat{\Phi}_I$-mesh translation quiver $\Gamma(\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}, \widehat{\Phi}_I)$.

Note that $\widehat{\Phi}_I(e_5) = (-1,-1,1,1,-1)$, $\widehat{\Phi}_I^2(e_5) = e_5$, $\widehat{\partial}_I(e_5) = 0$, and $\widehat{\partial}_I(\widehat{\Phi}_I(e_5)) = 0$. It follows that the two-element $\widehat{\Phi}_I$-orbits of e_5 and $-e_5$ lie in $\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}$. Moreover, the vectors

$$\begin{gathered}
u_+^{(1)} = (1,1,-1,-1,-1), \quad -u_+^{(1)}, \\
w^{(0)} = (1,0,0,-1,0), \\
w_+^{(0)} = \widehat{\Phi}_I\left(w^{(0)}\right) = (0,1,-1,0,0), \\
v^{(0)} = (0,1,0,-1,0), \\
v_+^{(0)} = \widehat{\Phi}_I\left(v^{(0)}\right) = (1,0,-1,0,0) \\
w_+^{(1)} = (2,1,-1,-2,0), \\
v_+^{(1)} = (1,2,-1,-2,0)
\end{gathered} \tag{96}$$

belong to $\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}$. It is easy to see that we have the following $\widehat{\Phi}_I^2$-mesh quivers of vectors in $\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}$:

$$\begin{tikzcd}
 & -u_+^{(1)} \arrow[dr] & \\
e_5 \arrow[ur] \arrow[dr] & \widehat{\Phi}_I(e_5) \arrow[dr] & e_5 \\
-e_5 \arrow[ur] \arrow[dr] & \widehat{\Phi}_I(-e_5) \arrow[ur] & -e_5 \\
 & u_+^{(1)} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & -w_+^{(1)} \arrow[dr] & \\
-w^{(0)} \arrow[ur] \arrow[dr] & \widehat{\Phi}_I(-w_+^{(0)}) \arrow[dr] & -w^{(0)} \\
w^{(0)} \arrow[ur] \arrow[dr] & \widehat{\Phi}_I(w_+^{(0)}) \arrow[ur] & w^{(0)} \\
 & w_+^{(1)} \arrow[ur] &
\end{tikzcd} \tag{97}$$

$$\begin{tikzcd}
 & -v_+^{(1)} \arrow[dr] & \\
-v^{(0)} \arrow[ur] \arrow[dr] & \widehat{\Phi}_I(-v_+^{(0)}) \arrow[dr] & -v^{(0)} \\
v^{(0)} \arrow[ur] \arrow[dr] & \widehat{\Phi}_I(v_+^{(0)}) \arrow[ur] & v^{(0)} \\
 & v_+^{(1)} \arrow[ur] &
\end{tikzcd}$$

Note that the $\widehat{\Phi}_I$-orbit of $u_+^{(1)}$ consists of the following two vectors:

$$u^{(1)} = (2,-2,-2,-2,1), \qquad u_+^{(1)} = \widehat{\Phi}_I\left(u^{(1)}\right). \tag{98}$$

By (92), a vector $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{Z}^5$ is a root of $\widehat{q}_I : \mathbb{Z}^5 \to \mathbb{Z}$ of defect zero if and only if

$$\begin{gathered}
(2v_1 + v_3 + v_4 - v_5)^2 + (2v_2 + v_3 + v_4 - v_5)^2 \\
+ 2(v_3 - v_4)^2 + 2v_5^2 = 4, \\
v_1 + v_2 + v_3 + v_4 = 0.
\end{gathered} \tag{99}$$

It follows that v or $-v$ belongs to any of the six series of roots presented in Table 2.

