Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 743873, 6 pages
doi:10.1155/2012/743873

Research Article

Transfer of the GPIT Property in Pullbacks

David E. Dobbs[1] and Jay Shapiro[2]

[1] *Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1320, USA*
[2] *Department of Mathematics, George Mason University, Fairfax, VA 22030-4444, USA*

Correspondence should be addressed to David E. Dobbs, dobbs@math.utk.edu

Received 6 September 2011; Accepted 15 November 2011

Academic Editor: Ayman Badawi

Let T be a commutative ring, I a prime ideal of T, D a subring of T/I, and R the pullback $T\times_{T/I}D$. Ascent and descent results are given for the transfer of the n-PIT and GPIT (generalized principal ideal theorem) properties between T and R. As a consequence, it follows that if I is a maximal ideal of both T and R, then R satisfies n-PIT (resp., GPIT) if and only if T satisfies n-PIT (resp., GPIT).

1. Introduction

All rings considered in this paper are commutative, with identity. If A is a ring, Spec(A) denotes the set of all prime ideals of A, and if $P \in$ Spec(A), then $\mathrm{ht}_A(P)$ denotes the height of P in A. As in [1], if n is a nonnegative integer, then we say that a ring A satisfies the n-PIT property if $\mathrm{ht}_A(P) \leq n$ whenever $P \in$ Spec(A) is minimal as a prime ideal of A containing a given n-generated ideal of A. Notice that each ring satisfies 0-PIT and that 1-PIT is the PIT (for "principal ideal theorem") property introduced in [2]. Also as in [1], we say that a ring A satisfies the GPIT (for "generalized principal ideal theorem") property in case A satisfies n-PIT for all $n \geq 0$. A well-known fundamental result states that each Noetherian ring satisfies GPIT (cf. [3, Theorem 152]). Moreover, GPIT figures implicitly in a characterization of Noetherian rings [2, Theorem 2.6 and Remark 5.3(a)]. The diversity of rings satisfying PIT [2, Corollaries 3.5, 3.11, and 6.6] and GPIT [1, Corollaries 2.3 and 4.3] has been exhibited by studying the transfer of these properties for various ring-theoretic constructions (cf. [2, Propositions 3.1(a), 4.1 and 6.3, Theorem 4.5 and Section 5], [1, Proposition 2.1, Theorems 3.3 and 4.2]). In particular, in view of the characterizations of Noetherian rings arising as certain pullbacks ([4, Proposition 1.8], [5, Proposition 1, Corollaire 1]), earlier papers developed transfer results for PIT and GPIT in pullbacks $D\times_{T/I}T$ in case (T, I) is a quasilocal domain and D is a subring of T/I (see [2, Corollary 3.2(c)], [1, Proposition 2.5(b)]). The main purpose of this note is to extend those results to the context in which T is not necessarily quasilocal or a domain. We thus enlarge the arena of applications in the same spirit in which the general $D + M$ construction studied by Brewer and Rutter [6] extended the classical $D + M$ construction (as in [7]) that had issued from a valuation (in particular, quasilocal) domain T.

It is convenient to organize our work so that the assertions regarding transfer of GPIT are consequences of transfer results on n-PIT. Our main descent (resp., ascent) result for these properties is Theorem 2.2 (resp., Theorem 2.5). Technical results dealing with new conditions on prime ideals in pullbacks are isolated in Lemmas 2.1 and 2.4. Our best "if and only if" transfer result appears in Corollary 2.6, with the special case for the $D + M$ construction in Corollary 2.7.

In addition to the notation and terminology introduced above, we let $\mathrm{Max}(A)$ denote the set of all maximal ideals of a ring A. Any other material is standard, as in [7] or [3].

2. Results

Henceforth, we adopt the following *standing hypotheses*: T is a ring, $I \in \mathrm{Spec}(T)$, D is a subring of T/I, and R is the pullback $T \times_{T/I} D$. Before giving our main descent result for the n-PIT and GPIT properties, we begin with a lemma that builds on a result of Cahen [5, Proposition 5], that compares the heights of I in T and R. An additional hypothesis is used in Lemma 2.1, since an example of Cahen [5, Exemple 3], shows that the standing hypotheses are not sufficient to imply that $\mathrm{ht}_R(I) = \mathrm{ht}_T(I)$.

