Research Article

On Certain Classes of Convex Functions

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For real numbers α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathcal{K}(\alpha, \beta)$ the class of normalized analytic functions which satisfy the following two sided-inequality: $\alpha < \Re\{1 + (zf''(z)/f'(z))\} < \beta$ $(z \in \mathbb{U})$, where $\mathbb{U}$ denotes the open unit disk. We find some relationships involving functions in the class $\mathcal{K}(\alpha, \beta)$. And we estimate the bounds of coefficients and solve the Fekete-Szegö problem for functions in this class. Furthermore, we investigate the bounds of initial coefficients of inverse functions or biunivalent functions.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\} \qquad (1)$$

which is normalized by

$$f(0) = 0, \qquad f'(0) = 1. \qquad (2)$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which is composed of functions which are univalent in $\mathbb{U}$. And, as usual, we denote by $\mathcal{K}$ the class of functions in $\mathcal{A}$ which are convex in $\mathbb{U}$.

We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, if and only if $f(z) = F(w(z))$ for some Schwarz function $w(z)$ such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}). \qquad (3)$$

If F is univalent in $\mathbb{U}$, then the subordination $f \prec F$ is equivalent to

$$f(0) = F(0), \qquad f(\mathbb{U}) \subset F(\mathbb{U}). \qquad (4)$$

Definition 1. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\alpha, \beta)$ if f satisfies the following inequality:

$$\alpha < \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{U}). \qquad (5)$$

It is clear that $\mathcal{K}(\alpha, \beta) \subset \mathcal{K}$. And we remark that, for given real numbers α and β $(0 \leq \alpha < 1 < \beta)$, $f \in \mathcal{K}(\alpha, \beta)$ if and only if f satisfies each of the following two subordination relationships:

$$
\begin{aligned}
1 + \frac{zf''(z)}{f'(z)} &\prec \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U}), \\
1 + \frac{zf''(z)}{f'(z)} &\prec \frac{1 + (1 - 2\beta)z}{1 - z} \quad (z \in \mathbb{U}).
\end{aligned}
\qquad (6)
$$

Now, we define an analytic function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}z}{1 - z}\right). \qquad (7)$$

The above function p was introduced by Kuroki and Owa [1], and they proved that p maps $\mathbb{U}$ onto a convex domain

$$\Lambda = \{w : \alpha < \Re\{w\} < \beta\} \qquad (8)$$

conformally. Using this fact and the definition of subordination, we can obtain the following lemma, directly.

Lemma 2. *Let* $f(z) \in \mathcal{A}$ *and* $0 \leq \alpha < 1 < \beta$. *Then* $f \in \mathcal{K}(\alpha, \beta)$ *if and only if*

$$1 + \frac{zf''(z)}{f'(z)}$$
$$\prec 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))} z}{1 - z}\right) \quad (z \in \mathbb{U}). \tag{9}$$

And we note that the function p, defined by (7), has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{10}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right) \quad (n \in \mathbb{N}). \tag{11}$$

For given real numbers α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathcal{K}_\sigma(\alpha, \beta)$ the class of biunivalent functions consisting of the functions in $\mathcal{A}$ such that

$$f \in \mathcal{K}(\alpha, \beta), \qquad f^{-1} \in \mathcal{K}(\alpha, \beta), \tag{12}$$

where f^{-1} is the inverse function of f.

In our present investigation, we first find some relationships for functions in bounded positive class $\mathcal{K}(\alpha, \beta)$. And we solve several coefficient problems including Fekete-Szegö problems for functions in the class. Furthermore, we estimate the bounds of initial coefficients of inverse functions and bi-univalent functions. For the coefficient bounds of functions in special subclasses of $\mathcal{S}$, the readers may be referred to the works [2–4].

2. Relations Involving Bounds on the Real Parts

In this section, we will find some relations involving the functions in $\mathcal{K}(\alpha, \beta)$. And the following lemma will be needed in finding the relations.

