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Research Article

Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

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The object of this paper is to study invariant submanifolds M of Sasakian manifolds $\widetilde{M}$ admitting a semisymmetric nonmetric connection, and it is shown that M admits semisymmetric nonmetric connection. Further it is proved that the second fundamental forms σ and $\overline{\sigma}$ with respect to Levi-Civita connection and semi-symmetric nonmetric connection coincide. It is shown that if the second fundamental form σ is recurrent, 2-recurrent, generalized 2-recurrent, semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel and M has parallel third fundamental form with respect to semisymmetric nonmetric connection, then M is totally geodesic with respect to Levi-Civita connection.

1. Semisymmetric Nonmetric Connection

The geometry of invariant submanifolds M of Sasakian manifolds $\widetilde{M}$ is carried out from 1970's by M. Kon [1], D. Chinea [2], K. Yano and M. Kon [3] and B.S. Anitha and C.S. Bagewadi [4]. The aurthor [1] has proved that invariant submanifold of Sasakian structure also carries Sasakian structure. In this paper we extend the results to invariant submanifolds M of Sasakian manifolds admitting Semisymmetric Nonmetric connection.

We know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M if $\nabla g = 0$; otherwise it is Nonmetric. Further it is said to be Semisymmetric if its torsion tensor $T(X, Y) = 0$; thatis, $T(X, Y) = w(Y)X - w(X)Y$, where w is a 1-form. A study of Semisymmetric connection on a Riemannian manifold was initiated by Yano [5]. In 1992, Agashe and Chafle [6] introduced the notion of Semisymmetric Nonmetric connection. If $\overline{\nabla}$ denotes Semisymmetric Nonmetric connection on a contact metric manifold, then it is given by [6]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \tag{1.1}$$

where $\eta(Y) = g(Y, \xi)$.

The covariant differential of the pth order, $p \geq 1$ of a $(0,k)$-tensor field T, $k \geq 1$ denoted by $\nabla^p T$, defined on a Riemannian manifold (M, g) with the Levi-Civita connection ∇. The tensor T is said to be *recurrent* [7], if the following condition holds on M:

$$(\nabla T)(X_1, \ldots, X_k; X)T(Y_1, \ldots, Y_k) = (\nabla T)(Y_1, \ldots, Y_k; X)T(X_1, \ldots, X_k), \tag{1.2}$$

respectively.

Consider

$$\left(\nabla^2 T\right)(X_1, \ldots, X_k; X, Y)T(Y_1, \ldots, Y_k) = \left(\nabla^2 T\right)(Y_1, \ldots, Y_k; X, Y)T(X_1, \ldots, X_k), \tag{1.3}$$

where $X, Y, X_1, Y_1, \ldots, X_k, Y_k \in TM$. From (1.2) it follows that at a point $x \in M$, if the tensor T is nonzero, then there exists a unique 1-form ϕ, respectively, a $(0,2)$-tensor ψ, defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d\left(\log \|T\|\right), \tag{1.4}$$

respectively.

The following

$$\nabla^2 T = T \otimes \psi \tag{1.5}$$

holds on U, where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T, T)$. The tensor T is said to be *generalized* 2*-recurrent* if

$$\begin{aligned} &\left(\left(\nabla^2 T\right)(X_1, \ldots, X_k; X, Y) - (\nabla T \otimes \phi)(X_1, \ldots, X_k; X, Y)\right)T(Y_1, \ldots, Y_k) \\ &\quad = \left(\left(\nabla^2 T\right)(Y_1, \ldots, Y_k; X, Y) - (\nabla T \otimes \phi)(Y_1, \ldots, Y_k; X, Y)\right)T(X_1, \ldots, X_k) \end{aligned} \tag{1.6}$$

holds on M, where ϕ is a 1-form on M. From this it follows that at a point $x \in M$ if the tensor

T is nonzero, then there exists a unique $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi \tag{1.7}$$

holds on U.

2. Isometric Immersion

Let $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M, g) into $(n+d)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$, respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A, and $\nabla^{\perp}$ are the second fundamental form, the shape operator, and the normal connection, respectively. If the second fundamental form σ is identically zero, then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N is related by

$$\widetilde{g}(\sigma(X, Y), N) = g(A_N X, Y), \tag{2.3}$$

for tangent vector fields X, Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$\left(\widetilde{\nabla}_X \sigma\right)(Y, Z) = \nabla^{\perp}_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.4}$$

$$\begin{aligned}\left(\widetilde{\nabla}^2 \sigma\right)(Z, W, X, Y) &= \left(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma\right)(Z, W) \\ &= \nabla^{\perp}_X\left(\left(\widetilde{\nabla}_Y \sigma\right)(Z, W)\right) - \left(\widetilde{\nabla}_Y \sigma\right)(\nabla_X Z, W) \\ &\quad - \left(\widetilde{\nabla}_X \sigma\right)(Z, \nabla_Y W) - \left(\widetilde{\nabla}_{\nabla_X Y} \sigma\right)(Z, W),\end{aligned} \tag{2.5}$$

respectively, where $\widetilde{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [8]. If $\widetilde{\nabla}\sigma = 0$, then M is said to have *parallel second fundamental form* [8]. We next define endomorphisms $R(X,Y)$ and $X\wedge_B Y$ of $\chi(M)$ by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$
$$(X\wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y, \tag{2.6}$$

respectively, where $X,Y,Z\in\chi(M)$ and B is a symmetric $(0,2)$-tensor.

Now, for a $(0,k)$-tensor field T, $k\geq 1$ and a $(0,2)$-tensor field B on (M,g), we define the tensor $Q(B,T)$ by

$$Q(B,T)(X_1,\dots,X_k;X,Y) = -(T(X\wedge_B Y)X_1,\dots,X_k) - \cdots - T(X_1,\dots,X_{k-1}(X\wedge_B Y)X_k). \tag{2.7}$$

Putting into consideration the previous formula "$B = g, S$ and $T=\sigma$," we obtain the tensors $Q(g,\sigma)$ and $Q(S,\sigma)$.