Lemma 2.1. *Let T, I, D, and R be as in the standing hypotheses. Assume also that if $Q \in \mathrm{Spec}(T)$ satisfies $\mathrm{ht}_T(Q) \geq \mathrm{ht}_T(I)$, then either Q is comparable to I (with respect to inclusion) or $Q + I = T$. Then, $\mathrm{ht}_R(I) = \mathrm{ht}_T(I)$.*

Proof. According to [5, Proposition 5], $\mathrm{ht}_T(I) \leq \mathrm{ht}_R(I)$. Thus, without loss of generality, we may assume that $n := \mathrm{ht}_T(I) < \infty$. The assertion can be proved by applying the fundamental gluing result on the spectra of pullbacks [4, Theorem 1.4]. We prefer the following somewhat more transparent argument.

Suppose that the assertion fails. Then, we can choose $q \in \mathrm{Spec}(R)$ such that $q \subset I$ and $\mathrm{ht}_R(q) \geq n$. By applying the isomorphism in [4, Corollary 1.5(3)], we see that there exists a (uniquely determined) $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = q$ and, moreover, that $\mathrm{ht}_T(Q) = \mathrm{ht}_R(q) \geq n$. In view of the hypothesis, there are three cases to consider.

If $Q \supseteq I$, then $q = Q \cap R \supseteq I \cap R = I$, a contradiction. On the other hand, if $Q \subset I$, then $\mathrm{ht}_T(I) \geq \mathrm{ht}_T(Q) + 1 \geq n + 1$, also a contradiction. We handle the remaining case, in which $Q + I = T$, by an argument that is reminiscent of a proof of Cahen [5, Lemme 5]. In this case, $\alpha + i = 1$, for some $\alpha \in Q$ and $i \in I$. Then,

$$\alpha = 1 - i \in Q \cap R = q \subset I \tag{2.1}$$

and so $1 = \alpha + i \in I + I \subseteq I$, the desired contradiction. □

Theorem 2.2. *Let T, I, D, and R be as in the standing hypotheses. Assume that if $Q \in \mathrm{Spec}(T)$ satisfies $\mathrm{ht}_T(Q) \geq \mathrm{ht}_T(I)$, then either Q is comparable to I or $Q + I = T$. Assume also that $I \in \mathrm{Max}(R)$. Then,*

(a) *if n is a positive integer and T satisfies n-*PIT*, then R satisfies n-*PIT*;*

(b) *if T satisfies* GPIT*, then R satisfies* GPIT*.*

Proof. The assertion in (b) follows from that in (a), by universal quantification on n. As for (a), consider an n-generated ideal $J = \sum_{i=1}^{n} Ra_i$ of R and let p be minimal among the prime ideals of R that contain J. Our task is to show that $\mathrm{ht}_R(p) \leq n$.

Suppose first that $p = I$. Then, every prime of T that is contained in I is actually a prime ideal of R, and so it follows easily that I is minimal as a prime ideal of T containing $JT = \sum_{i=1}^{n} Ta_i$. Since T satisfies n-PIT, $\mathrm{ht}_T(I) \leq n$. Therefore, by Lemma 2.1, $\mathrm{ht}_R(p) = \mathrm{ht}_R(I) = \mathrm{ht}_T(I) \leq n$.

In the remaining case, $p \neq I$. Since $I \in \mathrm{Max}(R)$, it follows that $I \nsubseteq p$. An appeal to [4, Corollary 1.5(3)] yields a (uniquely determined) $P \in \mathrm{Spec}(T)$ such that $P \cap R = p$. Clearly, $I \nsubseteq P$. Moreover, since P is uniquely determined, it follows easily that P is minimal among primes of T that contain JT. As T satisfies n-PIT, we have that $\mathrm{ht}_T(P) \leq n$. Furthermore, since $I \nsubseteq p$, [4, Theorem 1.4(c)] ensures that $R_p = T_P$. Accordingly

$$\mathrm{ht}_R(p) = \mathrm{ht}_{R_p}(pR_p) = \mathrm{ht}_{T_P}(PT_P) = \mathrm{ht}_T(P) \leq n. \tag{2.2}$$

□

Recall (cf. [8]) that a ring A is said to be *treed* in case no maximal ideal of A contains prime ideals of A that are incomparable.