Lemma 3 (see Miller and Mocanu [5, Theorem 2.3.i]). *Let* Ξ *be a set in the complex plane* $\mathbb{C}$ *and let* b *be a complex number such that* $\mathfrak{R}\{b\} > 0$. *Suppose that a function* $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ *satisfies the condition*

$$\psi(i\rho, \sigma; z) \notin \Xi \tag{13}$$

for all real $\rho, \sigma \leq -|b - i\rho|^2/(2\mathfrak{R}\{b\})$ *and all* $z \in \mathbb{U}$. *If the function* $p(z)$ *defined by* $p(z) = b + b_1 z + b_2 z^2 + \cdots$ *is analytic in* $\mathbb{U}$ *and if*

$$\psi(p(z), zp'(z); z) \in \Xi, \tag{14}$$

then $\mathfrak{R}\{p(z)\} > 0$ *in* $\mathbb{U}$.

Theorem 4. *Let* $f \in \mathcal{A}$, $0 \leq \alpha < 1$ *and*

$$\mathfrak{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}). \tag{15}$$

Then

$$\mathfrak{R}\left\{\sqrt{f'(z)}\right\} > \frac{1}{2 - \alpha} \quad (z \in \mathbb{U}). \tag{16}$$

Proof. First of all, we put $\gamma = 1/(2 - \alpha)$ and note that $1/2 \leq \gamma < 1$ for $0 \leq \alpha < 1$. Let

$$p(z) = \frac{1}{1 - \gamma}\left(\sqrt{f'(z)} - \gamma\right). \tag{17}$$

Differentiating (17), we can obtain

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2(1 - \gamma) zp'(z)}{(1 - \gamma) p(z) + \gamma} = \psi(p(z), zp'(z)), \tag{18}$$

where

$$\psi(r, s) = 1 + \frac{2(1 - \gamma) s}{(1 - \gamma) r + \gamma}. \tag{19}$$

Using (15), we have

$$\left\{\psi(p(z), zp'(z)) : z \in \mathbb{U}\right\} \subset \{w \in \mathbb{C} : \mathfrak{R}(w) > \alpha\} := \Omega_\alpha. \tag{20}$$

Now for all real $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$,

$$\mathfrak{R}\{\psi(i\rho, \sigma)\} = 1 + \frac{2\gamma(1 - \gamma)\sigma}{(1 - \gamma)^2 \rho^2 + \gamma^2}$$
$$\leq 1 - \frac{\gamma(1 - \gamma)(1 + \rho^2)}{(1 - \gamma)^2 \rho^2 + \gamma^2}. \tag{21}$$

Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(\rho) = \frac{1 + \rho^2}{(1 - \gamma)^2 \rho^2 + \gamma^2}. \tag{22}$$

Then g is a continuous even function and

$$g'(\rho) = \frac{2(2\gamma - 1)\rho}{\left((1 - \gamma)^2 \rho^2 + \gamma^2\right)^2}. \tag{23}$$

Hence $g'(0) = 0$ and g is increasing on $(0, \infty)$, since $1/2 \leq \gamma < 1$. Hence g satisfies that

$$\frac{1}{\gamma^2} \leq g(\rho) < \frac{1}{(1 - \gamma)^2}, \tag{24}$$

for all $\rho \in \mathbb{R}$. Therefore, by combining (21) and (24), we can get

$$\mathfrak{R}\{\psi(i\rho, \sigma)\} \leq 1 - \gamma(1 - \gamma) g(\rho) \leq 2 - \frac{1}{\gamma} = \alpha. \tag{25}$$

And this shows that $\mathfrak{R}\{\psi(i\rho, \sigma)\} \notin \Omega_\alpha$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$. By Lemma 3, we get $\mathfrak{R}\{p(z)\} > 0$ for all $z \in \mathbb{U}$, and this shows that the inequality (16) holds and the proof of Theorem 4 is completed. $\qquad\square$

Theorem 5. *Let $f \in \mathscr{A}$, $1 < \beta < 2$ and*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{U}). \qquad (26)$$

Then

$$\Re\left\{\sqrt{f'(z)}\right\} < \frac{1}{2-\beta} \quad (z \in \mathbb{U}). \qquad (27)$$

Proof. We put $\delta = 1/(2-\beta)$ and note that $\delta > 1$ for $1 < \beta < 2$. And let

$$p(z) = \frac{1}{1-\delta}\left(\sqrt{f'(z)} - \delta\right),$$
$$\psi(r,s) = 1 + \frac{2(1-\delta)s}{(1-\delta)r + \delta}. \qquad (28)$$