3. Sasakian Manifolds

An n-dimensional differential manifold M is said to have an almost contact structure (ϕ,ξ,η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ, and 1-form η on M, respectively, such that

$$\phi^2 = -I + \eta\otimes\xi, \qquad \eta(\xi)=1, \qquad \eta\circ\phi = 0, \qquad \phi\xi = 0. \tag{3.1}$$

Thus a manifold M equipped with this structure is called an almost contact manifold and is denoted by (M,ϕ,ξ,η). If g is a Riemannian metric on an almost contact manifold M such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X,\xi) = \eta(X), \tag{3.2}$$

where X,Y are vector fields defined on M, then M is said to have an almost contact metric structure (ϕ,ξ,η,g), and M with this structure is called an almost contact metric manifold and is denoted by (M,ϕ,ξ,η,g).

If on (M,ϕ,ξ,η,g) the exterior derivative of 1-form η satisfies

$$\Phi(X,Y) = d\eta(X,Y) = g(X,\phi Y), \tag{3.3}$$

then (ϕ,ξ,η,g) is said to be a contact metric structure and together with manifold M is called contact metric manifold and Φ is a 2-form. The contact metric structure (M,ϕ,ξ,η,g) is said to be normal if

$$[\phi,\phi](X,Y) + 2d\eta\otimes\xi = 0. \tag{3.4}$$

If the contact metric structure is normal, then it is called a Sasakian structure and M is called a Sasakian manifold. Note that an almost contact metric manifold defines Sasakian structure if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{3.5}$$

$$\nabla_X \xi = -\phi X. \tag{3.6}$$

Example of Sasakian Manifold

Consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}. \tag{3.7}$$

Let g be the Riemannian metric defined by

$$\begin{gathered} g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0, \\ g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1. \end{gathered} \tag{3.8}$$

The (ϕ, ξ, η) is given by

$$\begin{gathered} \eta = 2y\,dx + dz, \qquad \xi = E_3 = \frac{\partial}{\partial z}, \\ \phi E_1 = E_2, \qquad \phi E_2 = -E_1, \qquad \phi E_3 = 0. \end{gathered} \tag{3.9}$$

The linearity property of ϕ and g yields

$$\begin{gathered} \eta(E_3) = 1, \qquad \phi^2 U = -U + \eta(U)E_3, \\ g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W), \qquad g(U, \xi) = \eta(U), \end{gathered} \tag{3.10}$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_2] = 2E_3. \tag{3.11}$$

Let ∇ be the Levi-Civita connection with respect to previously mentioned metric g and be given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{3.12}$$

Then, we have

$$\begin{aligned} &\nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2 = E_3, \quad \nabla_{E_1}E_3 = -E_2, \\ &\nabla_{E_2}E_1 = -E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_2}E_3 = E_1, \\ &\nabla_{E_3}E_1 = -E_2, \quad \nabla_{E_3}E_2 = E_1, \quad \nabla_{E_3}E_3 = 0. \end{aligned} \tag{3.13}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3; that is, $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy (3.1), (3.2), (3.5), and (3.6). Thus M is a Sasakian manifold. Further the following relations hold:

$$\begin{aligned} R(X,Y)Z &= \{g(Y,Z)X - g(X,Z)Y\}, \\ R(X,Y)\xi &= \{\eta(Y)X - \eta(X)Y\}, \\ R(\xi,X)Y &= \{g(X,Y)\xi - \eta(Y)X\}, \end{aligned} \tag{3.14}$$

$$R(\xi,X)\xi = \{\eta(X)\xi - X\}, \tag{3.15}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{3.16}$$

$$Q\xi = (n-1)\xi, \tag{3.17}$$

for all vector fields, X, Y, Z and where ∇ denotes the operator of covariant differentiation with respect to g, ϕ is a $(1,1)$ tensor field, S is the Ricci tensor of type $(0,2)$, and R is the Riemannian curvature tensor of the manifold.

4. Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

If $\widetilde{M}$ is a Sasakian manifold with structure tensors $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$, then we know that its invariant submanifold M has the induced Sasakian structure (ϕ, ξ, η, g).

A submanifold M of a Sasakian manifold $\widetilde{M}$ with a Semisymmetric Nonmetric connection is called an invariant submanifold of $\widetilde{M}$ with a Semisymmetric Nonmetric connection, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. For an invariant submanifold of a Sasakian manifold with a Semisymmetric Nonmetric connection we have

$$\sigma(X,\xi) = 0, \tag{4.1}$$

for any vector X tangent to M.

Let $\widetilde{M}$ be a Sasakian manifold admitting a Semisymmetric Nonmetric connection $\widetilde{\nabla}$.

Lemma 4.1. *Let M be an invariant submanifold of contact metric manifold $\widetilde{M}$ which admits Semisymmetric Nonmetric connection $\overline{\tilde{\nabla}}$, and let σ and $\overline{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and Semisymmetric Nonmetric connection; then (1) M admits Semisymmetric Nonmetric connection and (2) the second fundamental forms with respect to $\tilde{\nabla}$ and $\overline{\tilde{\nabla}}$ are equal.*

Proof. We know that the contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\widetilde{M}$ induces (ϕ, ξ, η, g) on invariant submanifold. By virtue of (1.1), we get

$$\overline{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)X. \tag{4.2}$$

By using (2.1) in (4.2), we get

$$\overline{\tilde{\nabla}}_X Y = \nabla_X Y + \sigma(X, Y) + \eta(Y)X. \tag{4.3}$$

Now Gauss formula (2.1) with respect to Semisymmetric Nonmetric connection is given by

$$\overline{\tilde{\nabla}}_X Y = \overline{\nabla}_X Y + \overline{\sigma}(X, Y). \tag{4.4}$$

Equating (4.3) and (4.4), we get (1.1) and

$$\overline{\sigma}(X, Y) = \sigma(X, Y). \tag{4.5}$$

□

Now we introduce the definitions of semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel with respect to Semisymmetric Nonmetric connection.