Hence, we conclude that the $\widehat{\Phi}_I$-orbits in the set $\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I}$ form three $\widehat{\Phi}_I$-mesh quivers $\mathscr{T}_u$, $\mathscr{T}_w$, $\mathscr{T}_v$, and each of them has the form of infinite two-surface tube of rank 2:

$$\mathscr{T}_a \qquad \begin{tikzcd}
\vdots \arrow[d, dash, dashed] & \vdots & \vdots \arrow[d, dash, dashed] \\
-a^{(2)} \arrow[r, dash, dashed] \arrow[dr] \arrow[d, dash, dashed] & -a_+^{(2)} \arrow[r, dash, dashed] \arrow[dr] & -a^{(2)} \arrow[d, dash, dashed] \\
-a^{(1)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] \arrow[d, dash, dashed] & -a_+^{(1)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] & -a^{(1)} \arrow[d, dash, dashed] \\
-a^{(0)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] \arrow[d, dash, dashed] & -a_+^{(0)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] & -a^{(0)} \arrow[d, dash, dashed] \\
a^{(0)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] \arrow[d, dash, dashed] & a_+^{(0)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] & a^{(0)} \arrow[d, dash, dashed] \\
a^{(1)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] \arrow[d, dash, dashed] & a_+^{(1)} \arrow[r, dash, dashed] \arrow[ur] \arrow[dr] & a^{(1)} \arrow[d, dash, dashed] \\
a^{(2)} \arrow[r, dash, dashed] \arrow[ur] \arrow[d, dash, dashed] & a_+^{(2)} \arrow[r, dash, dashed] \arrow[ur] & a^{(2)} \arrow[d, dash, dashed] \\
\vdots & \vdots & \vdots
\end{tikzcd}, \tag{100}$$

where a is one of the vectors

$$\begin{gathered}
u = u^{(0)} = (0,0,0,0,-1), \\
w = w^{(0)} = (1,0,0,-1,0), \\
v = v^{(0)} = (0,1,0,-1,0).
\end{gathered} \tag{101}$$

(5) $\widehat{\Phi}_I$*-mesh quiver* $\Gamma(\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I} \cup \operatorname{Ker}\widehat{q}_I, \widehat{\Phi}_I)$. We recall that $\operatorname{Ker}\widehat{q}_I = \mathbb{Z}\cdot\widehat{\mathbf{h}}_I$, where $\widehat{\mathbf{h}}_I = (1,1,-1,-1,0)$. Note that

$$\widehat{\Phi}_I\left(\widehat{\mathbf{h}}_I\right) = \widehat{\mathbf{h}}_I, \quad \widehat{\Phi}_I\left(m\cdot\widehat{\mathbf{h}}_I\right) = m\cdot\widehat{\mathbf{h}}_I, \text{ for any } m \in \mathbb{Z}. \tag{102}$$

Obviously, the vectors lying in $\operatorname{Ker}\widehat{q}_I$ form the $\widehat{\Phi}$-mesh translation quiver $\widehat{\mathscr{T}}_{\widehat{\mathbf{h}}_I}$ presented in (108).

Now, we construct from the $\widehat{\Phi}_I$-orbits in the set $\widehat{\partial}_I^0\mathscr{R}_{\widehat{q}_I} \cup \operatorname{Ker}\widehat{q}_I$ an infinite $\widehat{\Phi}_I$-mesh translation quiver. For this purpose, we note that the following six vectors

$$-e_5, e_5, \widehat{\Phi}_I(-e_5), \widehat{\Phi}_I(e_5), \quad -\widehat{\mathbf{h}}_I, \widehat{\mathbf{h}}_I \tag{103}$$

form two $\widehat{\Phi}_I$-meshes of width 1. If we complete them by the three vectors

$$0, u_+^{(1)} := (1,1,-1,-1,-1), \quad -u_+^{(1)}, \tag{104}$$

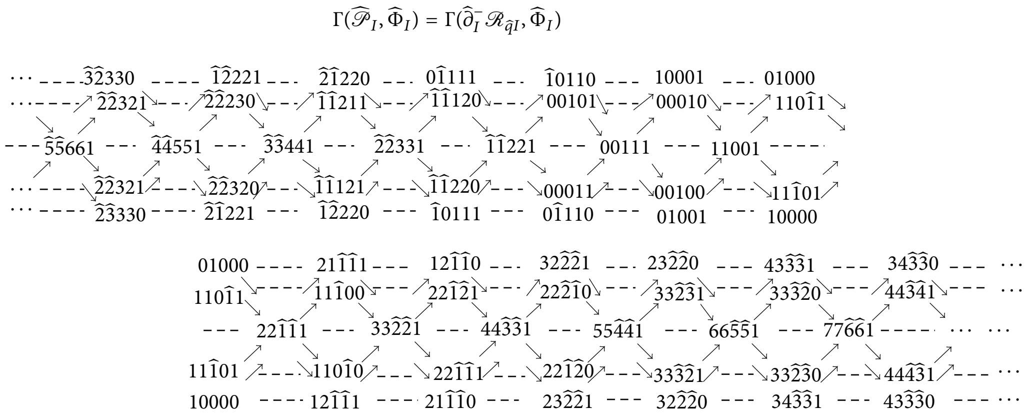
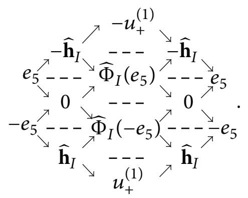