Corollary 2.3. *Let T, I, D, and R be as in the standing hypotheses. Assume also that $I \in \mathrm{Max}(R)$ and either $I \in \mathrm{Max}(T)$ or T is treed. Then,*

(a) *if n is a positive integer and T satisfies n-*PIT*, then R satisfies n-*PIT*;*

(b) *if T satisfies* GPIT*, then R satisfies* GPIT*.*

Proof. The assumptions ensure that if $Q \in \mathrm{Spec}(T)$, then either Q is comparable to I or $Q + I = T$. An application of Theorem 2.2 completes the proof. □

We next isolate a lemma of some independent interest. Notice that the assumption in Lemma 2.4 arose naturally in the proof of Corollary 2.3.

Lemma 2.4. *Let T, I, D, and R be as in the standing hypotheses. Assume that if $Q \in \mathrm{Spec}(T)$, then either Q is comparable to I or $Q + I = T$. If $P \in \mathrm{Spec}(R)$ and $P \subseteq I$, then $P \in \mathrm{Spec}(T)$.*

Proof. We adapt the proof of Lemma 2.1. Let $P \in \mathrm{Spec}(R)$ such that $P \subseteq I$. Since $I \in \mathrm{Spec}(T)$, we may assume that $P \subset I$; in particular, $P \nsupseteq I$. Therefore, by [4, Corollary 1.4(3)], $P = Q \cap R$ for some (uniquely determined) $Q \in \mathrm{Spec}(T)$. In view of the hypothesis, there are three cases to consider.

The case $Q \supseteq I$ is handled as in the proof of Lemma 2.1. On the other hand, if $Q \subset I$, then $Q \subset R$, and so $P = Q \cap R = Q \in \mathrm{Spec}(T)$. Finally, the ostensibly final case, $Q + I = T$, cannot actually arise, for otherwise, the proof of Lemma 2.1 would show that $1 \in I$, a contradiction. □

We next present our main ascent result for the n-PIT and GPIT properties.

Theorem 2.5. *Let T, I, D, and R be as in the standing hypotheses. Assume also that $I \in \mathrm{Max}(T)$. Then,*

(a) *if n is a positive integer and R satisfies n-*PIT*, then T satisfies n-*PIT*;*

(b) *if R satisfies* GPIT*, then T satisfies* GPIT*.*

Proof. As in the proof of Theorem 2.2, it suffices to establish (a). To that end, let P be minimal as a prime ideal of T containing a given n-generated ideal $J = \sum_{i=1}^{n} Ta_i$. We consider two cases.

Suppose first that $P \neq I$, that is, $P \not\supseteq I$. Choose $\alpha \in I \setminus P$. We claim that P is minimal as a prime ideal of T containing the n-generated ideal $H := \sum_{i=1}^{n} Ta_i\alpha$. If not, pick $Q \in \mathrm{Spec}(T)$ such that $H \subseteq Q \subset P$. By the minimality of P, $a_i \notin Q$ for some i, and so $\alpha \in Q$ since Q is prime. Then, $\alpha \in P$, a contradiction, thus proving the above claim.

Since $\alpha \in I \subset R$, it is clear that $p := P \cap R$ contains the elements $a_1\alpha, \ldots, a_n\alpha$. Moreover, it follows from the isomorphism in [4, Corollary 1.5(3)] and the minimality of P that p is minimal among the prime ideals of R containing the n-generated ideal $\sum_{i=1}^{n} Ra_i\alpha$. Furthermore, this isomorphism yields that $\mathrm{ht}_T(P) = \mathrm{ht}_R(p)$ and the assumption that R satisfies n-PIT yields that $\mathrm{ht}_R(p) \leq n$. Thus, $\mathrm{ht}_T(P) \leq n$, as desired.