Definition 4.2. An immersion is said to be semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel with respect to Semisymmetric Nonmetric connection, respectively, if the following conditions hold for all vector fields X, Y tangent to M:

$$\begin{aligned} \overline{\tilde{R}} \cdot \sigma &= 0, \\ \overline{\tilde{R}} \cdot \sigma &= L_1 Q(g, \sigma), \\ \overline{\tilde{R}} \cdot \sigma &= L_2 Q(S, \sigma), \end{aligned} \tag{4.6}$$

where $\overline{\tilde{R}}$ denotes the curvature tensor with respect to connection $\overline{\tilde{\nabla}}$. Here L_1 and L_2 are functions depending on σ.

Lemma 4.3. *Let M be an invariant submanifold of contact manifold $\widetilde{M}$ which admits Semisymmetric Nonmetric connection. Then Gauss and Weingarten formulae with respect to Semisymmetric Nonmetric connection are given by*

$$\begin{aligned}\tan\left(\bar{\tilde{R}}(X,Y)Z\right) = {} & R(X,Y)Z + \eta(\nabla_Y Z)X + \eta(Z)\nabla_X Y + \eta(Z)\eta(Y)X \\ & - \eta(\nabla_X Z)Y - \eta(Z)\nabla_Y X - \eta(Z)\eta(X)Y - \eta(Z)[X,Y] \\ & + \tan\left\{\bar{\tilde{\nabla}}_X\{\sigma(Y,Z)\} - \bar{\tilde{\nabla}}_Y\{\sigma(X,Z)\} - \bar{\tilde{\nabla}}_Y\eta(Z)X + \bar{\tilde{\nabla}}_X\eta(Z)Y\right\},\end{aligned} \tag{4.7}$$

$$\begin{aligned}\operatorname{nor}\left(\bar{\tilde{R}}(X,Y)Z\right) = {} & \sigma(X,\nabla_Y Z) + \eta(Z)\sigma(X,Y) - \sigma(Y,\nabla_X Z) - \eta(Z)\sigma(Y,X) - \sigma([X,Y],Z) \\ & + \operatorname{nor}\left\{\bar{\tilde{\nabla}}_X\{\sigma(Y,Z)\} - \bar{\tilde{\nabla}}_Y\{\sigma(X,Z)\} - \bar{\tilde{\nabla}}_Y\eta(Z)X + \bar{\tilde{\nabla}}_X\eta(Z)Y\right\}.\end{aligned} \tag{4.8}$$

Proof. The Riemannian curvature tensor $\tilde{R}$ on $\widetilde{M}$ with respect to Semisymmetric Nonmetric connection is given by

$$\bar{\tilde{R}}(X,Y)Z = \bar{\tilde{\nabla}}_X\bar{\tilde{\nabla}}_Y Z - \bar{\tilde{\nabla}}_Y\bar{\tilde{\nabla}}_X Z - \bar{\tilde{\nabla}}_{[X,Y]}Z. \tag{4.9}$$

Using (1.1) and (2.1) in (4.9), we get

$$\begin{aligned}\bar{\tilde{R}}(X,Y)Z = {} & R(X,Y)Z + \sigma(X,\nabla_Y Z) + \eta(\nabla_Y Z)X + \bar{\tilde{\nabla}}_X\{\sigma(Y,Z)\} + \bar{\tilde{\nabla}}_X\eta(Z)Y \\ & + \eta(Z)\nabla_X Y + \eta(Z)\sigma(X,Y) + \eta(Z)\eta(Y)X - \sigma(Y,\nabla_X Z) - \eta(\nabla_X Z)Y \\ & - \bar{\tilde{\nabla}}_Y\{\sigma(X,Z)\} - \bar{\tilde{\nabla}}_Y\eta(Z)X - \eta(Z)\nabla_Y X \\ & - \eta(Z)\sigma(Y,X) - \eta(Z)\eta(X)Y - \sigma([X,Y],Z) - \eta(Z)[X,Y].\end{aligned} \tag{4.10}$$

Comparing tangential and normal part of (4.10), we obtain Gauss and Weingarten formulae (4.7) and (4.8). □

Lemma 4.4. *Let M be an invariant submanifold of contact manifold $\widetilde{M}$ which admits Semisymmetric Nonmetric connection. If σ is semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel with* *respect to Semisymmetric Nonmetric connection, then we have*

