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Research Article

Boundary Value Problems for a Super-Sublinear Asymmetric Oscillator: The Exact Number of Solutions

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Properties of asymmetric oscillator described by the equation $x'' = -\lambda(x^+)^p + \mu(x^-)^q$ (i), where $p \geq 1$ and $0 < q \leq 1$, are studied. A set of (λ, μ) such that the problem (i), $x(0) = 0 = x(1)$ (ii), and $|x'(0)| = \alpha$ (iii) have a nontrivial solution, is called α-spectrum. We give full description of α-spectra in terms of solution sets and solution surfaces. The exact number of nontrivial solutions of the two-parameter Dirichlet boundary value problem (i), and (ii) is given.

1. Introduction

Asymmetric oscillators were studied intensively starting from the works by Kufner and Fučík; see [1] and references therein. Simple equations like (2) given with the boundary conditions allow for complete investigation of spectra. It is known that the spectrum of the problem (2), (4) is a set of hyperbola looking curves in the (λ, μ)-plane. On the other hand, there is a plenty of works devoted to one-parameter case of equations $x'' + \lambda f(x) = 0$ given together with the two-point boundary conditions. Due to nonlinearity of f one should consider solutions $x(t; \alpha)$ with different $\alpha = x'(0)$. Bifurcation diagrams in terms of α and λ, or $\|x\|$ and λ, can serve then to evaluate the number of solutions [2–4].

In this paper we consider differential equations of the form

$$x'' = -\lambda f(x^+) + \mu g(x^-), \tag{1}$$

where $f = x^p$, $p \geq 1$, and $g = x^q$, $0 < q \leq 1$. Here λ and μ are nonnegative parameters, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. This equation describes asymmetric oscillator with different nonlinear restoring forces on both sides of $x = 0$. If $f = g = x$, then equation becomes famous Fučík equation

$$x'' = -\lambda x^+ + \mu x^-. \tag{2}$$

Properties of the Fučík spectrum are well known (the Fučík spectrum is a set of all pairs (λ, μ) where $\lambda, \mu \geq 0$, such that the Dirichlet problem—(2) with boundary conditions $x(0) = 0 = x(1)$—has a non-trivial solution).

The aim of our study in this paper is to describe properties of the spectrum of the problem

$$x'' = -\lambda(x^+)^p + \mu(x^-)^q, \quad 0 < q \leq 1,\ 1 \leq p, \tag{3}$$

$$x(0) = 0, \qquad x(1) = 0. \tag{4}$$

For this we study first the time maps for the related functions (Section 2), then we give the analytical description of the spectrum (Section 3), and formulate the properties of the spectrum (Section 4), including the asymptotics. In Section 5 we consider the solution sets and solution surfaces which bear information on multiplicity of solutions to the problem. Analysis of properties of solution surfaces (Section 6) can give us estimations of the number of solutions to the problem (Section 7). These estimations contain also information of properties of solutions such as the number of zeros and evaluations of $x'(0)$.

This paper continues series of publications by the authors devoted to nonlinear asymmetric oscillations [5–8].

2. Time Maps

Consider the Cauchy problem

$$x'' = -\lambda x^r, \qquad x(0) = 0, \qquad x'(0) = \alpha, \tag{5}$$

where $r > 0$. Solutions of this problem for α and λ positive have a zero. This zero will be denoted $T_r(\alpha, \lambda)$ and called time map function; more on time maps can be found in [9, 10].

Proposition 1. *Suppose $r, \alpha, \lambda > 0$, $p > 1$ and $0 < q < 1$.*

(1) For any $\alpha, \lambda > 0$ the formula is valid:

$$T_r(\alpha, \lambda) = \frac{1}{\sqrt{\lambda}} t_r\left(\frac{\alpha}{\sqrt{\lambda}}\right), \tag{6}$$

where $t_r(\gamma) = T_r(\gamma, 1)$.

(2) The function $T_r(\alpha, \lambda)$ for the problem (5) *is*

$$T_r(\alpha, \lambda) = 2^{r/(r+1)}(r+1)^{1/(r+1)} A(r)\, \alpha^{-(r-1)/(r+1)} \lambda^{-1/(r+1)}, \tag{7}$$

where $A(r) = \int_0^1 (ds/\sqrt{1 - s^{r+1}})$.

(3) The function $T_r(\alpha, \lambda)$ for fixed r and α is strictly decreasing function of λ and possesses the properties

$$\lim_{\lambda \to 0+} T_r(\alpha, \lambda) = +\infty, \qquad \lim_{\lambda \to +\infty} T_r(\alpha, \lambda) = 0. \tag{8}$$

(4) The function $T_p(\alpha, \lambda)$ for fixed p and λ is decreasing function of α.

(5) The function $T_q(\alpha, \lambda)$ for fixed q and λ is increasing function of α.

Proof. By standard computations. □

Remark 2. $T_r(\alpha, \lambda) = \pi/\sqrt{\lambda}$ for $r = 1$, irrespective of α.

3. Spectrum

Suppose that the problems (3) and (4) are considered with the additional condition

$$\left|x'(0)\right| = \alpha > 0. \tag{9}$$

Definition 3. For given $\alpha > 0$ a set of all nonnegative (λ, μ) for which the problems (3), (4), and (9) have a nontrivial solution is called α-spectrum.

Remark 4. A solution of (3) is a C^2-function. Therefore $x'(t)$ is continuous. If z_1 and z_2 are two consecutive zeros of $x(t)$, then $|x'(z_1)| = |x'(z_2)|$ since a solution in the interval (z_1, z_2) is symmetric with respect to the middle point. Therefore $|x'(z)| = \alpha$ for any zero point z and signs of $x'(t)$ alternate.

Theorem 5. *The α-spectrum for the problem*

$$\begin{gathered} x'' = -\lambda(x^+)^p + \mu(x^-)^q, \qquad x(0) = 0, \\ x(1) = 0, \qquad \left|x'(0)\right| = \alpha > 0 \end{gathered} \tag{10}$$

consists of the following α-branches:

$$\begin{aligned} F_0^+(\alpha) &= \left\{(\lambda, \mu) : T_p(\alpha, \lambda) = 1,\ \mu \ge 0\right\}, \\ F_0^-(\alpha) &= \left\{(\lambda, \mu) : \lambda \ge 0,\ T_q(\alpha, \mu) = 1\right\}, \\ F_{2i-1}^+(\alpha) &= \left\{(\lambda; \mu) : iT_p(\alpha, \lambda) + iT_q(\alpha, \mu) = 1\right\}, \\ F_{2i-1}^-(\alpha) &= \left\{(\lambda; \mu) : iT_q(\alpha, \lambda) + iT_p(\alpha, \mu) = 1\right\}, \\ F_{2i}^+(\alpha) &= \left\{(\lambda; \mu) : (i+1) T_p(\alpha, \lambda) + iT_q(\alpha, \mu) = 1\right\}, \\ F_{2i}^-(\alpha) &= \left\{(\lambda; \mu) : (i+1) T_q(\alpha, \mu) + iT_p(\alpha, \lambda) = 1\right\}. \end{aligned} \tag{11}$$

The notation $F_i^+(\alpha)$, respectively: $F_i^-(\alpha)$, refers to solutions x which satisfy the initial conditions $x(0) = 0$, $x'(0) = \alpha > 0$ (resp.: $x'(0) = -\alpha < 0$) and have exactly i zeros in the interval $(0, 1)$.