It remains only to consider the case $P = I$. Since $I \in \mathrm{Max}(T)$, Lemma 2.4 may be applied, with the upshot that $\mathrm{ht}_T(I) = \mathrm{ht}_R(I)$. Suppose that $G := \sum_{i=1}^{n} Ra_i$ and $P_0 \in \mathrm{Spec}(R)$ are such that $G \subseteq P_0 \subseteq P = I$. Then, by Lemma 2.4 and the minimality of P, we have that $J \subseteq P_0 \in \mathrm{Spec}(T)$ and $P_0 = P$. Hence, P is minimal as a prime ideal of R containing G. Since R satisfies n-PIT, $\mathrm{ht}_R(P) \leq n$. Thus, $\mathrm{ht}_T(P) = \mathrm{ht}_T(I) = \mathrm{ht}_R(P) \leq n$, which completes the proof. □

Combining Corollary 2.3 and Theorem 2.5, we obtain the following sufficient conditions for the GPIT property in R to be equivalent to the GPIT property in T.

Corollary 2.6. *Let T, I, D, and R be as in the standing hypotheses. Assume also that $I \in \mathrm{Max}(R)$ and $I \in \mathrm{Max}(T)$. Then*

(a) *for any positive integer n, R satisfies n-*PIT *if and only if T satisfies n-*PIT*;*

(b) *R satisfies* GPIT *if and only T satisfies* GPIT*.*

We next state a special case of the previous corollary that generalizes [2, Corollary 3.2(c)] and [1, Proposition 2.5(b)] to the general $D + M$ context of [6] in which T need not be quasilocal.

Corollary 2.7. *Let $B = K + M$ be a ring, where K is a field and $M \in \mathrm{Max}(B)$. Let k be a subfield of K, and put $A := k + M$. Then,*

(a) *for any positive integer n, A satisfies n-*PIT *if and only if B satisfies n-*PIT*;*

(b) *A satisfies* GPIT *if and only B satisfies* GPIT*.*

Remark 2.8. (a) It was stated in Section 1 that our results generalize transfer results for PIT and GPIT that had been given in [1, 2] for the case in which (T, I) is quasilocal. In those earlier results, the PIT (resp., GPIT) property for R was shown to be equivalent to the condition that $I \in \mathrm{Max}(R)$ and the n-PIT (resp., GPIT) property for T. However, the condition that $I \in \mathrm{Max}(R)$ appears as a hypothesis, rather than a conclusion, in Theorem 2.2. Accordingly, we should underscore that, for each $n > 0$, the GPIT property for T does *not* imply the n-PIT property for R under our standing hypothesis, even if $I \in \mathrm{Max}(T)$.

To see this, begin by taking D to be an n-dimensional Noetherian unique factorization domain. Let L denote the quotient field of D. Set $T := L[X] = L + I$, with $I := XT$, and

$R := T\times_{T/I}D = T\times_L D = D + XL[X]$. Observe that the standing hypotheses are satisfied. Moreover, $I \in \mathrm{Max}(T)$ and, since T is a Noetherian ring, T satisfies GPIT (and, hence, n-PIT). However, R does not satisfy n-PIT.

Indeed, pick a prime element p of D, so that $\mathrm{ht}_D(pD) = 1$ by the classical principal ideal theorem. We have $I \subseteq pR$, essentially since $X = p(p^{-1}X)$. Moreover, $pR \in \mathrm{Spec}(R)$, since $R/pR \cong D/pD$. Then, the order-theoretic impact of the fundamental gluing result for pullbacks [4, Theorem 1.4] yields that $\mathrm{ht}_R(pR) = n + 1$, even though pR is an n-generated ideal of R, and so R does not satisfy n-PIT.

(b) Apropos of generalizing the pullback-theoretic results on PIT and GPIT in [1, 2], we do not know whether the standing hypotheses, coupled with the additional assumptions that $I \in \mathrm{Max}(T)$ and R satisfies n-PIT with $n > 0$, implies that $I \in \mathrm{Max}(R)$. However, we do have the following positive result along these lines for the $D+M$ context. Let $B = K + M$ be a domain, where K is a field and $0 \neq M \in \mathrm{Max}(B)$. Let D be a subring of K, and put $A := D+M$. If A satisfies PIT, then D is a field.