$$\begin{aligned}
\left(\overline{\widetilde{R}}(X,Y)\cdot\sigma\right)(U,V) = {} & R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V) \\
& - \nabla_X A_{\sigma(U,V)}Y + \nabla_Y A_{\sigma(U,V)}X - A_{\nabla^{\perp}_{Y}\sigma(U,V)}X + A_{\nabla^{\perp}_{X}\sigma(U,V)}Y \\
& + A_{\sigma(U,V)}[X,Y] - \sigma\left(X, A_{\sigma(U,V)}Y\right) + \sigma\left(Y, A_{\sigma(U,V)}X\right) \\
& - \eta\left(A_{\sigma(U,V)}Y\right)X + \eta\left(A_{\sigma(U,V)}X\right)Y - \eta(\nabla_Y U)\sigma(X,V) \\
& - \eta(U)\sigma(\nabla_X Y,V) - \eta(U)\eta(Y)\sigma(X,V) + \eta(\nabla_X U)\sigma(Y,V) \\
& + \eta(U)\sigma(\nabla_Y X,V) + \eta(U)\eta(X)\sigma(Y,V) + \eta(U)\sigma([X,Y],V) \\
& - \sigma\left(\overline{\widetilde{\nabla}}_X \eta(U)Y,V\right) + \sigma\left(\overline{\widetilde{\nabla}}_Y \eta(U)X,V\right) - \sigma\left(\overline{\widetilde{\nabla}}_X\{\sigma(Y,U)\},V\right) \\
& + \sigma\left(\overline{\widetilde{\nabla}}_Y\{\sigma(X,U)\},V\right) - \sigma(\sigma(X,\nabla_Y U),V) - \eta(U)\sigma(\sigma(X,Y),V) \\
& + \sigma(\sigma(Y,\nabla_X U),V) + \eta(U)\sigma(\sigma(Y,X),V) + \sigma(\sigma([X,Y],U),V) \\
& - \eta(\nabla_Y V)\sigma(U,X) - \eta(V)\sigma(U,\nabla_X Y) - \eta(V)\eta(Y)\sigma(U,X) \\
& + \eta(\nabla_X V)\sigma(U,Y) + \eta(V)\sigma(U,\nabla_Y X) + \eta(V)\eta(X)\sigma(U,Y) \\
& + \eta(V)\sigma(U,[X,Y]) - \sigma\left(U,\overline{\widetilde{\nabla}}_X \eta(V)Y\right) + \sigma\left(U,\overline{\widetilde{\nabla}}_Y \eta(V)X\right) \\
& - \sigma\left(U,\overline{\widetilde{\nabla}}_X\{\sigma(Y,V)\}\right) + \sigma\left(U,\overline{\widetilde{\nabla}}_Y\{\sigma(X,V)\}\right) - \sigma(U,\sigma(X,\nabla_Y V)) \\
& - \eta(V)\sigma(U,\sigma(X,Y)) + \sigma(U,\sigma(Y,\nabla_X V)) \\
& + \eta(V)\sigma(U,\sigma(Y,X)) + \sigma(U,\sigma([X,Y],V)),
\end{aligned} \tag{4.11}$$

for all vector fields X, Y, U, and V tangent to M, where

$$R^{\perp}(X,Y) = \left[\nabla^{\perp}_{X},\nabla^{\perp}_{Y}\right] - \nabla^{\perp}_{[X,Y]}. \tag{4.12}$$

Proof. We know, from tensor algebra, that

$$\left(\overline{\widetilde{R}}(X,Y)\cdot\sigma\right)(U,V) = \overline{\widetilde{R}}(X,Y)\sigma(U,V) - \sigma\left(\overline{\widetilde{R}}(X,Y)U,V\right) - \sigma\left(U,\overline{\widetilde{R}}(X,Y)V\right). \tag{4.13}$$

Replacing Z by $\sigma(U,V)$ in (4.9), we get

$$\overline{\widetilde{R}}(X,Y)\sigma(U,V) = \overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y\sigma(U,V) - \overline{\widetilde{\nabla}}_Y\overline{\widetilde{\nabla}}_X\sigma(U,V) - \overline{\widetilde{\nabla}}_{[X,Y]}\sigma(U,V). \tag{4.14}$$

In view of (1.1), (2.1), and (2.2), we have the following equalities:

$$\begin{aligned}\overline{\tilde{\nabla}}_X\overline{\tilde{\nabla}}_Y\sigma(U,V) &= \overline{\tilde{\nabla}}_X\Big(-A_{\sigma(U,V)}Y + \nabla^{\perp}_Y\sigma(U,V)\Big),\\ &= -\nabla_X A_{\sigma(U,V)}Y - \eta\big(A_{\sigma(U,V)}Y\big)X - \sigma\big(X, A_{\sigma(U,V)}Y\big)\\ &\quad - A_{\nabla^{\perp}_Y\sigma(U,V)}X + \nabla^{\perp}_X\nabla^{\perp}_Y\sigma(U,V).\end{aligned} \tag{4.15}$$

Similarly

$$\begin{aligned}&\overline{\tilde{\nabla}}_Y\overline{\tilde{\nabla}}_X\sigma(U,V)\\ &\quad= -\nabla_Y A_{\sigma(U,V)}X - \eta\big(A_{\sigma(U,V)}X\big)Y - \sigma\big(Y, A_{\sigma(U,V)}X\big) - A_{\nabla^{\perp}_X\sigma(U,V)}Y + \nabla^{\perp}_Y\nabla^{\perp}_X\sigma(U,V),\end{aligned} \tag{4.16}$$

$$\overline{\tilde{\nabla}}_{[X,Y]}\sigma(U,V) = -A_{\sigma(U,V)}[X,Y] + \nabla^{\perp}_{[X,Y]}\sigma(U,V). \tag{4.17}$$

Substituting (4.15), (4.16) and (4.17) into (4.14), we get

$$\begin{aligned}\overline{\tilde{R}}(X,Y)\sigma(U,V) &= R^{\perp}(X,Y)\sigma(U,V) - \nabla_X A_{\sigma(U,V)}Y + \nabla_Y A_{\sigma(U,V)}X - A_{\nabla^{\perp}_Y\sigma(U,V)}X\\ &\quad + A_{\nabla^{\perp}_X\sigma(U,V)}Y + A_{\sigma(U,V)}[X,Y] - \sigma\big(X, A_{\sigma(U,V)}Y\big) + \sigma\big(Y, A_{\sigma(U,V)}X\big)\\ &\quad - \eta\big(A_{\sigma(U,V)}Y\big)X + \eta\big(A_{\sigma(U,V)}X\big)Y.\end{aligned} \tag{4.18}$$