As in the proof of Theorem 4, we can get

$$\left\{\psi\left(p(z), zp'(z)\right) : z \in \mathbb{U}\right\} \subset \left\{w \in \mathbb{C} : \Re(w) < \beta\right\} := \Omega_\beta, \qquad (29)$$

by (26). And for all real ρ, σ with $\sigma \leq -(1+\rho^2)/2$,

$$\Re\left\{\psi(i\rho, \sigma)\right\} = 1 + \frac{2\delta(1-\delta)\sigma}{(1-\delta)^2\rho^2 + \delta^2}$$
$$\geq 1 - \frac{\delta(1-\delta)\left(1+\rho^2\right)}{(1-\delta)^2\rho^2 + \delta^2} \qquad (30)$$
$$= 1 - \delta(1-\delta)g(\rho),$$

where $g(\rho)$ is given by

$$g(\rho) = \frac{1+\rho^2}{(1-\delta)^2\rho^2 + \delta^2}. \qquad (31)$$

Since $\delta > 1$, g satisfies the inequality

$$\frac{1}{\delta^2} \leq g(\rho) < \frac{1}{(1-\delta)^2}, \qquad (32)$$

for all $\rho \in \mathbb{R}$. Therefore,

$$\Re\left\{\psi(i\rho, \sigma)\right\} \geq 1 - \delta(1-\delta)g(\rho) \geq 2 - \frac{1}{\delta} = \beta. \qquad (33)$$

And this shows that $\Re\{\psi(i\rho, \sigma)\} \notin \Omega_\beta$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1+\rho^2)/2$. By Lemma 3, we get $\Re\{p(z)\} > 0$ for all $z \in \mathbb{U}$, and this shows that the inequality (27) holds and the proof of Theorem 5 is completed. □

By combining Theorems 4 and 5, we can obtain the following result.

Theorem 6. *Let $f \in \mathscr{A}$, $0 \leq \alpha < 1 < \beta < 2$ and*

$$\alpha < \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{U}). \qquad (34)$$

Then

$$\frac{1}{2-\alpha} < \Re\left\{\sqrt{f'(z)}\right\} < \frac{1}{2-\beta} \quad (z \in \mathbb{U}). \qquad (35)$$

3. Coefficient Problems Involving Functions in $\mathscr{K}(\alpha, \beta)$

In the present section, we will solve some coefficient problems involving functions in the class $\mathscr{K}(\alpha, \beta)$. And our first result on the coefficient estimates involves the function class $\mathscr{K}(\alpha, \beta)$ and the following lemma will be needed.

Lemma 7 (see Rogosinski [6, Theorem 10]). *Let $p(z) = \sum_{n=1}^{\infty} C_n z^n$ be analytic and univalent in $\mathbb{U}$ and suppose that $p(z)$ maps $\mathbb{U}$ onto a convex domain. If $q(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies the following subordination:*

$$q(z) \prec p(z) \quad (z \in \mathbb{U}), \qquad (36)$$

then

$$|A_n| \leq |C_1| \quad (n = 1, 2, \ldots). \qquad (37)$$

Theorem 8. *Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. If the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{K}(\alpha, \beta)$, then*

$$|a_n| \leq \begin{cases} \dfrac{1}{2}|B_1|, & n = 2, \\[2mm] \dfrac{|B_1|}{n(n-1)}\displaystyle\prod_{k=1}^{n-2}\left(1 + \dfrac{|B_1|}{k}\right), & n = 3, 4, 5, \ldots, \end{cases} \qquad (38)$$

where $|B_1|$ is given by

$$|B_1| = \frac{2(\beta - \alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta - \alpha}. \qquad (39)$$

Proof. Let us define

$$q(z) = 1 + \frac{zf''(z)}{f'(z)}, \qquad (40)$$

$$p(z) = 1 + \frac{\beta - \alpha}{\pi}i\log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}z}{1 - z}\right). \qquad (41)$$

Then, the subordination (9) can be written as follows:

$$q(z) \prec p(z) \quad (z \in \mathbb{U}). \qquad (42)$$

Note that the function $p(z)$ defined by (41) is convex in $\mathbb{U}$ and has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \qquad (43)$$

where

$$B_n = \frac{\beta - \alpha}{n\pi}i\left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right). \qquad (44)$$