Proof. We prove the theorem only for solutions which have exactly one zero in $(0, 1)$ and satisfy the initial condition $|x'(0)| = \alpha > 0$.

Consider the case $x'(0) = \alpha > 0$. The first zero of $x(t)$ appears at $t = T_p$ and $x'(T_p) = -\alpha$. The second zero is at $t = T_p + T_q$. One has that for solutions with exactly one zero in $(0, 1)$ the relation $T_p + T_q = 1$ holds, which defines the branch $F_1^+(\alpha)$.

Suppose $x'(0) = -\alpha$. The first zero now is at $t = T_q$. The second one is at $t = T_q + T_p$ and therefore $T_q + T_p = 1$.

The branches $F_1^+(\alpha)$ and $F_1^-(\alpha)$ are given by the equivalent relations $T_p + T_q = 1$ and $T_q + T_p = 1$, respectively, and therefore coincide.

Proof for solutions with different nodal structure is similar. □

Remark 6. If $p = q = 1$, then for any $\alpha > 0$ the α-spectrum is

$$\begin{aligned} F_0^+(\alpha) &= \left\{(\lambda, \mu) : \frac{\pi}{\sqrt{\lambda}} = 1,\ \mu \ge 0\right\}, \\ F_0^-(\alpha) &= \left\{(\lambda, \mu) : \lambda \ge 0,\ \frac{\pi}{\sqrt{\mu}} = 1\right\}, \\ F_{2i-1}^+(\alpha) &= \left\{(\lambda; \mu) : i\frac{\pi}{\sqrt{\lambda}} + i\frac{\pi}{\sqrt{\mu}} = 1\right\}, \\ F_{2i-1}^-(\alpha) &= \left\{(\lambda; \mu) : i\frac{\pi}{\sqrt{\mu}} + i\frac{\pi}{\sqrt{\lambda}} = 1\right\}, \\ F_{2i}^+(\alpha) &= \left\{(\lambda; \mu) : (i+1)\frac{\pi}{\sqrt{\lambda}} + i\frac{\pi}{\sqrt{\mu}} = 1\right\}, \\ F_{2i}^-(\alpha) &= \left\{(\lambda; \mu) : (i+1)\frac{\pi}{\sqrt{\mu}} + i\frac{\pi}{\sqrt{\lambda}} = 1\right\}. \end{aligned} \tag{12}$$

It is classical Fučík spectrum for the problems (2) and (4), see [1].

4. Properties of the α-Spectrum

Proposition 7.

(1) *The branches $F^+_{2i-1}(\alpha)$ and $F^-_{2i-1}(\alpha)$ coincide.*

(2) *The branches $F^\pm_{2i-1}(\alpha)$ and $F^\pm_{2i-1}(\alpha)$ do not intersect unless $i \neq j$.*

(3) *The branches $F^+_{2i}(\alpha)$ and $F^+_{2j}(\alpha)$ do not intersect unless $i \neq j$.*

(4) *The branches $F^-_{2i}(\alpha)$ and $F^-_{2j}(\alpha)$ do not intersect unless $i \neq j$.*

(5) *Any branch $F^s_i(\alpha)$ where $i \geq 1$ and s is either "+" or "−" is a graph of monotonically decreasing function $\mu = \mu(\lambda)$.*

(6) *The branches $F^+_{2i}(\alpha)$ and $F^-_{2i}(\alpha)$ intersect once.*

Proof. (1) It follows from (11) that $F^+_{2i-1}(\alpha)$ coincides with $F^-_{2i-1}(\alpha)$, since both sets (branches) are defined by symmetric relations:

$$iT_p(\alpha,\lambda) + iT_q(\alpha,\mu) = 1, \qquad iT_q(\alpha,\mu) + iT_p(\alpha,\lambda) = 1. \tag{13}$$

(2), (3) and (4) follows from (11), but (5) follows from the relations (11) and (7). (6) Indeed, any point of intersection satisfies the system

$$\begin{gathered}(i+1)\,T_p(\alpha,\lambda) + iT_q(\alpha,\mu) = 1,\\ (i+1)\,T_q(\alpha,\mu) + iT_p(\alpha,\lambda) = 1.\end{gathered} \tag{14}$$

It follows that

$$T_p(\alpha,\lambda) = T_q(\alpha,\mu) \tag{15}$$

or

$$\begin{aligned}&2^{p/(p+1)}(p+1)^{1/(p+1)}A(p)\,\alpha^{-(p-1)/(p+1)}\lambda^{-1/(p+1)}\\ &\quad = 2^{q/(q+1)}(q+1)^{1/(q+1)}A(q)\,\alpha^{-(q-1)/(q+1)}\mu^{-1/(q+1)}.\end{aligned} \tag{16}$$

The above relation defines a curve which is a graph of monotonically increasing function $\mu = C(p,q,\alpha)\lambda^{(q+1)/(p+1)}$, where $C(p,q,\alpha)$ is a constant computable from (16). This curve emanates from the origin and intersects the graph of monotonically decreasing from $+\infty$ to 0 function $\mu = \mu(\lambda)$ from (5) only once. □

Let $\lambda_i(\alpha,p)$ be a unique solution of the equation $T_p(\alpha,\lambda) = 1/i$; similarly, let $\mu_i(\alpha,q)$ be a unique solution of the equation $T_q(\alpha,\mu) = 1/i$.

Proposition 8. *Suppose $\alpha,\lambda,\mu > 0$.*

(1) The branch $F^\pm_{2i-1}(\alpha)(i \in \mathbb{N})$ is located in the sector

$$\{(\lambda,\mu) : \lambda > \lambda_i(\alpha,p),\ \mu > \mu_i(\alpha,q)\} \tag{17}$$

and is a hyperbola looking curve in (λ,μ) plane with vertical asymptote $\lambda = \lambda_i(\alpha,p)$ and horizontal asymptote $\mu = \mu_i(\alpha,q)$.

(2) The branch $F^+_{2i}(\alpha)$ $(i \in \mathbb{N})$ is located in the sector

$$\{(\lambda,\mu) : \lambda > \lambda_{i-1}(\alpha,p),\ \mu > \mu_i(\alpha,q)\} \tag{18}$$

and is a hyperbola looking curve in (λ,μ) plane with vertical asymptote $\lambda = \lambda_{i-1}(\alpha,p)$ and horizontal asymptote $\mu = \mu_i(\alpha,q)$.

(3) The branch $F^-_{2i}(\alpha)$ $(i \in \mathbb{N})$ is located in the sector

$$\{(\lambda,\mu) : \lambda > \lambda_i(\alpha,p),\ \mu > \mu_{i-1}(\alpha,q)\} \tag{19}$$

and is a hyperbola looking curve in (λ,μ) plane with vertical asymptote $\lambda = \lambda_i(\alpha,p)$ and horizontal asymptote $\mu = \mu_{i-1}(\alpha,q)$.

