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Research Article

Faber Polynomial Coefficient Estimates for Meromorphic Bi-Starlike Functions

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We consider meromorphic starlike univalent functions that are also bi-starlike and find Faber polynomial coefficient estimates for these types of functions. A function is said to be bi-starlike if both the function and its inverse are starlike univalent.

Consider the function

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \tag{1}$$

where the coefficients $(b_0, b_1, b_2, \ldots, b_n, \ldots)$ are in the submanifold M on C^N such that $g(z)$ is univalent in $\Delta := \{z : 1 < |z| < \infty\}$. Therefore

$$\frac{zg'(z)}{g(z)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, b_2, \ldots, b_n) \frac{1}{z^{n+1}}, \tag{2}$$

where $F_{n+1}(b_0, b_1, b_2, \ldots, b_n)$ is a Faber polynomial of degree $n+1$. (Also see [1, 2].) We note that

$$\begin{aligned} F_1 &= -b_0, \\ F_2 &= b_0^2 - 2b_1, \\ F_3 &= -b_0^3 + 3b_1 b_0 - 3b_2, \\ F_4 &= b_0^4 - 4b_0^2 b_1 + 4b_0 b_2 + 2b_1^2 - 4b_3, \\ F_5 &= -b_0^5 + 5b_0^3 b_1 - 5b_0^2 b_2 - 5b_0 b_1^2 + 5b_1 b_2 + 5b_0 b_3 - 5b_4, \\ F_6 &= b_0^6 + 3b_2^2 + 6b_0^3 b_2 - 12 b_0 b_1 b_2 - 6b_0^4 b_1 - 2b_1^3 \\ &\quad + 9b_0^2 b_1^2 + 6b_0 b_4 + 6b_1 b_3 - 6b_0^2 b_3 - 6b_5. \end{aligned} \tag{3}$$

In general (also see Bouali [3, page 52])

$$F_{n+1}(b_0, b_1, \ldots, b_n) = \sum_{i_1 + 2i_2 + \cdots + (n+1)i_{n+1} = n+1} A(i_1, i_2, \ldots, i_{n+1}) b_0^{i_1} b_1^{i_2} \cdots b_n^{i_{n+1}}, \tag{4}$$

where

$$A(i_1, i_2, \ldots, i_{n+1}) := (-1)^{(n+1)+2i_1+\cdots+(n+2)i_{n+1}} \times \frac{(i_1 + i_2 + \cdots + i_{n+1} - 1)!\,(n+1)}{(i_1!)(i_2!)\cdots(i_{n+1}!)}. \tag{5}$$

The coefficients of g^{-1}, the inverse map of g are given by

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n \geq 1} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}, \tag{6}$$

where

$$\begin{aligned}K_{n+1}^{n} &= nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 \\ &\quad + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}\left(b_3 + b_1^2\right) \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}\left(b_4 + 3b_1b_2\right) \\ &\quad + \sum_{j\geq 5} b_0^{n-j}V_j\end{aligned} \quad (7)$$

and V_j with $5 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, \ldots, b_n$. (Also see [1, page 349].)

Similarly

$$\frac{wh'(w)}{h(w)} = 1 + \sum_{n=1}^{\infty} F_n\left(B_0, B_1, B_2, \ldots, B_n\right)\frac{1}{w^n}, \quad (8)$$

where $F_n(B_0, B_1, B_2, \ldots, B_n)$ is a Faber polynomial of degree n and

$$\begin{aligned}F_n = {}& \frac{-n(n-(n+1))!}{n!(n-2n)!}B_0^n - \frac{n(n-(n+1))!}{(n-2)!(n-(2n-1))!}B_0^{n-2}B_1 \\ & - \frac{n(n-(n+1))!}{(n-3)!(n-(2n-2))!}B_0^{n-3}B_2 \\ & - \frac{n(n-(n+1))!}{(n-4)!(n-(2n-3))!}B_0^{n-4}\left(B_3 + \frac{n-(2n-3)}{2}B_1^2\right) \\ & - \sum_{j\geq 5} B_0^{n-j}K_j,\end{aligned} \quad (9)$$

where K_j for $5 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $B_1, B_2, \ldots, B_{n-1}$.

The Faber polynomials introduced by Faber [4] play an important role in various areas of mathematical sciences, especially in geometric function theory (e.g., see Gong [5] and Schiffer [6]). The recent interest in the calculus of the Faber polynomials, especially when it involves the function $h = g^{-1}$, the inverse map of g (see [2, page 186]) beautifully fits the case for the meromorphic bi-univalent functions.