If we let

$$q(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \qquad (45)$$

then by Lemma 7, we see that the subordination (42) implies that

$$|A_n| \le |B_1| \quad (n = 1, 2, \ldots), \tag{46}$$

where

$$|B_1| = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}. \tag{47}$$

Now, equality (40) implies that

$$f'(z) + zf''(z) = q(z)f'(z) \quad (z \in \mathbb{U}). \tag{48}$$

Then, the coefficients of z^{n-1} in both sides lead to

$$a_n = \frac{1}{n(n-1)} \left(A_{n-1} + 2a_2 A_{n-2} + 3a_3 A_{n-3} \right.$$
$$\left. + \cdots + (n-1)a_{n-1}A_1 \right). \tag{49}$$

A simple calculation combined with the inequality (46) yields that

$$|a_n| = \frac{1}{n(n-1)}$$
$$\times \left| A_{n-1} + 2a_2 A_{n-2} + 3a_3 A_{n-3} \right.$$
$$\left. + \cdots + (n-1)a_{n-1}A_1 \right|$$
$$\le \frac{1}{n(n-1)} \tag{50}$$
$$\times \left(|A_{n-1}| + 2|a_2| |A_{n-2}| + 3|a_3| |A_{n-3}| \right.$$
$$\left. + \cdots + (n-1)|a_{n-1}| |A_1| \right)$$
$$\le \frac{|B_1|}{n(n-1)} \sum_{k=2}^{n} (k-1)|a_{k-1}|,$$

where $|B_1|$ is given by (47) and $|a_1| = 1$. Hence, we have $|a_2| \le |B_1|/2$. To prove the assertion of the theorem, we need to show that

$$|a_n| \le \frac{|B_1|}{n(n-1)} \sum_{k=2}^{n} (k-1)|a_{k-1}|$$
$$\le \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k} \right) \quad (n = 3, 4, 5, \ldots). \tag{51}$$

We now use the mathematical induction for the proof of the theorem. For the case $n = 3$, it is clear. We assume that the

inequality (51) holds for $n = m$. Then, some calculation gives us that

$$|a_{m+1}| \le \frac{|B_1|}{(m+1)m} \sum_{k=2}^{m+1} (k-1)|a_{k-1}|$$
$$\le \frac{|B_1|}{(m+1)m} \prod_{k=1}^{m-2} \left(1 + \frac{|B_1|}{k} \right)$$
$$+ \frac{|B_1|^2}{m(m+1)(m-1)} \prod_{k=1}^{m-2} \left(1 + \frac{|B_1|}{k} \right) \tag{52}$$
$$= \frac{|B_1|}{(m+1)m} \prod_{k=1}^{m-1} \left(1 + \frac{|B_1|}{k} \right),$$

which implies that the inequality (51) is true for $n = m + 1$. Hence, by the mathematical induction, we prove that

$$|a_n| \le \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k} \right) \quad (n = 3, 4, 5, \ldots), \tag{53}$$

where $|B_1|$ is given by (47). This completes the proof of Theorem 8. □

And now, we will solve the Fekete-Szegö problem for $f \in \mathcal{K}(\alpha, \beta)$, and we will need the following lemma.

Lemma 9 (see Keogh and Merkes [7]). *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be a function with positive real part in $\mathbb{U}$. Then, for any complex number ν,*

$$\left| c_2 - \nu c_1^2 \right| \le 2 \max \{1, |1 - 2\nu|\}. \tag{54}$$

Now, the following result holds for the coefficient of $f \in \mathcal{K}(\alpha, \beta)$.

Theorem 10. *Let $0 \le \alpha < 1 < \beta$ and let the function $f(z)$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{K}(\alpha, \beta)$. Then, for a complex number μ,*

$$\left| a_3 - \mu a_2^2 \right|$$
$$\le \frac{\beta - \alpha}{3\pi} \sin \left(\frac{1 - \alpha}{\beta - \alpha} \pi \right)$$
$$\cdot \max \left\{ 1; \left| \frac{1}{2} + \left(1 - \frac{3}{2}\mu \right) \frac{\beta - \alpha}{\pi} i \right. \right. \tag{55}$$
$$+ \left(\frac{1}{2} - \left(1 - \frac{3}{2}\mu \right) \frac{\beta - \alpha}{\pi} i \right)$$
$$\left. \left. \times e^{2\pi i((1-\alpha)/(\beta-\alpha))} \right| \right\}.$$

The result is sharp.