Proof. Follows from the relations (11) and Proposition 7. □

Remark 9. So the positive part of the α-spectrum

$$F^+(\alpha) = \bigcup_{i=0}^{+\infty} F^+_i(\alpha) \tag{20}$$

in the extended (λ,μ)-plane may be schematically described by the chain

$$\begin{aligned}&(\lambda_1,0) \xrightarrow{F^+_0(\alpha)} (\lambda_1,+\infty) \xrightarrow{F^+_1(\alpha)} (+\infty,\mu_1) \xrightarrow{F^+_2(\alpha)} (\lambda_2,+\infty)\\ &\quad \xrightarrow{F^+_3(\alpha)} (+\infty,\mu_3) \xrightarrow{F^+_4(\alpha)} (\lambda_3,+\infty)\cdots.\end{aligned} \tag{21}$$

Similarly, the negative part of the α-spectrum

$$F^-(\alpha) = \bigcup_{i=0}^{+\infty} F^-_i(\alpha) \tag{22}$$

in the extended (λ,μ)-plane may be described as follows:

$$\begin{aligned}&(0,\mu_1) \xrightarrow{F^-_0(\alpha)} (+\infty,\mu_1) \xrightarrow{F^-_1(\alpha)} (\lambda_1,+\infty) \xrightarrow{F^-_2(\alpha)} (+\infty,\mu_2)\\ &\quad \xrightarrow{F^-_3(\alpha)} (\lambda_2,+\infty) \xrightarrow{F^-_4(\alpha)} (+\infty,\mu_3)\cdots.\end{aligned} \tag{23}$$

Proposition 10. *Let $p > 1$ and $0 < q < 1$ be fixed. It is true that*

$$\begin{gathered}\lim_{\alpha\to+0}\lambda_1(\alpha,p) = +\infty, \qquad \lim_{\alpha\to+\infty}\lambda_1(\alpha,p) = +0,\\ \lim_{\alpha\to+0}\mu_1(\alpha,p) = +0, \qquad \lim_{\alpha\to+\infty}\mu_1(\alpha,p) = +\infty.\end{gathered} \tag{24}$$

Proof. Follows from Propositions 1 and 10. □

5. Solution Sets and Solution Surfaces

Definition 11. A solution set of the problems (3) and (4) is a set F of all triples (λ,μ,α) $(\lambda \geq 0,\ \mu \geq 0,\ \alpha > 0)$ such that there exists a nontrivial solution of the problem.

Let us distinguish between solutions of the problems (3) and (4) with different number of zeros in the interval $(0,1)$.

Let F_i^+ be a set of all triples (λ, μ, α) such that there exists a nontrivial solution of the respective problems (3) and (4), $x'(0) = \alpha > 0$ which has exactly i zeros in $(0, 1)$, $i = 0, 1, \ldots$, but F_i^- be a set of all triples (λ, μ, α) such that there exists a nontrivial solution of the respective problems (3) and (4), $x'(0) = -\alpha < 0$ which has exactly i zeros in $(0, 1)$, $i = 0, 1, \ldots$

Definition 12. F_i^+ will be called a positive i-solution surface, but F_i^--a negative i-solution surface.

5.1. Description and Properties of a Solution Set. We will identify the cross section of a solution surface F_i^s with the plane $\alpha = \alpha_0 > 0$ (α_0 is fixed) with its projection to the (λ, μ)-plane. This projection is in fact the respective branch $F_i^s(\alpha_0)$ of the spectrum of the problem (3), (4), and (9).

Theorem 13.

(1) Solution surfaces are unions of respective α-branches:

$$F_i^+ = \bigcup_{\alpha>0} F_i^+(\alpha), \qquad F_i^- = \bigcup_{\alpha>0} F_i^-(\alpha). \tag{25}$$

(2) A solution set F of the problems (3) *and* (4) *is a union of all i-solution surfaces and is a union of all α-branches:*

$$F = \bigcup_{i=1}^{\infty} F_i^{\pm} = \bigcup_{\alpha>0} F_i^{\pm}(\alpha). \tag{26}$$

(3) Solution surfaces F_{2i-1}^+ and F_{2i-1}^- coincide.

(4) Solution surfaces $F_{2i-1}^{\pm}$ and $F_{2j-1}^{\pm}$ do not intersect unless $i = j$.

(5) For given $i \neq j$ the solution surfaces $F_{2i-1}^{\pm}$ and $F_{2j-1}^{\pm}$ are centroaffine equivalent under the mapping $\Phi_{i,j} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$(\lambda, \mu, \alpha) \xmapsto{\Phi_{i,j}} (\overline{\lambda}, \overline{\mu}, \overline{\alpha}), \tag{27}$$

where $\overline{\lambda} = (j/i)^2\lambda$, $\overline{\mu} = (j/i)^2\mu$, $\overline{\alpha} = (j/i)\alpha$.

Proof. (1), (2), (3), and (4) follow from definitions of F, F_i^+, F_i^-, $F_i^+(\alpha)$, $F_i(\alpha)^-$ and Proposition 7.

(5) First observe that for $\alpha, \beta, \lambda > 0$ we have by making use of the formula (6) that

$$\begin{aligned} T(\beta, \lambda) &= \frac{1}{\sqrt{\lambda}} T\left(\frac{\beta}{\sqrt{\lambda}}, 1\right) \\ &= \frac{1}{\sqrt{\lambda}} T\left(\frac{\alpha}{\sqrt{\lambda}}\frac{\beta}{\alpha}, 1\right) = \frac{1}{\sqrt{\lambda}} T\left(\frac{\alpha}{\sqrt{\lambda\alpha^2/\beta^2}}, 1\right) \\ &= \frac{1}{\sqrt{\lambda\alpha^2/\beta^2}}\frac{\alpha}{\beta} T\left(\frac{\alpha}{\sqrt{\lambda\alpha^2/\beta^2}}, 1\right) = \frac{\alpha}{\beta} T\left(\alpha, \lambda\frac{\alpha^2}{\beta^2}\right). \end{aligned} \tag{28}$$

The above formula is applicable to T_p and to T_q.

For given $i \neq j$ and $\alpha > 0$ set $\overline{\alpha} = (j/i)\alpha$. Suppose $(\lambda, \mu, \alpha) \in F_{2i-1}^{\pm}$:

$$iT_p(\alpha, \lambda) + iT_q(\alpha, \mu) = 1. \tag{29}$$

Applying the rescaling formula (28), where $\overline{\alpha}$ replaces β, to the previous equation one has

$$\begin{gathered} i\frac{\overline{\alpha}}{\alpha} T_p\left(\overline{\alpha}, \lambda\frac{\overline{\alpha}^2}{\alpha^2}\right) + i\frac{\overline{\alpha}}{\alpha} T_q\left(\overline{\alpha}, \mu\frac{\overline{\alpha}^2}{\alpha^2}\right) = 1, \\ jT_p\left(\overline{\alpha}, \left(\frac{j}{i}\right)^2\lambda\right) + jT_q\left(\overline{\alpha}, \left(\frac{j}{i}\right)^2\mu\right) = 1, \\ jT_p(\overline{\alpha}, \overline{\lambda}) + jT_q(\overline{\alpha}, \overline{\mu}) = 1, \end{gathered} \tag{30}$$

therefore $(\overline{\lambda}, \overline{\mu}, \overline{\alpha}) \in F_{2j-1}^+$. Since $\Phi_{i,j}^{-1} = \Phi_{j,i}$, one has $\Phi_{i,j}(F_{2i-1}^{\pm}) = F_{2j-1}^{\pm}$, and the surfaces $F_{2i-1}^{\pm}$ and $F_{2j-1}^{\pm}$ are centroaffine equivalent under the mapping $\Phi_{i,j}$. □

Remark 14. Since solution surfaces $F_{2i-1}^{\pm}$ and $F_{2j-1}^{\pm}$ $(i \neq j)$ are centro-affine equivalent, they have similar shape. Therefore it is enough to study properties of the solution surface $F_1^{\pm}$, in order to know properties of other odd-numbered solution surfaces. The same is true for F_{2i-1}^-.