The function g is said to be meromorphic bi-univalent in Δ if both g and its inverse $h = g^{-1}$ are meromorphic univalent in Δ. By the same token, the function g is said to be meromorphic bi-starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ if both g and its inverse map $h = g^{-1}$ are meromorphic starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ, that is,

$$\begin{aligned}&\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > \alpha \quad (z \in \Delta), \\ &\operatorname{Re}\left(\frac{wh'(w)}{h(w)}\right) > \alpha \quad (w \in \Delta).\end{aligned} \quad (10)$$

Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature. For example, Schiffer [6] obtained the estimate $|b_2| \leq 2/3$ for meromorphic univalent functions g with $b_0 = 0$ and Duren ([7] or [8, Theorem 4.9, page 139]) proved that if $b_1 = b_2 = \cdots = b_k = 0$ for $1 \leq k < (1/2)n$ then $|b_n| \leq 2/(n+1)$. He then proved that this bound also holds for meromorphic starlike univalent functions g of order zero (Duren [8, Theorem 4.8, page 137]). So far, the latest known results are given by the following two articles. Kapoor and Mishra [9] found sharp bounds for the coefficients of starlike univalent functions of order α; $0 \leq \alpha < 1$ in Δ and for its inverse functions they obtained the bound $2(1-\alpha)/(n+1)$ when $((n-1)/n) \leq \alpha < 1$. More recently, Srivastava et al. [10] found sharp bounds for the coefficients of starlike univalent functions of order α, $0 \leq \alpha < 1$, having m-fold gaps in their series representation in Δ and also for their inverse functions. The above two articles settled the coefficient bounds for starlike functions and their inverses but they have not considered the bi-starlike case. The problem arises when the bi-univalency condition is imposed on the meromorphic functions g. The bi-univalency requirement makes the task of finding bounds for the coefficients of g and its inverse map $h = g^{-1}$ more involved. In this paper, for the first time, we use the Faber polynomial expansions to study the coefficients of meromorphic bi-starlike functions. As a result, we are able to prove.

Theorem 1. *Let $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n(1/z^n)$ be meromorphic bi-starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ. If $b_1 = b_2 = \cdots = b_{n-1} = 0$ for n being odd or if $b_0 = b_1 = \cdots = b_{n-1} = 0$ for n being even, then*

$$\left|b_n\right| \leq \frac{2(1-\alpha)}{n+1}; \quad 0 \leq \alpha < 1, \ n \in \mathbb{N}. \quad (11)$$

Proof. Suppose that the function g is a meromorphic bi-starlike function of order $\alpha : (0 \leq \alpha < 1)$ in Δ. Then both g and its inverse $h = g^{-1}$ are starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ. Therefore, by definition, there exist two functions p and q with positive real parts in Δ of the form

$$\begin{aligned}p(z) &= 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta), \\ q(w) &= 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta),\end{aligned} \quad (12)$$

so that

$$\begin{aligned}\frac{zg'(z)}{g(z)} &= \alpha + (1-\alpha)p(z) \\ &= 1 + \frac{(1-\alpha)c_1}{z} + \frac{(1-\alpha)c_2}{z^2} + \frac{(1-\alpha)c_3}{z^3} + \cdots, \\ \frac{wh'(w)}{h(w)} &= \alpha + (1-\alpha)q(w) \\ &= 1 + \frac{(1-\alpha)d_1}{w} + \frac{(1-\alpha)d_2}{w^2} + \frac{(1-\alpha)d_3}{w^3} + \cdots.\end{aligned} \quad (13)$$

Note that, according to the Caratheodory lemma (see Duren [8, page 41]), $|c_n| \leq 2$ and $|d_n| \leq 2$ for $n = 1, 2, 3, \ldots$. On the other hand, comparing the corresponding coefficients of the functions g and $h = g^{-1}$, we obtain

$$B_0 = -b_0, \qquad B_n = -\frac{1}{n}K_{n+1}^{n}. \tag{14}$$

Now, from $F_{n+1}(b_0, b_1, \ldots, b_n)$ and $F_{n+1}(B_0, B_1, B_2, \ldots, B_n)$, upon noting that there are just two choices of $i_1 = n$ and $i_2 = i_3 = \cdots = i_n = 0$ or $i_1 = i_2 = \cdots = i_{n-1} = 0$ and $i_n = 1$, we obtain

$$\begin{gathered} F_{n+1} = \begin{cases} b_0^{n+1} - (n+1)\, b_n & n = \text{odd} \\ -b_0^{n+1} - (n+1)\, b_n, & n = \text{even}, \end{cases} \\ F_{n+1} = \begin{cases} B_0^{n+1} - (n+1)\, B_n & n = \text{odd} \\ -B_0^{n+1} - (n+1)\, B_n, & n = \text{even}. \end{cases} \end{gathered} \tag{15}$$