By virtue of (4.10) in $\sigma(\overline{\tilde{R}}(X,Y)U,V)$ and $\sigma(U,\overline{\tilde{R}}(X,Y)V)$, we get

$$\begin{aligned}\sigma\Big(\overline{\tilde{R}}(X,Y)U,V\Big) &= \sigma(R(X,Y)U,V) + \eta(\nabla_Y U)\sigma(X,V) + \eta(U)\sigma(\nabla_X Y,V)\\ &\quad + \eta(U)\eta(Y)\sigma(X,V) - \eta(\nabla_X U)\sigma(Y,V) - \eta(U)\sigma(\nabla_Y X,V)\\ &\quad - \eta(U)\eta(X)\sigma(Y,V) - \eta(U)\sigma([X,Y],V) + \sigma\Big(\overline{\tilde{\nabla}}_X\eta(U)Y,V\Big)\\ &\quad - \sigma\Big(\overline{\tilde{\nabla}}_Y\eta(U)X,V\Big) + \sigma\Big(\overline{\tilde{\nabla}}_X\{\sigma(Y,U)\},V\Big) - \sigma\Big(\overline{\tilde{\nabla}}_Y\{\sigma(X,U)\},V\Big)\\ &\quad + \sigma(\sigma(X,\nabla_Y U),V) + \eta(U)\sigma(\sigma(X,Y),V) - \sigma(\sigma(Y,\nabla_X U),V)\\ &\quad - \eta(U)\sigma(\sigma(Y,X),V) - \sigma(\sigma([X,Y],U),V),\end{aligned} \tag{4.19}$$

$$\begin{aligned}\sigma\Big(U,\overline{\widetilde{R}}(X,Y)V\Big) ={}& \sigma(U,R(X,Y)V)+\eta(\nabla_Y V)\sigma(U,X)+\eta(V)\sigma(U,\nabla_X Y)\\ &+\eta(V)\eta(Y)\sigma(U,X)-\eta(\nabla_X V)\sigma(U,Y)-\eta(V)\sigma(U,\nabla_Y X)\\ &-\eta(V)\eta(X)\sigma(U,Y)-\eta(V)\sigma(U,[X,Y])+\sigma\Big(U,\overline{\widetilde{\nabla}}_X\eta(V)Y\Big)\\ &-\sigma\Big(U,\overline{\widetilde{\nabla}}_Y\eta(V)X\Big)+\sigma\Big(U,\overline{\widetilde{\nabla}}_X\{\sigma(Y,V)\}\Big)-\sigma\Big(U,\overline{\widetilde{\nabla}}_Y\{\sigma(X,V)\}\Big)\\ &+\sigma(U,\sigma(X,\nabla_Y V))+\eta(V)\sigma(U,\sigma(X,Y))-\sigma(U,\sigma(Y,\nabla_X V))\\ &-\eta(V)\sigma(U,\sigma(Y,X))-\sigma(U,\sigma([X,Y],V)).\end{aligned} \tag{4.20}$$

Substituting (4.18), (4.19) and (4.20) into (4.13), we get (4.11). □

5. Recurrent Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

We consider invariant submanifolds of a Sasakian manifold when σ is recurrent, 2-recurrent, and generalized 2-recurrent and M has parallel third fundamental form with respect to Semisymmetric Nonmetric connection. We write (2.4) and (2.5) with respect to Semisymmetric Nonmetric connection, and they are given by

$$\Big(\overline{\widetilde{\nabla}}_X\sigma\Big)(Y,Z)=\overline{\nabla}^{\perp}_X(\sigma(Y,Z))-\sigma\Big(\overline{\nabla}_X Y,Z\Big)-\sigma\Big(Y,\overline{\nabla}_X Z\Big), \tag{5.1}$$

$$\begin{aligned}\Big(\overline{\widetilde{\nabla}}^2\sigma\Big)(Z,W,X,Y)&=\Big(\overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y\sigma\Big)(Z,W)\\ &=\overline{\nabla}^{\perp}_X\Big(\Big(\overline{\widetilde{\nabla}}_Y\sigma\Big)(Z,W)\Big)-\Big(\overline{\widetilde{\nabla}}_Y\sigma\Big)\Big(\overline{\nabla}_X Z,W\Big)\\ &\quad-\Big(\overline{\widetilde{\nabla}}_X\sigma\Big)\Big(Z,\overline{\nabla}_Y W\Big)-\Big(\overline{\widetilde{\nabla}}_{\overline{\nabla}_X Y}\sigma\Big)(Z,W).\end{aligned} \tag{5.2}$$

We prove the following theorems.

Theorem 5.1. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then σ is recurrent with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be recurrent with respect to Semisymmetric Nonmetric connection; from (1.4) we get

$$\Big(\overline{\widetilde{\nabla}}_X\sigma\Big)(Y,Z)=\phi(X)\sigma(Y,Z), \tag{5.3}$$

where ϕ is a 1-form on M; in view of (5.1) and putting $Z = \xi$ in the above equation, we have

$$\overline{\nabla}_X^{\perp}\sigma(Y,\xi) - \sigma\left(\overline{\nabla}_X Y,\xi\right) - \sigma\left(Y,\overline{\nabla}_X\xi\right) = \phi(X)\sigma(Y,\xi). \tag{5.4}$$

By virtue of (4.1) in (5.4), we get

$$-\sigma\left(\overline{\nabla}_X Y,\xi\right) - \sigma\left(Y,\overline{\nabla}_X\xi\right) = 0. \tag{5.5}$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.5), we get

$$\sigma\left(Y,\phi X\right) - \sigma(Y,X) = 0. \tag{5.6}$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.6), we get

$$-\sigma(Y,X) - \sigma\left(Y,\phi X\right) = 0. \tag{5.7}$$

Adding (5.6) and (5.7), we obtain $\sigma(X,Y) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Theorem 5.2. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then M has parallel third fundamental form with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M have parallel third fundamental form with respect to Semisymmetric Nonmetric connection. Then we have

$$\left(\overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,W) = 0. \tag{5.8}$$