5.2. Cross-Sections of Solution Surfaces with the Planes $\alpha = Const$. A cross-section of any solution surface F_i^+ or F_i^- by the plane $\alpha = \text{const} > 0$ locates in the sector $Q_1(\alpha) = \{(\lambda, \mu) : \lambda > \lambda_1(\alpha), \mu > \mu_1(\alpha)\}$.

Irrespective of the choice of α no oscillatory (with at least one zero in $(0, 1)$) solution of the problems (3), (4), and (9) exists for (λ, μ) in the "dead zone" below the envelope (see Figure 1).

The analytical description of envelopes, corresponding to branches of spectra, follows.

6. Envelopes of Solution Surfaces

Any solution surface for (3) is defined by one of the following relations:

$$\begin{aligned} F_{2i-1}^{\pm} &= \left\{(\lambda, \mu, \alpha) :\ iT_f(\alpha, \lambda) + iT_g(\alpha, \mu) = 1\right\} \\ &\qquad (i = 1, 2, \ldots), \\ F_{2i}^{+} &= \left\{(\lambda, \mu, \alpha) : (i+1)T_f(\alpha, \lambda) + iT_g(\alpha, \mu) = 1\right\} \\ &\qquad (i = 0, 1, \ldots), \\ F_{2i}^{-} &= \left\{(\lambda, \mu, \alpha) : iT_f(\alpha, \lambda) + (i+1)T_g(\alpha, \mu) = 1\right\} \\ &\qquad (i = 0, 1, \ldots). \end{aligned} \tag{31}$$

We use the unifying formula

$$\mathscr{F} : kT_f(\alpha, \lambda) + mT_g(\alpha, \mu) = 1 \tag{32}$$

and consider

$$H(\lambda, \mu, \alpha) := kT_f(\alpha, \lambda) + mT_g(\alpha, \mu) - 1 = 0. \tag{33}$$

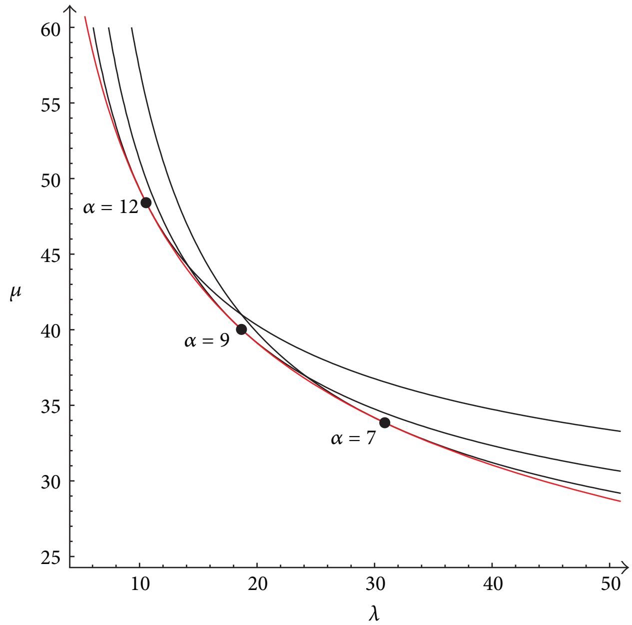


Figure 1: "Dead zone" beneath the envelope (in red) of branches $F_1^{\pm}(\alpha)$ of the spectrum of the problems (3) and (4) for $p = 3$, $q = 1/3$ and $\alpha = 7, 9, 12$.

Treat α as a parameter. The family of envelopes for $H(\lambda, \mu, \alpha) = 0$ can be determined from the system

$$H = 0, \qquad \frac{\partial H}{\partial \alpha} = 0. \tag{34}$$

For the case of (3) one gets

$$\begin{aligned} H(\lambda, \mu, \alpha) = {} & kh(p)\,\alpha^{-(p-1)/(p+1)}\lambda^{-1/(p+1)} \\ & + mh(q)\,\alpha^{-(q-1)/(q+1)}\,\mu^{-1/(q+1)} - 1, \end{aligned} \tag{35}$$

where

$$\begin{aligned} h(p) &= 2^{p/(p+1)}(p+1)^{1/(p+1)}A(p), \\ A(p) &= \int_0^1 \frac{ds}{\sqrt{1 - s^{p+1}}}, \\ \frac{\partial H}{\partial \alpha} &= -k\frac{p-1}{p+1}h(p)\,\alpha^{-2p/(p+1)}\lambda^{-1/(p+1)} \\ & \quad - m\frac{q-1}{q+1}h(q)\,\alpha^{-2q/(q+1)}\mu^{-1/(q+1)}. \end{aligned} \tag{36}$$

Consider the system

$$\begin{aligned} & kh(p)\,\alpha^{-(p-1)/(p+1)}\lambda^{-1/(p+1)} \\ & \quad + mh(q)\,\alpha^{-(q-1)/(q+1)}\mu^{-1/(q+1)} = 1, \\ & k\frac{p-1}{p+1}h(p)\,\alpha^{-2p/(p+1)}\lambda^{-1/(p+1)} \\ & \quad + m\frac{q-1}{q+1}h(q)\,\alpha^{-2q/(q+1)}\mu^{-1/(q+1)} = 0 \end{aligned} \tag{37}$$

and exclude the parameter α. One obtains that

$$\mu = \left(\frac{\psi(p,q)m^{(q+1)(p-1)}}{\psi(q,p)k^{(p+1)(q-1)}}\right)^{1/(p-1)} \lambda^{(q-1)/(p-1)}, \tag{38}$$

where

$$\psi(p,q) = \left(\frac{(q-1)(p+1)}{2(q-p)h(p)}\right)^{(p+1)(q-1)}. \tag{39}$$

(1) For

$$F_{2i-1}^{\pm} : iT_f(\alpha, \lambda) + iT_g(\alpha, \mu) = 1, \tag{40}$$

$k = i$, $m = i$, and the equation of the envelope is

$$\begin{aligned} \mathscr{E}_{2i-1}^{\pm} : \mu &= \omega_{2i-1}^{\pm}(\lambda) \\ &:= \left(\frac{\psi(p,q)\,i^{(q+1)(p-1)}}{\psi(q,p)\,i^{(p+1)(q-1)}}\right)^{1/(p-1)} \lambda^{(q-1)/(p-1)}. \end{aligned} \tag{41}$$