Since $B_n = -b_n$ for the second system of equation we can write

$$F_{n+1} = \begin{cases} b_0^{n+1} + (n+1)\, b_n & n = \text{odd} \\ b_0^{n+1} + (n+1)\, b_n, & n = \text{even}. \end{cases} \tag{16}$$

Therefore, for odd n, we obtain the system of equations

$$\begin{gathered} b_0^{n+1} - (n+1)\, b_n = (1-\alpha)\, c_{n+1}, \\ b_0^{n+1} + (n+1)\, b_n = (1-\alpha)\, d_{n+1}. \end{gathered} \tag{17}$$

Hence

$$2\,(n+1)\, b_n = (1-\alpha)\left(d_{n+1} - c_{n+1}\right). \tag{18}$$

Applying the Caratheodory Lemma yields

$$\left|b_n\right| \le \frac{4\,(1-\alpha)}{2\,(n+1)} = \frac{2\,(1-\alpha)}{n+1}. \tag{19}$$

Similarly, for even n with $(b_0 = b_1 = \cdots = b_{n-1} = 0)$, we obtain

$$\begin{gathered} (n+1)\, b_n = -(1-\alpha)\, c_{n+1}, \\ (n+1)\, b_n = +(1-\alpha)\, d_{n+1}. \end{gathered} \tag{20}$$

Hence

$$2\,(n+1)\, b_n = (1-\alpha)\left(d_{n+1} - c_{n+1}\right), \tag{21}$$

which upon applying the Caratheodory Lemma, we obtain

$$\left|b_n\right| \le \frac{2\,(1-\alpha)}{n+1}. \tag{22}$$

Relaxing the coefficient restrictions imposed on Theorem 1, we can prove the following. □

Theorem 2. *Let $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n(1/z^n)$ be meromorphic bi-starlike of order $\alpha : (0 \le \alpha < 1)$ in Δ. Then*

(i) $|b_0| \le \sqrt[2]{2(1-\alpha)}$,

(ii) $|b_1| \le (1-\alpha)$.

Proof. Comparing the corresponding coefficients of

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= 1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} - \cdots, \\ \frac{zg'(z)}{g(z)} &= \alpha + (1-\alpha)\, p(z) \\ &= 1 + \frac{(1-\alpha)\, c_1}{z} + \frac{(1-\alpha)\, c_2}{z^2} + \cdots, \end{aligned} \tag{23}$$

we obtain

$$\begin{gathered} -b_0 = (1-\alpha)\, c_1, \\ b_0^2 - 2b_1 = (1-\alpha)\, c_2. \end{gathered} \tag{24}$$

Similarly, comparing the corresponding coefficients of

$$\begin{aligned} \frac{wh'(w)}{h(w)} &= 1 + \frac{b_0}{w} + \frac{b_0^2 + 2b_1}{w^2} + \cdots, \\ \frac{wh'(w)}{h(w)} &= \alpha + (1-\alpha)\, q(w) \\ &= 1 + \frac{(1-\alpha)\, d_1}{w} + \frac{(1-\alpha)\, d_2}{w^2} + \cdots, \end{aligned} \tag{25}$$

we obtain

$$\begin{gathered} b_0 = (1-\alpha)\, d_1, \\ b_0^2 + 2b_1 = (1-\alpha)\, d_2. \end{gathered} \tag{26}$$

Adding $b_0^2 - 2b_1 = (1-\alpha)c_2$ and $b_0^2 + 2b_1 = (1-\alpha)d_2$, we obtain

$$2b_0^2 = (1-\alpha)\left(c_2 + d_2\right) \tag{27}$$

which, upon applying the Caratheodory Lemma, yields

$$\left|b_0\right| \le \sqrt[2]{2\,(1-\alpha)}. \tag{28}$$

On the other hand, subtracting $b_0^2 - 2b_1 = (1-\alpha)c_2$ from $b_0^2 + 2b_1 = (1-\alpha)d_2$, we obtain $4b_1 = (1-\alpha)(d_2 - c_2)$ which upon, applying the Caratheodory Lemma, yields

$$\left|b_1\right| \le 1 - \alpha. \tag{29}$$

□

Remark 3. For the estimates of the first two coefficients of certain subclasses of analytic and bi-univalent functions, also see recent publications by Srivastava et al. [11] and Frasin and Aouf [12].

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