Proof. Let us consider a function $q(z)$ given by $q(z) = 1 + zf''(z)/f'(z)$. Then, since $f \in \mathcal{K}(\alpha, \beta)$, we have $q(z) \prec p(z)(z \in \mathbb{U})$, where

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}z}{1 - z}\right)$$
$$= 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{56}$$

where B_n is given by (11). Let

$$h(z) = \frac{1 + p^{-1}(q(z))}{1 - p^{-1}(q(z))} = 1 + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{57}$$

Then h is analytic and has positive real part in the open unit disk $\mathbb{U}$. We also have

$$q(z) = p\left(\frac{h(z) - 1}{h(z) + 1}\right) \quad (z \in \mathbb{U}). \tag{58}$$

We find from (58) that

$$a_2 = \frac{1}{4} B_1 h_1,$$
$$a_3 = \frac{1}{12} B_1 h_2 - \frac{1}{24} B_1 h_1^2 + \frac{1}{24} B_2 h_1^2 + \frac{1}{24} B_1^2 h_1^2, \tag{59}$$

which imply that

$$a_3 - \mu a_2^2 = \frac{1}{12} B_1 \left(h_2 - \nu h_1^2\right), \tag{60}$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - B_1 + \frac{3}{2}\mu B_1\right). \tag{61}$$

Applying Lemma 9, we can obtain

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{12}|B_1|\left|h_2 - \nu h_1^2\right|$$
$$\leq \frac{1}{6}|B_1| \cdot \max\{1; |1 - 2\nu|\}. \tag{62}$$

And substituting

$$B_1 = \frac{\beta - \alpha}{\pi} i\left(1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\right), \tag{63}$$
$$B_2 = \frac{\beta - \alpha}{2\pi} i\left(1 - e^{4\pi i((1-\alpha)/(\beta-\alpha))}\right) \tag{64}$$

in (62), we can obtain the result as asserted. The estimate is sharp for the function $f : \mathbb{U} \to \mathbb{C}$ defined by

$$f(z) = \int_0^z \left\{\exp\left(\int_0^\zeta \frac{p(\xi) - 1}{\xi} d\xi\right)\right\} d\zeta, \tag{65}$$

where the function p is given by (7). Hence the proof of Theorem 10 is completed. □

Using Theorem 10, we can get the following result.

Corollary 11. *Let $0 \leq \alpha < 1 < \beta$ and let the function f, given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{K}(\alpha, \beta)$. Also let the function f^{-1}, defined by*

$$f^{-1}(f(z)) = z = f(f^{-1}(z)), \tag{66}$$

be the inverse of f. If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad \left(|w| < r_0; r_0 > \frac{1}{4}\right), \tag{67}$$

then

$$|b_2| \leq \frac{\beta - \alpha}{\pi} \sin\left(\frac{1 - \alpha}{\beta - \alpha}\pi\right),$$
$$|b_3| \leq \frac{\beta - \alpha}{3\pi} \sin\left(\frac{1 - \alpha}{\beta - \alpha}\pi\right)$$
$$\cdot \max\left\{1; \left|\frac{1}{2} - \frac{2(\beta - \alpha)}{\pi}i\right.\right.$$
$$\left.\left. + \left(\frac{1}{2} + \frac{2(\beta - \alpha)}{\pi}i\right)e^{2\pi i((1-\alpha)/(\beta-\alpha))}\right|\right\}. \tag{68}$$

Proof. The relations (66) and (67) give

$$b_2 = -a_2, \qquad b_3 = 2a_2^2 - a_3. \tag{69}$$

Thus, we can get the estimate for $|b_2|$ by

$$|b_2| = |a_2| \leq \frac{1}{2}|B_1| = \frac{\beta - \alpha}{\pi} \sin\left(\frac{1 - \alpha}{\beta - \alpha}\pi\right), \tag{70}$$

immediately. Furthermore, an application of Theorem 10 (with $\mu = 2$) gives the estimates for $|b_3|$; hence, the proof of Corollary 11 is completed. □

Finally, we will estimate some initial coefficients for the bi-univalent functions $f \in \mathcal{K}_\sigma(\alpha, \beta)$.