(2) For

$$F_{2i}^{+} : (i+1)T_f(\alpha, \lambda) + iT_g(\alpha, \mu) = 1, \tag{42}$$

$k = i + 1$, $m = i$, the equation of the envelope is

$$\begin{aligned} \mathscr{E}_{2i}^{+} : \mu &= \omega_{2i}^{+}(\lambda) \\ &:= \left(\frac{\psi(p,q)\,i^{(q+1)(p-1)}}{\psi(q,p)\,(i+1)^{(p+1)(q-1)}}\right)^{1/(p-1)} \lambda^{(q-1)/(p-1)}. \end{aligned} \tag{43}$$

(3) For

$$F_{2i}^{-} : iT_f(\alpha,\lambda) + (i+1)T_g(\alpha,\mu) = 1, \tag{44}$$

$k = i$, $m = i+1$, the equation of the envelope is

$$\mathscr{E}_{2i}^{-} : \mu = \omega_{2i}^{-}(\lambda) := \left(\frac{\psi(p,q)(i+1)^{(q+1)(p-1)}}{\psi(q,p)\, i^{(p+1)(q-1)}} \right)^{1/(p-1)} \lambda^{(q-1)/(p-1)}. \tag{45}$$

Proposition 15. *For given $p > 1$ and $0 < q < 1$ the location of the envelopes is as follows:*

(1) $\mathscr{E}_{2i-1}^{\pm} \prec \mathscr{E}_{2i}^{+}$,
(2) $\mathscr{E}_{2i-1}^{\pm} \prec \mathscr{E}_{2i}^{-}$,
(3)

(a) *if $pq - 1 < 0$, then $\mathscr{E}_{2i}^{-} \prec \mathscr{E}_{2i}^{+}$,*
(b) *if $pq - 1 = 0$, then $\mathscr{E}_{2i}^{-} = \mathscr{E}_{2i}^{+}$,*
(c) *if $pq - 1 > 0$, then $\mathscr{E}_{2i}^{+} \prec \mathscr{E}_{2i}^{-}$,*

(4) $\mathscr{E}_{2i}^{+} \prec \mathscr{E}_{2(i+1)-1}^{\pm}$,
(5) $\mathscr{E}_{2i}^{-} \prec \mathscr{E}_{2(i+1)-1}^{\pm}$,

where $\mathscr{E}_n^s \prec \mathscr{E}_m^t$ means that $\omega_n^s(\lambda) < \omega_m^t(\lambda)$ for any $\lambda > 0$.

Proof. (1) First consider

$$\mathscr{E}_{2i-1}^{\pm} : \mu = \omega_{2i-1}^{\pm}(\lambda) := \left(\frac{\psi(p,q)\, i^{(q+1)(p-1)}}{\psi(q,p)\, i^{(p+1)(q-1)}} \right)^{1/(p-1)} \lambda^{(q-1)/(p-1)},$$

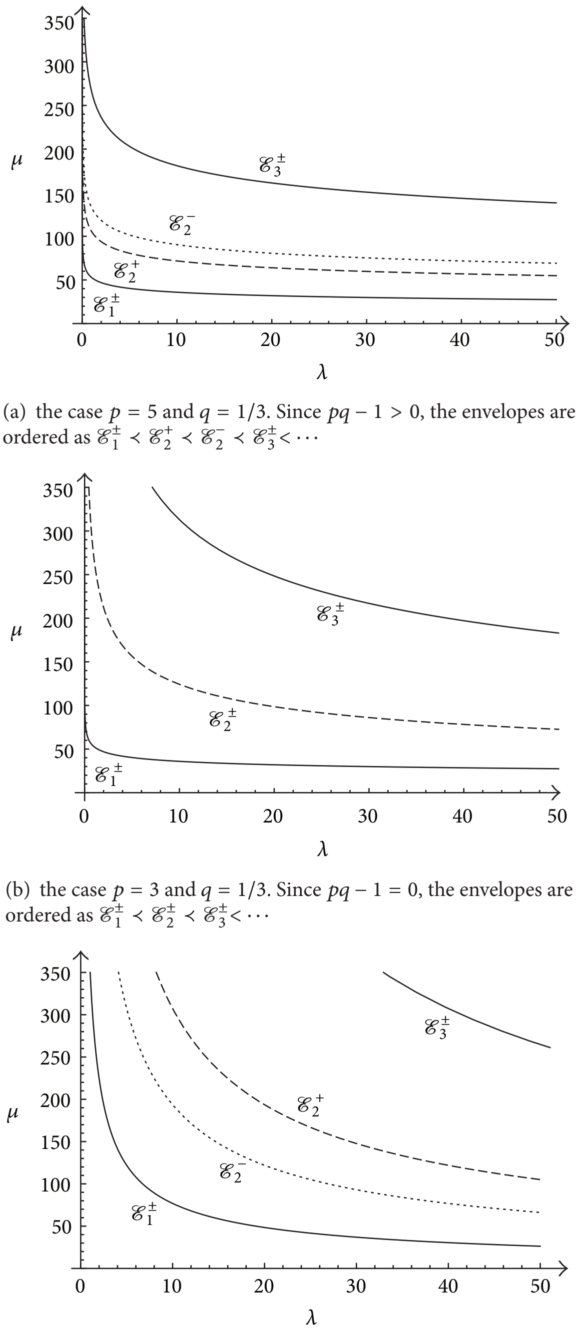
$$\mathscr{E}_{2i}^{+} : \mu = \omega_{2i}^{+}(\lambda) := \left(\frac{\psi(p,q)\, i^{(q+1)(p-1)}}{\psi(q,p)(i+1)^{(p+1)(q-1)}} \right)^{1/(p-1)} \lambda^{(q-1)/(p-1)}. \tag{46}$$

For any $\lambda > 0$,

$$\frac{\omega_{2i}^{+}(\lambda)}{\omega_{2i-1}^{\pm}(\lambda)} = \frac{i^{(q+1)(p-1)} i^{(p+1)(q-1)}}{(i+1)^{(p+1)(q-1)} i^{(q+1)(p-1)}} = \left(\frac{i}{i+1} \right)^{(p+1)(q-1)} > 1, \tag{47}$$

then $\mathscr{E}_{2i-1}^{\pm} \prec \mathscr{E}_{2i}^{+}$. □

Remark 16. Layout of envelopes depends on sign$(pq-1)$ (see Figure 2).



(a) the case $p = 5$ and $q = 1/3$. Since $pq - 1 > 0$, the envelopes are ordered as $\mathscr{E}_1^{\pm} \prec \mathscr{E}_2^{+} \prec \mathscr{E}_2^{-} \prec \mathscr{E}_3^{\pm} < \cdots$

(b) the case $p = 3$ and $q = 1/3$. Since $pq - 1 = 0$, the envelopes are ordered as $\mathscr{E}_1^{\pm} \prec \mathscr{E}_2^{\pm} \prec \mathscr{E}_3^{\pm} < \cdots$

(c) the case $p = 2$ and $q = 1/3$. Since $pq - 1 < 0$, the envelopes are ordered as $\mathscr{E}_1^{\pm} \prec \mathscr{E}_2^{-} \prec \mathscr{E}_2^{+} \prec \mathscr{E}_3^{\pm} < \cdots$

Figure 2: Layout of envelopes depends on sign$(pq - 1)$.