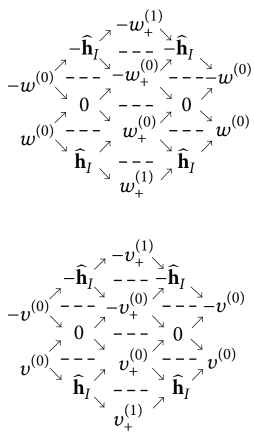
Figure 2

we get the $\widehat{\Phi}_I$-mesh quiver



(105)

Analogously, we construct the following two $\widehat{\Phi}_I$-mesh quivers



(106)

where

$$u^{(1)} = (2, -2, -2, -2, 1),$$

$$u_+^{(1)} = \widehat{\Phi}_I\left(u^{(1)}\right),$$

$$w^{(0)} = (1, 0, 0, -1, 0),$$

$$w_+^{(0)} = \widehat{\Phi}_I\left(w^{(0)}\right) = (0, 1, -1, 0, 0),$$

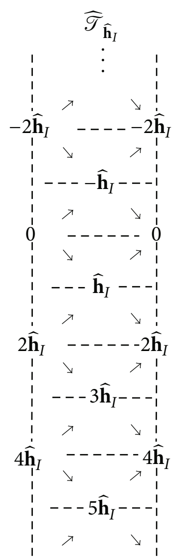
$$v^{(0)} = (0, 1, 0, -1, 0),$$

$$v_+^{(0)} = \widehat{\Phi}_I\left(v^{(0)}\right) = (1, 0, -1, 0, 0),$$

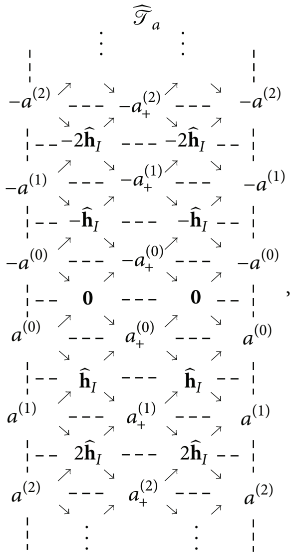
$$w_+^{(1)} = (2, 1, -1, -2, 0),$$

$$v_+^{(1)} = (1, 2, -1, -2, 0). \tag{107}$$

We recall that if $v \in \widehat{\partial}_I^0 \mathscr{R}_{\widehat{q}_I}$, then v or $-v$ is one of the vectors presented in Table 2. It follows that the $\widehat{\Phi}_I$-orbits in $\widehat{\partial}_I^0 \mathscr{R}_{\widehat{q}_I} \cup \mathbb{Z} \cdot \widehat{\mathbf{h}}_I$ form three infinite $\widehat{\Phi}_I$-mesh sand-glass tubes $\widehat{\mathscr{T}}_u$, $\widehat{\mathscr{T}}_w$, $\widehat{\mathscr{T}}_v$ of rank $(2, 1)$, and each of them has the shape presented in (109)



(108)



$$, \tag{109}$$

where a is one of the vectors

$$u = u^{(0)} = (0,0,0,0,-1),$$
$$w = w^{(0)} = (1,0,0,-1,0), \tag{110}$$
$$v = v^{(0)} = (0,1,0,-1,0).$$