Theorem 12. *For given α and β such that $0 \leq \alpha < 1 < \beta$, let f given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{K}_\sigma(\alpha, \beta)$. Then*

$$|a_2| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{2|B_1^2 - 2B_1 + 2B_2|}}, \tag{71}$$
$$|a_3| \leq \frac{1}{2}(|B_1| + |B_2 - B_1|), \tag{72}$$

where B_1 and B_2 are given by (63) and (64).

Proof. If $f \in \mathcal{K}_\sigma(\alpha, \beta)$, then $f \in \mathcal{K}(\alpha, \beta)$ and $g \in \mathcal{K}(\alpha, \beta)$, where $g = f^{-1}$. Hence

$$Q(z) := 1 + \frac{zf''(z)}{f'(z)} \prec p(z) \quad (z \in \mathbb{U}),$$
$$L(z) := 1 + \frac{zg''(z)}{g'(z)} \prec p(z) \quad (z \in \mathbb{U}), \tag{73}$$

where $p(z)$ is given by (7). Let

$$
\begin{aligned}
h(z) &= \frac{1 + p^{-1}(Q(z))}{1 - p^{-1}(Q(z))} \\
&= 1 + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U}), \\
k(z) &= \frac{1 + p^{-1}(L(z))}{1 - p^{-1}(L(z))} \\
&= 1 + k_1 z + k_2 z^2 + \cdots \quad (z \in \mathbb{U}).
\end{aligned}
\tag{74}
$$

Then h and k are analytic and have positive real part in $\mathbb{U}$. Also, we have

$$
\begin{aligned}
Q(z) &= p\left(\frac{h(z) - 1}{h(z) + 1}\right) \quad (z \in \mathbb{U}), \\
L(z) &= p\left(\frac{k(z) - 1}{k(z) + 1}\right) \quad (z \in \mathbb{U}).
\end{aligned}
\tag{75}
$$

By suitably comparing coefficients, we get

$$
a_2 = \frac{1}{4} B_1 h_1,
\tag{76}
$$

$$
3a_3 - 2a_2^2 = \frac{1}{4} B_1 h_2 - \frac{1}{8} B_1 h_1^2 + \frac{1}{8} B_2 h_1^2,
\tag{77}
$$

$$
b_2 = \frac{1}{4} B_1 k_1,
\tag{78}
$$

$$
3b_3 - 2b_2^2 = \frac{1}{4} B_1 k_2 - \frac{1}{8} B_1 k_1^2 + \frac{1}{8} B_2 k_1^2,
\tag{79}
$$

where B_1 and B_2 are given by (63) and (64), respectively. Now, considering (76) and (78), we get

$$
h_1 = -k_1.
\tag{80}
$$

Also, from (77), (78), (79), and (80), we find that

$$
a_2^2 = \frac{B_1^3 (h_2 + k_2)}{8 (B_1^2 - 2B_1 + 2B_2)}.
\tag{81}
$$

Therefore, we have

$$
\begin{aligned}
\left| a_2^2 \right| &\le \frac{|B_1|^3}{8 |B_1^2 - 2B_1 + 2B_2|} \left(|h_2| + |k_2| \right) \\
&\le \frac{|B_1|^3}{2 |B_1^2 - 2B_1 + 2B_2|}.
\end{aligned}
\tag{82}
$$

This gives the bound on $|a_2|$ as asserted in (71). Now, further computations from (77), (79), (80), and (81) lead to

$$
a_3 = \frac{1}{12} B_1 (2h_2 + k_2) + \frac{1}{8} h_1^2 (B_2 - B_1).
\tag{83}
$$

This equation, together with the well-known estimates

$$
|h_1| \le 2, \qquad |h_2| \le 2, \qquad |k_2| \le 2,
\tag{84}
$$

leads us to the inequality (72). Therefore, the proof of Theorem 12 is completed. $\qquad \square$

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