7. The Number of Solutions by Geometrical Analysis of Solution Surfaces

We can detect the precise number of solutions to the problem for given positive (λ,μ). We can evaluate the initial values $x'(0)$ for solutions on a basis of geometrical analysis of solution surfaces and the respective envelopes. The nodal structure of solutions can be described also.

Let $\mu = c^2\lambda$ be the family of rays, covering the first quadrant of the (λ,μ)-plane. Consider the cross-section of a solution surface (32) by the plane $\mu = c^2\lambda$ in the (λ, μ, α)-space, for example, the curve $\mathcal{F}|_{\mu=c^2\lambda}$, which is defined by the relations $\mu = c^2\lambda$ and

$$kT_p(\alpha, \lambda) + mT_q(\alpha, c^2\lambda) = 1. \tag{48}$$

This 3D curve $\mathcal{F}|_{\mu=c^2\lambda}$ can be regularly parameterized as

$$\begin{aligned} \lambda &= \lambda(r) := \left[\frac{ckt_f(cr) + mt_g(r)}{c}\right]^2, \\ \mu &= \mu(r) := \left[ckt_f(cr) + mt_g(r)\right]^2, \\ \alpha &= \alpha(r) := r\left[ckt_f(cr) + mt_g(r)\right], \end{aligned} \tag{49}$$

where $r > 0$. These formulas define homeomorphism $\mathbb{R}_+ \to \mathcal{F}|_{\mu=c^2\lambda}$, where $\mathbb{R}_+ = (0, +\infty)$.

Let $\mathcal{G}_c$ be projection of $\mathcal{F}|_{\mu=c^2\lambda}$ to the (λ, α) plane, that is:

$$\mathcal{G}_c = \left\{(\lambda, \alpha) \in \mathbb{R}_+^2 : kT_f(\alpha, \lambda) + mT_g\left(\alpha, c^2\lambda\right) = 1\right\}. \tag{50}$$

The curve $\mathcal{G}_c$ can be regularly [11] parameterized as

$$\begin{aligned} \lambda &= \lambda(r) := \left[\frac{ckt_f(cr) + mt_g(r)}{c}\right]^2, \\ \alpha &= \alpha(r) := r\left[ckt_f(cr) + mt_g(r)\right], \end{aligned} \tag{51}$$

where $r > 0$. These formulas define homeomorphism $\mathbb{R}_+ \to \mathcal{G}_c$.

Proposition 17. *There exists a unique parameter $r_0 > 0$ such that the line $\lambda = \lambda_* > 0$*

(a) *does not intersect the curve $\mathcal{G}_c$ if $0 < \lambda_* < \lambda(r_0)$,*

(b) *intersects the curve $\mathcal{G}_c$ only once if $\lambda_* = \lambda(r_0)$,*

(c) *intersects the curve $\mathcal{G}_c$ exactly at two points if $\lambda_* > \lambda(r_0)$.*

Proof. (a) Consider equation $\lambda'(r) = kc^2t_p'(cr) + mt_q'(r) = 0$, which turns to

$$\begin{aligned} &kc^2\frac{p-1}{p+1}h(p)c^{-2p/(p+1)}r^{-2p/(p+1)} \\ &\qquad + m\frac{q-1}{q+1}h(q)r^{-2q/(q+1)} = 0. \end{aligned} \tag{52}$$

It can be found from the above that

$$r_0 = \left[-\frac{m(q-1)(p+1)h(q)}{k(p-1)(q+1)h(p)c^{2/(p+1)}}\right]^{-(p+1)(q+1)/2(p-q)} \tag{53}$$

is the only root of the equation $\lambda'(r) = 0$. Hence at the point $(\lambda_0, \alpha_0) = (\lambda(r_0), \alpha(r_0))$ the curve $\mathcal{G}_c$ has a unique tangent line parallel to the $O\alpha$ axis.

(b) Since the given parametrization of the curve $\mathcal{G}_c$ is regular, then $\alpha'(r_0) \neq 0$ and in some neighborhood of the point (λ_0, α_0) the curve $\mathcal{G}_c$ can be represented as the graph of the function $\lambda = \varphi(\alpha)$. Now find

$$\begin{aligned} \varphi''(r_0) &= \frac{\lambda''(r_0)}{(\alpha'(r_0))^2} \\ &= \frac{2\left[ckt_f(cr) + mt_g(r)\right]\left[kc^3t_f''(cr_0) + mt_g''(r_0)\right]}{c^2\left[\alpha'(r_0)\right]^2}. \end{aligned} \tag{54}$$

The next step is to detect the sign of the expression

$$\begin{aligned} &kc^3t_p''(cr) + mt_q''(r) \\ &= kc^3\frac{2p(p-1)}{(p+1)^2}h(p)c^{-(3p+1)/(p+1)}r^{-(3p+1)/(p+1)} \\ &\quad + m\frac{2q(q-1)}{(q+1)^2}h(q)r^{-(3q+1)/(q+1)} \\ &= k\frac{2p(p-1)}{(p+1)^2}h(p)c^{3-(3p+1)/(p+1)}r^{-(3p+1)/(p+1)} \\ &\quad + m\frac{2q(q-1)}{(q+1)^2}h(q)r^{-(3q+1)/(q+1)} \\ &= r^{-(3q+1)/(q+1)} \\ &\quad \times \left(k\frac{2p(p-1)}{(p+1)^2}h(p)c^{3-(3p+1)/(p+1)}\right. \\ &\qquad \times r^{-(3(p+1)/(p+1))+((3q+1)/(q+1))} \\ &\qquad \left. + m\frac{2q(q-1)}{(q+1)^2}h(q)\right). \end{aligned} \tag{55}$$

The routine calculations show that the expression in parentheses is equal to

$$\begin{aligned} &-m\frac{q-1}{q+1}h(q)\left(\frac{2p}{p+1} - \frac{2q}{q+1}\right) \\ &\quad = -m\,\frac{q-1}{q+1}\,h(q)\,\frac{2(p-q)}{(p+1)(q+1)} > 0. \end{aligned} \tag{56}$$

Hence the function $\lambda = \varphi(\alpha)$ has the strict local minimum at the point $\alpha = \alpha_0$.

(c) It follows from the relations

$$\begin{aligned} &\lim_{r\to 0+} t_p(r) = +\infty, \qquad \lim_{r\to +\infty} t_p(r) = 0+, \\ &\lim_{r\to 0+} rt_p(r) = 0+, \qquad \lim_{r\to +\infty} rt_p(r) = +\infty, \\ &\lim_{r\to 0+} t_q(r) = 0+, \qquad \lim_{r\to +\infty} t_q(r) = +\infty, \\ &\lim_{r\to 0+} rt_q(r) = 0+, \qquad \lim_{r\to +\infty} rt_q(r) = +\infty \end{aligned} \tag{57}$$

that if a parameter $r > 0$ goes from 0 to $+\infty$, then a point $(\lambda, \alpha) \in \mathscr{G}_c$ goes from the point $(+\infty, 0+)$ to the point $(+\infty, +\infty)$.