Taking $W = \xi$ and using (5.2) in the above equation, we have

$$\overline{\nabla}_X^{\perp}\left(\left(\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,\xi)\right) - \left(\overline{\widetilde{\nabla}}_Y\sigma\right)\left(\overline{\nabla}_X Z,\xi\right) - \left(\overline{\widetilde{\nabla}}_X\sigma\right)\left(Z,\overline{\nabla}_Y\xi\right) - \left(\overline{\widetilde{\nabla}}_{\overline{\nabla}_X Y}\sigma\right)(Z,\xi) = 0. \tag{5.9}$$

In view of (4.1) and by virtue of (5.1) in (5.9), we get

$$\begin{aligned} 0 = & -\overline{\nabla}_X^{\perp}\left\{\sigma\left(\overline{\nabla}_Y Z,\xi\right) + \sigma\left(Z,\overline{\nabla}_Y\xi\right)\right\} - \overline{\nabla}_Y^{\perp}\sigma\left(\overline{\nabla}_X Z,\xi\right) + \sigma\left(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi\right) \\ & + 2\sigma\left(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi\right) - \overline{\nabla}_X^{\perp}\sigma\left(Z,\overline{\nabla}_Y\xi\right) + \sigma\left(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi\right) + \sigma\left(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi\right) + \sigma\left(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi\right). \end{aligned} \tag{5.10}$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.10), we get

$$\begin{aligned}0 = {} & 2\overline{\nabla}_X^{\perp}\sigma(Z,\phi Y) - 2\overline{\nabla}_X^{\perp}\sigma(Z,Y) - 2\eta(Z)\sigma(X,\phi Y) + 2\sigma(\nabla_X Z,Y) \\ & + 2\eta(Z)\sigma(X,Y) - \sigma(Z,\nabla_X\phi Y) - \sigma(Z,\phi\nabla_X Y) - \eta(Y)\sigma(Z,\phi X) \\ & + 2\sigma(Z,\nabla_X Y) + 2\eta(Y)\sigma(Z,X) - 2\sigma(\nabla_X Z,\phi Y).\end{aligned} \tag{5.11}$$

Putting $Y = \xi$ and using (3.1), (3.6), and (4.1) in (5.11), we get

$$0 = \sigma(Z,X) - 3\sigma(Z,\phi X). \tag{5.12}$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.12), we get

$$0 = \sigma(Z,\phi X) + 3\sigma(Z,X). \tag{5.13}$$

Multiplying (5.12) by 1 and (5.13) by 3 and adding these two equations, we obtain $\sigma(X,Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Corollary 5.3. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then σ is 2-recurrent with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be 2-recurrent with respect to Semisymmetric Nonmetric connection; from (1.5), we have

$$\left(\overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,W) = \sigma(Z,W)\phi(X,Y). \tag{5.14}$$

Taking $W = \xi$ and using (5.2) in the above equation, we have

$$\begin{aligned}&\overline{\nabla}_X^{\perp}\left(\left(\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,\xi)\right) - \left(\overline{\widetilde{\nabla}}_Y\sigma\right)\left(\overline{\nabla}_X Z,\xi\right) - \left(\overline{\widetilde{\nabla}}_X\sigma\right)\left(Z,\overline{\nabla}_Y\xi\right) - \left(\overline{\widetilde{\nabla}}_{\overline{\nabla}_X Y}\sigma\right)(Z,\xi) \\ &\quad = \sigma(Z,\xi)\phi(X,Y).\end{aligned} \tag{5.15}$$

In view of (4.1) and by virtue of (5.1) in (5.15), we get

$$\begin{aligned}0 = {} & -\overline{\nabla}_X^{\perp}\left\{\sigma\left(\overline{\nabla}_Y Z,\xi\right) + \sigma\left(Z,\overline{\nabla}_Y\xi\right)\right\} - \overline{\nabla}_Y^{\perp}\sigma\left(\overline{\nabla}_X Z,\xi\right) + \sigma\left(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi\right) \\ & + 2\sigma\left(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi\right) - \overline{\nabla}_X^{\perp}\sigma\left(Z,\overline{\nabla}_Y\xi\right) + \sigma\left(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi\right) + \sigma\left(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi\right) + \sigma\left(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi\right).\end{aligned} \tag{5.16}$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.16), we get

$$\begin{aligned}0 &= 2\overline{\nabla}^{\perp}_{X}\sigma(Z,\phi Y) - 2\overline{\nabla}^{\perp}_{X}\sigma(Z,Y) - 2\eta(Z)\sigma(X,\phi Y) + 2\sigma(\nabla_X Z, Y)\\ &\quad + 2\eta(Z)\sigma(X,Y) - \sigma(Z,\nabla_X\phi Y) - \sigma(Z,\phi\nabla_X Y) - \eta(Y)\sigma(Z,\phi X)\\ &\quad + 2\sigma(Z,\nabla_X Y) + 2\eta(Y)\sigma(Z,X) - 2\sigma(\nabla_X Z,\phi Y).\end{aligned} \tag{5.17}$$

Putting $Y = \xi$ and using (3.1), (3.6), (4.1) in (5.17), we get

$$0 = \sigma(Z,X) - 3\sigma(Z,\phi X). \tag{5.18}$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.18), we get

$$0 = \sigma(Z,\phi X) + 3\sigma(Z,X). \tag{5.19}$$

Multiplying (5.18) by 1 and (5.19) by 3 and adding these two equations, we obtain $\sigma(X,Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Theorem 5.4. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then σ is generalized 2-recurrent with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Letting σ be generalized 2-recurrent with respect to Semisymmetric Nonmetric connection, from (1.7), we have

$$\left(\overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,W) = \psi(X,Y)\sigma(Z,W) + \phi(X)\left(\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,W), \tag{5.20}$$

where ψ and ϕ are 2-recurrent and 1-form, respectively. Taking $W = \xi$ in (5.20) and using (4.1), we get

$$\left(\overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,\xi) = \phi(X)\left(\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,\xi). \tag{5.21}$$