Construct the disjoint union $\widehat{\mathcal{T}}_u \cup \widehat{\mathcal{T}}_w \cup \widehat{\mathcal{T}}_v$ of the tubes $\widehat{\mathcal{T}}_u$, $\widehat{\mathcal{T}}_w$, $\widehat{\mathcal{T}}_v$, and note that each of them contains the tube $\widehat{\mathcal{T}}_{\widehat{\mathbf{h}}_I}$. By making the identification of the vectors $m \cdot \widehat{\mathbf{h}}_I$, with $m \in \mathbb{Z}$, lying in the corresponding $\widehat{\Phi}_I$-orbits, we get the quotient $\widehat{\Phi}_I$-mesh translation quiver

$$\Gamma\left(\widehat{\partial}_I^0 \mathscr{R}_{\widehat{q}_I} \cup \operatorname{Ker} \widehat{q}_I, \widehat{\Phi}_I\right) = \frac{\widehat{\mathcal{T}}_u \cup \widehat{\mathcal{T}}_w \cup \widehat{\mathcal{T}}_v}{\simeq} \tag{111}$$

that has a shape of a threefold sand-glass tube of rank $(2,2,2,1)$ in the sense of [40]. It is obtained from the disjoint union of three copies of the onefold sand-glass tube of rank $(2,1)$ presented in Figure 3 (see also [34, Figure 5.8]) by making an obvious identification of their waist vectors.

(6) *A $\mathbb{Z}$-congruence of the bigraph $\widehat{\Delta}_I$ with the Euclidean diagram $\widetilde{\mathbb{D}}_4$*. Since we have $\mathrm{cox}_I(t) = F_{\widetilde{\mathbb{D}}_4}(t) = t^5 + t^4 - 2t^3 - 2t^2 + t + 1$ and $\mathbf{specc}_I = \mathbf{specc}_{\widetilde{\mathbb{D}}_4}$, the Euclidean diagram $\widetilde{\mathbb{D}}_4$ is the diagram DI associated to I. A technique developed in [2, 17, 18, 34, 40] allows us to construct a $\mathbb{Z}$-invertible matrix $B \in \mathrm{Gl}(5,\mathbb{Z})$ such that the following diagrams are commutative:

$$\begin{tikzcd} \mathbb{Z}^5 \times \mathbb{Z}^5 \arrow[r, "\widehat{b}_I"] & \mathbb{Z} \\ \mathbb{Z}^5 \times \mathbb{Z}^5 \arrow[u, "h_B \times h_B", "\cong"'] \arrow[ur, "b_{\widetilde{\mathbb{D}}_4}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}^5 \arrow[r, "\widehat{q}_I"] & \mathbb{Z} \\ \mathbb{Z}^5 \arrow[u, "h_B", "\cong"'] \arrow[ur, "q_{\widetilde{\mathbb{D}}_4}"'] & \end{tikzcd} \tag{112}$$

where $b_{\widetilde{\mathbb{D}}_4}$ and $q_{\widetilde{\mathbb{D}}_4}$ are the forms of the Euclidean diagram

$$\widetilde{\mathbb{D}}_4: \quad \begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 5 & 3 \arrow[l] \\ & 4. \arrow[u] & \end{tikzcd} \tag{113}$$

defined by the formulae $b_{\widetilde{\mathbb{D}}_4}(x,y) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 - (x_1+x_2+x_3+x_4)y_5 = x \cdot \check{G}_{\widetilde{\mathbb{D}}_4} \cdot y^{\mathrm{tr}}$, $q_{\widetilde{\mathbb{D}}_4}(x) = b_{\widetilde{\mathbb{D}}_4}(x,x)$, for $x, y \in \mathbb{Z}^5$, $h_B : \mathbb{Z}^5 \to \mathbb{Z}^5$ is the group automorphism defined by the formula $h_B(x) = x \cdot B$, and

$$B = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\check{G}_{\widetilde{\mathbb{D}}_4} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{114}$$

It is easy to check that the equality $\check{G}_{\widetilde{\mathbb{D}}_4} = B \cdot \widehat{C}_I \cdot B^{\mathrm{tr}}$ holds, and therefore the diagrams (112) are commutative. Furthermore, by the same technique, we construct another matrix

$$B_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{115}$$

such that the equality $\check{G}_{\widetilde{\mathbb{D}}_4} = B_1 \cdot \widehat{C}_I \cdot B_1^{\mathrm{tr}}$ holds.