(d) It follows from the above argument that if the parameter $r > 0$ goes from 0 to $+\infty$ then a point $(\lambda, \alpha) \in \mathscr{G}_c$ goes from $(+\infty, 0+)$ to (λ_0, α_0), then turns to the right, and goes to $(+\infty, +\infty)$. Since the parametrization of the curve $\mathscr{G}_c$ is without self-intersection points, one can deduce that the curve $\mathscr{G}_c$ is a union of two branches (i.e., graphs of the functions $\alpha = \psi_1(\lambda)$ and $\alpha = \psi_2(\lambda)$, where $(\lambda_0 < \lambda < +\infty)$ which do not intersect and are continuously "glued" at the point (λ_0, α_0). □

Before presenting "the exact number of solutions" result we make the following conventions:

(1) solution means a nontrivial solution of the problems (3) and (4),

(2) $\mathscr{E}_n^s$ means the envelope, where $n \in \mathbb{N}$ and s is either + or − or ±,

(3) we mean by s-solution a solution of the problem (3), (4): (a) a solution with $x'(0) > 0$ if $s = +$; (b) a solution with $x'(0) < 0$ if $s = -$; (c) two solutions with $x'(0) = \alpha > 0$ and $x'(0) = -\alpha < 0$ if $s = \pm$.

Proposition 18. *Consider an envelope*

$$\mathscr{E}_n^s = \left\{(\lambda, \mu) \in \mathbb{R}_+^2 : \mu = \omega_n^s(\lambda)\right\}, \quad n > 0. \tag{58}$$

The problems (3) *and* (4) *have*

(a) *no s-solutions with n zeroes in* $(0, 1)$ *if* $\mu < \omega_n^s(\lambda)$,

(b) *exactly one s-solution with even number n zeroes in* $(0, 1)$ *if* $s = +$ *or* $s = -$ *and* $\mu = \omega_n^s(\lambda)$,

(c) *exactly two s-solutions with n zeroes in* $(0, 1)$ *if*

n is odd, $\mu = \omega_n^s(\lambda)$ *and* $s = \pm$

or

n is even, $\mu > \omega_n^s(\lambda)$ *and* $s = +$ *or* $s = -$,

(d) *exactly four s-solutions with odd number n of zeroes in* $(0, 1)$ *if* $s = \pm$ *and* $\mu > \omega_n^s(\lambda)$.

Proof. It can be verified analytically that $(\lambda_0, \mu_0) \in \mathscr{E}_n^s$, where

$$\lambda_0 = \lambda(r_0) = \frac{1}{c^2}\left(ckh(p)(cr_0)^{-(p-1)/(p+1)} + mh(q)\,r_0^{-(q-1)/(q+1)}\right)^2. \tag{59}$$

and $\mu_0 = c^2\lambda(r_0)$.

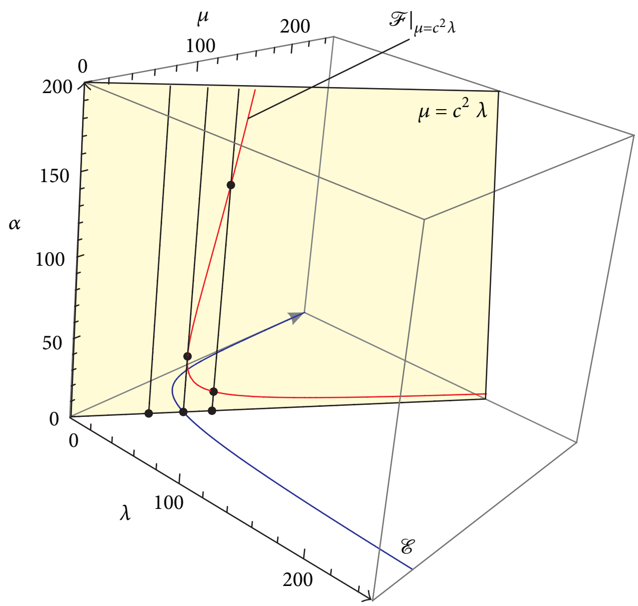


Figure 3: The case $p = 3$, $q = 1/2$, $k = 2$, $m = 1$ and $c = 1.2$.

Recall that the curve $\mathscr{G}_c$ is the projection of the 3D curve $\mathscr{F}|_{\mu=c^2\lambda}$ to the (λ, α)-plane. Taking in mind Proposition 17 we can assert that there exists a unique parameter $r_0 > 0$, see (53), such that the line parallel to the $O\alpha$ axis in the (λ, μ, α)-space going through the point (λ, μ), where $\lambda > 0$ and $\mu = c^2\lambda$, and the curve $\mathscr{F}|_{\mu=c^2\lambda}$ (hence the solution surface $\mathscr{F}$ also)

(i) does not intersect if $\mu < \omega_n^s(\lambda)$,

(ii) intersects only once if $\mu = \omega_n^s(\lambda)$,

(iii) intersects exactly at two points if $\mu > \omega_n^s(\lambda)$,

(see Figure 3).

Depending on the value of s one can detect the number of s-solutions as the theorem states. For example, in the case (d): if n is odd, $\mu > \omega_n^s(\lambda)$ and $s = \pm$, then the mentioned above line intersects the solution surface F_n^+ exactly twice. Since $F_n^- = F_n^+$ the mentioned above line intersects the solution surface F_n^- exactly twice also. Hence the problem (3), (4) has exactly four solutions with n zeros in $(0; 1)$. □

Now we are able to prove the main theorem. In this theorem we suppose that $\omega_0^-(\lambda) = 0$ and $\omega_0^+(\lambda) = 0$ for all $\lambda > 0$ and will use auxiliary envelopes $\mathscr{E}_0^- : \mu = \omega_0^-(\lambda)$ and $\mathscr{E}_0^+ : \mu = \omega_0^+(\lambda)$.

Theorem 19. *Suppose* $0 < q < 1 < p$.

(1) *If* $pq - 1 > 0$ *and* $i \in \mathbb{N}$, *then the exact number of nontrivial solutions to the problems* (3) *and* (4) *is*

(a) $8i - 6$ *if* $\omega_{2i-2}^-(\lambda) < \mu < \omega_{2i-1}^\pm(\lambda)$,
(b) $8i - 4$ *if* $\mu = \omega_{2i-1}^\pm(\lambda)$,
(c) $8i - 2$ *if* $\omega_{2i-1}^\pm(\lambda) < \mu < \omega_{2i}^+(\lambda)$,
(d) $8i - 1$ *if* $\mu = \omega_{2i}^+(\lambda)$,
(e) $8i$ *if* $\omega_{2i}^+(\lambda) < \mu < \omega_{2i}^-(\lambda)$,
(f) $8i + 1$ *if* $\mu = \omega_{2i}^-(\lambda)$.

(2) *If* $pq - 1 = 0$ *and* $i \in \mathbb{N}$, *then the exact number of nontrivial solutions to the problems* (3) *and* (4) *is*

(a) $8i - 6$ *if* $\omega^{+}_{2i-2}(\lambda) < \mu < \omega^{\pm}_{2i-1}(\lambda)$,
(b) $8i - 4$ *if* $\mu = \omega^{\pm}_{2i-1}(\lambda)$,
(c) $8i - 2$ *if* $\omega^{\pm}_{2i-1}(\lambda) < \mu < \omega^{+}_{2i}(\lambda)$,
(d) $8i$ *if* $\mu = \omega^{+}_{2i}(\lambda)$.