For an indirect proof, suppose that we can pick a nonzero nonunit $d \in D$. Then, for all $\alpha \in M$, we have $\alpha = d(d^{-1}\alpha) \in d(KM) \subseteq dM \subseteq dA$, whence $M \subseteq dA$. Moreover, $M \neq dA$ (the point being that $d \notin M$ since $M \cap D \subseteq M \cap K = 0$). As d is a nonunit of A, we can choose P minimal among the prime ideals of A containing dA. Then, $\mathrm{ht}_A(P) \geq \mathrm{ht}_A(M) + 1 \geq 2$, contrary to A satisfying PIT.

(c) We close by answering a question of the referee that asked for additional examples of a ring T satisfying the hypothesis of Theorem 2.2. Specifically, we show that for each positive integer $n \geq 2$, there exists a nontreed non-quasilocal n-dimensional domain T and a nonmaximal nonzero prime ideal I of T such that, whenever $Q \in \mathrm{Spec}(T)$ satisfies $\mathrm{ht}_T(Q) \geq \mathrm{ht}_T(I)$, then either Q is comparable to I (with respect to inclusion) or $Q + I = T$.

For the details of the construction, suppose first that $n \geq 3$. Then, a suitable T can be found that possesses exactly $n + 3$ prime ideals. Indeed, consider the $(n + 3)$-element poset $S = \{0 = P_0, P_1, \ldots, I = P_{n-1}, N = P_n, M_1, M = M_2\}$, where the only nontrivial relations are given by $P_i < P_j$ if and only if $0 \leq i < j \leq n$; $P_0 < M_1 < M$; and $M_1 < N$. It is known (cf. [9, Theorem 2.10]) that there exists a ring B such that $\mathrm{Spec}(B)$ is order-isomorphic to S, since S is a finite poset. Then, taking T to be the associated reduced ring of B suffices, for the only prime ideal $Q \neq I$ of T that has height at least that of I is N, which is comparable to I. Note that T has exactly two maximal ideals, namely, N and M. The reader is invited to augment the above construction and thus produce an example T with any desired finite number ≥ 3 of maximal ideals.

Finally, in case $n = 2$, one can produce a suitable T by arguing as above with the 6-element poset $S = \{0, I, N, P, Q, M\}$, where the only nontrivial relations are given by $0 < I < N$, $0 < P < M$, and $0 < Q < M$. The simple verification in this case is left to the reader.

(d) We note that the paper [10] touches on some related matters. Indeed, [10, Theorem 2.7] shows that each ϕ-Noetherian ring satisfies GPIT, while [10, Theorem 2.2] characterizes a ϕ-Noetherian ring as a ϕ-ring R such that $\phi(R)$ is the pullback of a certain type of diagram.

References

[1] D. F. Anderson, D. E. Dobbs, P. M. Eakin, Jr., and W. J. Heinzer, "On the generalized principal ideal theorem and Krull domains," *Pacific Journal of Mathematics*, vol. 146, no. 2, pp. 201–215, 1990.

[2] V. Barucci, D. F. Anderson, and D. E. Dobbs, "Coherent Mori domains and the principal ideal theorem," *Communications in Algebra*, vol. 15, no. 6, pp. 1119–1156, 1987.

[3] I. Kaplansky, *Commutative Rings*, The University of Chicago Press, Chicago, Ill, USA, 1974.

[4] M. Fontana, "Topologically defined classes of commutative rings," *Annali di Matematica Pura ed Applicata*, vol. 123, pp. 331–355, 1980.

[5] P.-J. Cahen, "Couples d'anneaux partageant un idéal," *Archiv der Mathematik*, vol. 51, no. 6, pp. 505–514, 1988.

[6] J. W. Brewer and E. A. Rutter, "*D+M* constructions with general overrings," *The Michigan Mathematical Journal*, vol. 23, no. 1, pp. 33–42, 1976.

[7] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, NY, USA, 1972, Pure and Applied Mathematics, No. 1.

[8] D. E. Dobbs, "On going down for simple overrings. II," *Communications in Algebra*, vol. 1, pp. 439–458, 1974.

[9] W. J. Lewis, "The spectrum of a ring as a partially ordered set," *Journal of Algebra*, vol. 25, pp. 419–434, 1973.

[10] A. Badawi and T. G. Lucas, "Rings with prime nilradical," in *Arithmetical Properties of Commutative Rings and Monoids*, vol. 241 of *Lect. Notes Pure Appl. Math.*, pp. 198–212, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2005.