6. Concluding Remarks

6.1 It follows from Lemma 3 and the results obtained recently in [3, 4] that for any connected positive (resp.,

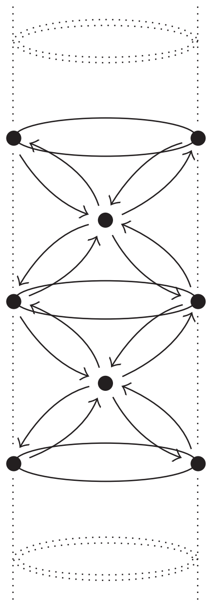


Figure 3: Sand-glass tube of rank $(2, 1)$.

principal) poset J, there exists a simply laced Dynkin diagram $DJ \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ (resp., a simply laced Euclidean diagram DJ), uniquely determined by J, such that the symmetric Gram matrices G_J, G_{DJ} are $\mathbb{Z}$-congruent.

Analogous Coxeter spectral classification of one-peak posets I, with almost P-critical Tits form $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$, is obtained in [33] by a reduction to computer calculations.

6.2. Although the Coxeter spectral classification problem for arbitrary finite posets remains unsolved, we have a solution for positive one-peak posets. Indeed, it follows from the results in [17] that for any one-peak positive poset J, there exists a simply laced Dynkin diagram $DJ \in \{\mathbb{A}_s, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ (uniquely determined by J) such that $\mathbf{specc}_J = \mathbf{specc}_{DJ}$, the nonsymmetric Gram matrices $\check{G}_J, \check{G}_{DJ}$ are $\mathbb{Z}$-congruent, and the symmetric Gram matrices G_J, G_{DJ} are $\mathbb{Z}$-congruent.

6.3. We can determine the diagram DJ as follows. Fix an upper-triangular numbering $\{a_1, \dots, a_m\}$ of elements of J. Then, the incidence matrix $C_J \in \mathbb{M}_m(\mathbb{Z})$ is upper-triangular, and the Euler matrix $\overline{C}_J := C_J^{-1}$ is also upper triangular. Then, the Euler edge-bipartite graph $\overline{\Delta}_J$ (33) is loop-free, and we have $C_{\overline{\Delta}_J} = \overline{C}_J$. Hence, the symmetric Gram matrices $G_{\overline{\Delta}_J}, \overline{G}_J$ coincide, and, by Lemma 3, the poset J is positive (resp., principal) if and only if the bigraph Δ_J is positive (resp., principal). By applying to $\overline{\Delta}_J$ the inflation algorithm constructed in [4, 21] (see also [45]), we get (in a finite number of steps) an edge-bipartite graph $D\Delta_J$ such that the symmetric Gram matrix $G_{\overline{\Delta}_J} = \overline{G}_J$ is $\mathbb{Z}$-congruent with the symmetric Gram matrix $G_{D\Delta_J}$, and the edge-bipartite graph $D\Delta_J$ has no dotted edges; that is, $D\Delta_J$ is a (multi) graph. We set $DJ := D\Delta_J$. It follows from the results in [3, 4] that DJ is a simply laced Dynkin diagram, if J is positive, and DJ is a simply laced Euclidean diagram, if J is principal. Moreover, the matrix $\overline{G}_J$ is $\mathbb{Z}$-congruent with G_{DJ}. Since the incidence Gram matrix G_J of J is $\mathbb{Z}$-congruent with the matrix $\overline{G}_J$ (by Proposition 5), then the matrices G_J and G_{DJ} are $\mathbb{Z}$-congruent.

6.4. Although we can apply in 6.3 the inflation algorithm to the incidence edge-bipartite graph Δ_J, we use in the construction of DJ the Euler edge-bipartite graph $\overline{\Delta}_J$, because the number of nonzero entries in the Euler matrix $\overline{C}_J := C_J^{-1}$ does not increase the number for the matrix C_J; see [28, Proposition 2.12]. It follows that the number of the dotted edges in $\overline{\Delta}_J$ does not increase the number of the dotted edges in Δ_J, and the use in 6.3 the bigraph $\overline{\Delta}_J$ reduces the time of calculation in the procedure $\overline{\Delta}_J \mapsto DJ$.

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