(3) *If* $pq - 1 < 0$ *and* $i \in \mathbb{N}$ *then the exact number of nontrivial solutions to the problems* (3) *and* (4) *is*

(a) $8i - 6$ *if* $\omega^{+}_{2i-2}(\lambda) < \mu < \omega^{\pm}_{2i-1}(\lambda)$,
(b) $8i - 4$ *if* $\mu = \omega^{\pm}_{2i-1}(\lambda)$,
(c) $8i - 2$ *if* $\omega^{\pm}_{2i-1}(\lambda) < \mu < \omega^{-}_{2i}(\lambda)$,
(d) $8i - 1$ *if* $\mu = \omega^{-}_{2i}(\lambda)$,
(e) $8i$ *if* $\omega^{-}_{2i}(\lambda) < \mu < \omega^{+}_{2i}(\lambda)$,
(f) $8i + 1$ *if* $\mu = \omega^{+}_{2i}(\lambda)$.

Proof. First notice that if $\lambda > 0$ and $\mu > 0$ then there exists exactly one $\pm$-solution without zeros in $(0, 1)$, in fact, accordingly to conventions, two solutions without zeros in $(0, 1)$, and of opposite sign values of $x'(0)$.

We consider only the case $pq - 1 > 0$. Similarly two other possible cases $pq - 1 < 0$ and $pq - 1 = 0$ can be treated.

It follows from Proposition 15 that the envelopes are ordered as

$$\begin{aligned} &\mathcal{E}^{-}_{0} \prec \mathcal{E}^{\pm}_{1} \prec \mathcal{E}^{+}_{2} \prec \mathcal{E}^{-}_{2} \prec \mathcal{E}^{\pm}_{3} \prec \cdots \prec \mathcal{E}^{\pm}_{2i-1} \\ &\quad \prec \mathcal{E}^{+}_{2i} \prec \mathcal{E}^{-}_{2i} \prec \mathcal{E}^{\pm}_{2i+1} \prec \cdots . \end{aligned} \tag{60}$$

Let us consider the case (1), when $pq - 1 > 0$, and the subcase (a): $\omega^{-}_{2i-2}(\lambda) < \mu < \omega^{\pm}_{2i-1}(\lambda)$ for $i = 1, 2, \ldots$.

(i) if $i = 1$ then there are $2 = 8 \cdot 1 - 6$ solutions without zeros;

(ii) if $i = 2$, then there are *two* solutions without zeros, two $\pm$-solutions with 1 zero (that is, *four* solutions with 1 zero), *two* "+"-solution with 2 zeros and *two* "−"-solutions with 2 zeros; totally there are $10 = 8 \cdot 2 - 6$ (nontrivial) solutions;

(iii) if $i = 3$, then in addition to 10 solutions mentioned in the previous step, there are also two $\pm$-solutions with 3 zeros (that is, *four* solutions with 3 zeros), *two* "+"-solutions with 4 zeros and *two* "−"-solution with 4 zeros; totally there are $18 = 8 \cdot 3 - 6$ (nontrivial) solutions and so on;

(iv) if $i \in \mathbb{N}$, then totally there are $8i - 6$ (nontrivial) solutions.

The other cases can be considered analogously. □

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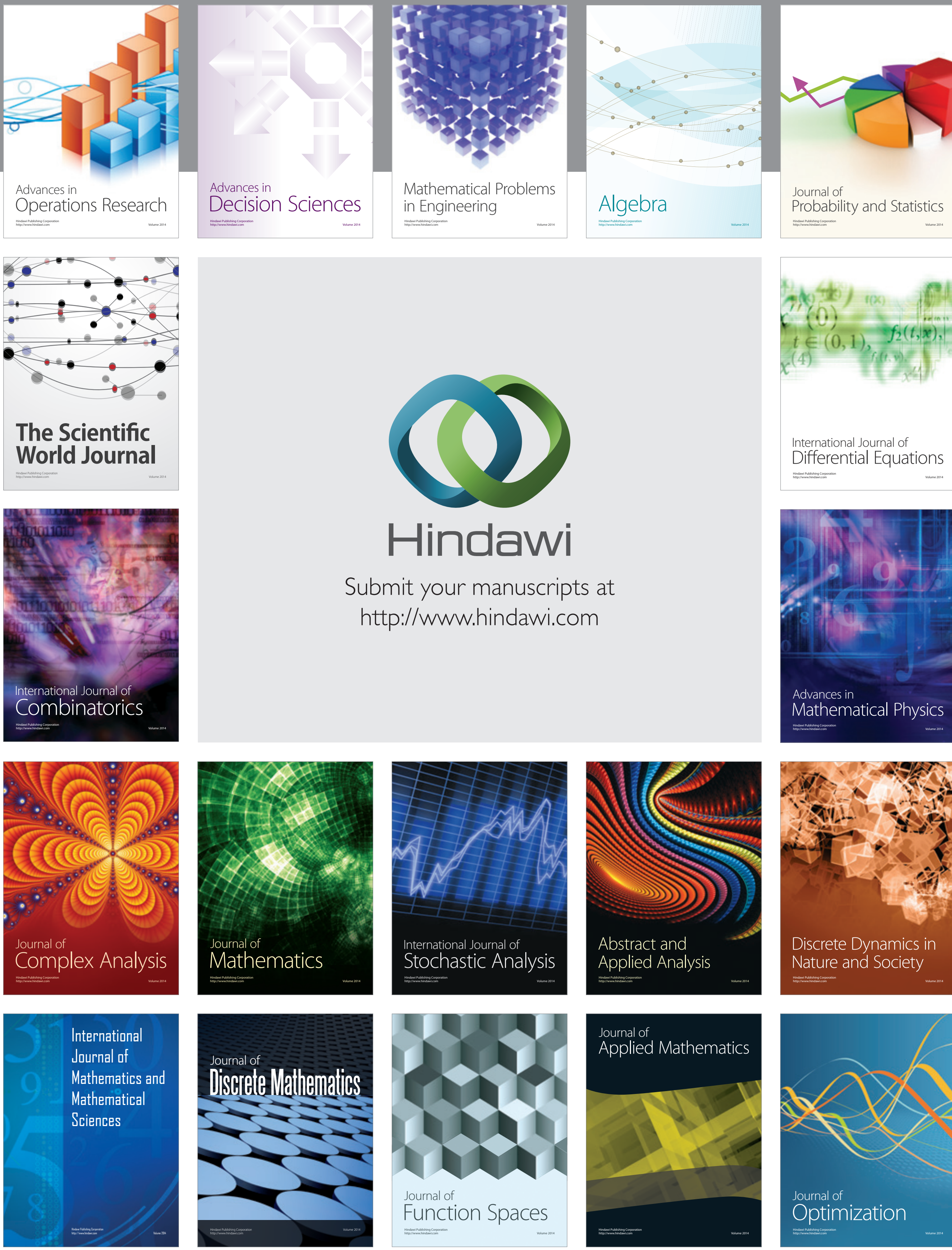
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