Using (4.1) and (5.2) in above equation, we get

$$\begin{aligned}&\overline{\nabla}^{\perp}_{X}\left(\left(\overline{\widetilde{\nabla}}_Y\sigma\right)(Z,\xi)\right) - \left(\overline{\widetilde{\nabla}}_Y\sigma\right)\left(\overline{\nabla}_X Z,\xi\right) - \left(\overline{\widetilde{\nabla}}_X\sigma\right)\left(Z,\overline{\nabla}_Y\xi\right) - \left(\overline{\widetilde{\nabla}}_{\overline{\nabla}_X Y}\sigma\right)(Z,\xi)\\ &\quad = -\phi(X)\left\{\sigma\left(\overline{\nabla}_Y Z,\xi\right) + \sigma\left(Z,\overline{\nabla}_Y\xi\right)\right\}.\end{aligned} \tag{5.22}$$

In view of (4.1) and by virtue of (5.1) in (5.22), we get

$$\begin{aligned} &-\overline{\nabla}_X^{\perp}\left\{\sigma\left(\overline{\nabla}_Y Z,\xi\right)+\sigma\left(Z,\overline{\nabla}_Y\xi\right)\right\}-\overline{\nabla}_Y^{\perp}\sigma\left(\overline{\nabla}_X Z,\xi\right)+\sigma\left(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi\right)\\ &\quad+2\sigma\left(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi\right)-\overline{\nabla}_X^{\perp}\sigma\left(Z,\overline{\nabla}_Y\xi\right)+\sigma\left(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi\right)+\sigma\left(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi\right)+\sigma\left(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi\right)\\ &\qquad=-\phi(X)\left\{\sigma\left(\overline{\nabla}_Y Z,\xi\right)+\sigma\left(Z,\overline{\nabla}_Y\xi\right)\right\}. \end{aligned} \tag{5.23}$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.23), we get

$$\begin{aligned} 0&=2\overline{\nabla}_X^{\perp}\sigma(Z,\phi Y)-2\overline{\nabla}_X^{\perp}\sigma(Z,Y)-2\eta(Z)\sigma(X,\phi Y)+2\sigma(\nabla_X Z,Y)+2\eta(Z)\sigma(X,Y)\\ &\quad-\sigma(Z,\nabla_X\phi Y)-\sigma(Z,\phi\nabla_X Y)-\eta(Y)\sigma(Z,\phi X)\\ &\quad+2\sigma(Z,\nabla_X Y)+2\eta(Y)\sigma(Z,X)-2\sigma(\nabla_X Z,\phi Y)\\ &=-\phi(X)\{-\sigma(Z,\phi Y)+\sigma(Z,Y)\}. \end{aligned} \tag{5.24}$$

Putting $Y=\xi$ and using (3.1), (3.6), (4.1) in (5.24), we get

$$0=\sigma(Z,X)-3\sigma(Z,\phi X). \tag{5.25}$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.25), we get

$$0=\sigma(Z,\phi X)+3\sigma(Z,X). \tag{5.26}$$

Multiplying (5.25) by 1 and (5.26) by 3 and adding these two equations, we obtain $\sigma(X,Z)=0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

6. Semiparallel, Pseudoparallel, and Ricci-Generalized Pseudoparallel Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

We consider invariant submanifolds of Sasakian manifolds admitting Semisymmetric Nonmetric connection satisfying the conditions $\overline{\overline{R}}\cdot\sigma=0$, $\overline{\overline{R}}\cdot\sigma=L_1Q(g,\sigma)$, $\overline{\overline{R}}\cdot\sigma=L_2Q(S,\sigma)$.

Theorem 6.1. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then we prove that M is semiparallel with respect to Semisymmetric Nonmetric connection if and only if* $6=2\overline{\phi}+\xi$.

Proof. Let M be semiparallel $\overline{\tilde{R}} \cdot \sigma = 0$. Putting $X = V = \xi$ and by virtue of (3.1), (3.6), and (4.1) in (4.11), we get

$$0 = -\sigma(U, R(\xi, Y)\xi) - \sigma\left(\overline{\tilde{\nabla}}_{\xi}\eta(U)Y, \xi\right) + \sigma\left(\overline{\tilde{\nabla}}_{Y}\eta(U)\xi, \xi\right) - \sigma\left(\overline{\tilde{\nabla}}_{\xi}\sigma(Y, U), \xi\right)$$
$$- \sigma(U, \nabla_{\xi}Y) + \sigma(U, \nabla_{Y}\xi) + \sigma(U, [\xi, Y]) - \sigma\left(U, \overline{\tilde{\nabla}}_{\xi}Y\right) + \sigma\left(U, \overline{\tilde{\nabla}}_{Y}\xi\right) + \sigma(U, Y). \tag{6.1}$$

Using (1.1), (2.1), (3.6), (3.15), (4.1), and (5.1) in (6.1), we get

$$0 = 3\sigma(U, Y) - \sigma\left(\overline{\tilde{\nabla}}_{\xi}\eta(U)Y, \xi\right) - \sigma(U, \phi Y) - \sigma(U, \nabla_{\xi}Y). \tag{6.2}$$

By definition σ is a vector-valued covariant tensor, and so $\sigma(U, Y)$ is a vector. Therefore $\overline{\tilde{\nabla}}_{\xi}\sigma(Y, U)$ is a vector, and hence by (4.1), we have

$$\sigma\left(\overline{\tilde{\nabla}}_{\xi}\sigma(Y, U), \xi\right) = 0. \tag{6.3}$$

Then from (6.2), we get

$$3\sigma(U, Y) = \overline{\phi}\sigma(U, Y) + \sigma(U, \nabla_{\xi}Y). \tag{6.4}$$

Interchanging Y and U in (6.4), we get

$$3\sigma(Y, U) = \overline{\phi}\sigma(Y, U) + \sigma(U, \nabla_{\xi}Y). \tag{6.5}$$

Adding these tow equations, (6.4) and (6.5), we get

$$6 = 2\overline{\phi} + \xi. \tag{6.6}$$

□

Theorem 6.2. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then we prove that M is pseudoparallel with respect to Semisymmetric Nonmetric connection if and only if $L_1 = \overline{\phi} + \xi/2 - 3$.*

Proof. Let M be pseudoparallel $\overline{\tilde{R}} \cdot \sigma = L_1 Q(g, \sigma)$. Putting $X = V = \xi$ and by virtue of (3.1), (3.6), and (4.1) in (2.7), (4.11), we get

$$-\sigma(U, R(\xi, Y)\xi) - \sigma\left(\overline{\tilde{\nabla}}_{\xi}\eta(U)Y, \xi\right) + \sigma\left(\overline{\tilde{\nabla}}_{Y}\eta(U)\xi, \xi\right) - \sigma\left(\overline{\tilde{\nabla}}_{\xi}\sigma(Y, U), \xi\right) - \sigma(U, \nabla_{\xi}Y)$$
$$+ \sigma(U, \nabla_{Y}\xi) + \sigma(U, [\xi, Y]) - \sigma\left(U, \overline{\tilde{\nabla}}_{\xi}Y\right) + \sigma\left(U, \overline{\tilde{\nabla}}_{Y}\xi\right) + \sigma(U, Y) = -L_1\sigma(U, Y). \tag{6.7}$$

Using (1.1), (2.1), (3.6), (3.15), (4.1), and (5.1) in (6.7), we get

$$3\sigma(U,Y)-\sigma\left(\overline{\overline{\tilde{\nabla}}}_{\xi}\eta(U)Y,\xi\right)-\sigma(U,\phi Y)-\sigma(U,\nabla_{\xi}Y)=-L_1\sigma(U,Y). \tag{6.8}$$

Now by using (6.3) in (6.8), we get

$$(3+L_1)\sigma(U,Y)=\overline{\phi}\sigma(U,Y)+\sigma(U,\nabla_{\xi}Y). \tag{6.9}$$

Interchanging Y and U in (6.9), we get

$$(3+L_1)\sigma(Y,U)=\overline{\phi}\sigma(Y,U)+\sigma(Y,\nabla_{\xi}U). \tag{6.10}$$

Adding (6.9) and (6.10), we get

$$L_1=\overline{\phi}+\frac{\xi}{2}-3. \tag{6.11}$$

□

Theorem 6.3. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then we prove that M is Ricci-generalized pseudoparallel with respect to Semisymmetric Nonmetric connection if and only if $L_2=(1/(n-1))[\overline{\phi}+\xi/2-3]$.*

Proof. Let M be Ricci-generalized pseudoparallel $\overline{\overline{\tilde{R}}}\cdot\sigma=L_2Q(S,\sigma)$. Putting $X=V=\xi$ and by virtue of (3.1), (3.6), (3.16), and (4.1) in (2.7), (4.11), we get

$$\begin{aligned}&-\sigma(U,R(\xi,Y)\xi)-\sigma\left(\overline{\overline{\tilde{\nabla}}}_{\xi}\eta(U)Y,\xi\right)+\sigma\left(\overline{\overline{\tilde{\nabla}}}_{Y}\eta(U)\xi,\xi\right)-\sigma\left(\overline{\overline{\tilde{\nabla}}}_{\xi}\sigma(Y,U),\xi\right)-\sigma(U,\nabla_{\xi}Y)\\&+\sigma(U,\nabla_{Y}\xi)+\sigma(U,[\xi,Y])-\sigma\left(U,\overline{\overline{\tilde{\nabla}}}_{\xi}Y\right)+\sigma\left(U,\overline{\overline{\tilde{\nabla}}}_{Y}\xi\right)+\sigma(U,Y)=-L_2(n-1)\sigma(U,Y).\end{aligned} \tag{6.12}$$

Using (1.1), (2.1), (3.6), (3.15), (4.1), and (5.1) in (6.12), we get

$$3\sigma(U,Y)-\sigma\left(\overline{\overline{\tilde{\nabla}}}_{\xi}\eta(U)Y,\xi\right)-\sigma(U,\phi Y)-\sigma(U,\nabla_{\xi}Y)=-L_2(n-1)\sigma(U,Y). \tag{6.13}$$

Now by using (6.3) in (6.13), we get

$$(3+L_2(n-1))\sigma(U,Y)=\overline{\phi}\sigma(U,Y)+\sigma(U,\nabla_{\xi}Y). \tag{6.14}$$

Interchanging Y and U in (6.14), we get

$$(3+L_2(n-1))\sigma(Y,U)=\overline{\phi}\sigma(Y,U)+\sigma(Y,\nabla_{\xi}U). \tag{6.15}$$

Adding (6.14) and (6.15), we get

$$2(3 + L_2(n-1))\sigma(U, Y) = 2\overline{\phi}\sigma(U, Y) + \nabla_\xi \sigma(U, Y). \tag{6.16}$$

Writting the above equation, we have

$$L_2 = \frac{1}{(n-1)}\left[\overline{\phi} + \frac{\xi}{2} - 3\right]. \tag{6.17}$$

□

Remark 6.4. Let M be an invariant submanifold of a Sasakian manifold which admits Semisymmetric Nonmetric connection. If M is semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel, then we have obtained conditions connecting ϕ, ξ, L_1, and L_2. These conditions need further investigation and are to be interpreted geometrically.

Using Theorems 5.1 to 5.4 and corollary 5.3, we have the following result.

Corollary 6.5. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then the following statements are equivalent:*

(1) *$\sigma$is recurrent,*

(2) *σ is 2-recurrent,*

(3) *σ is generalized 2-recurrent,*

(4) *M has parallel third fundamental form.*

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