*Research Article*

# Derivation of Identities Involving Bernoulli and Euler Numbers

**Imju Lee and Dae San Kim**

*Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*

Correspondence should be addressed to Dae San Kim, dskim@sogang.ac.kr

We derive some new and interesting identities involving Bernoulli and Euler numbers by using some polynomial identities and $p$-adic integrals on $\mathbb{Z}_p$.

## 1. Introduction and Preliminaries

Let $p$ be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will, respectively, denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value $|\ |_p$ on $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. Let $\mathbb{Z}_{>0}$ be the set of natural numbers and $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{>0} \cup \{0\}$.

As is well known, the Bernoulli polynomials $B_n(x)$ are defined by the generating function as follows:

$$F(t, x) = \frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.1}$$

with the usual convention of replacing $B(x)^n$ by $B_n(x)$.

In the special case, $x = 0$, $B_n(0) = B_n$ is referred to as the $n$th Bernoulli number. That is, the generating function of Bernoulli numbers is given by

$$F(t) = F(t, 0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = e^{Bt}, \tag{1.2}$$

with the usual convention of replacing $B^n$ by $B_n$, (cf. [1–23]).

From (1.2), we see that the recurrence formula for the Bernoulli numbers is

$$(B + 1)^n - B_n = \delta_{1,n}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}, \tag{1.3}$$

where $\delta_{k,n}$ is the Kronecker symbol.

By (1.1) and (1.2), we easily get the following:

$$B_n(x) = (B + x)^n = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.4}$$

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic $p$-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x), \tag{1.5}$$

(cf. [12]). Then it is easy to see that

$$I(f_1) = I(f) + f'(0), \tag{1.6}$$

where $f_1(x) = f(x + 1)$ and $f'(0) = df(x)/dx|_{x=0}$.

By (1.6), we have the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.7}$$

(cf. [12–14]). From (1.7), we can derive the Witt's formula for the $n$th Bernoulli polynomial as follows:

$$\int_{\mathbb{Z}_p} (x + y)^n d\mu(y) = B_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.8}$$

By (1.1), we have the following:

$$B_n(1 - x) = (-1)^n B_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.9}$$

Thus, from (1.3), (1.4), and (1.9), we have the following:

$$B_n(1) = B_n + \delta_{1,n} = (-1)^n B_n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.10}$$

By (1.4), we have the following:

$$B_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.11}$$

Especially, for $x = 1$ and $y = 1$,

$$B_n(2) = \sum_{k=0}^{n} \binom{n}{k} B_k(1) = \sum_{k=0}^{n} \binom{n}{k}(B_k + \delta_{1,k}), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.12}$$

Therefore, from (1.9), (1.10), and (1.12), we can derive the following relation. For $n \in \mathbb{Z}_{\geq 0}$,

$$(-1)^n B_n(-1) = B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n} = n + (-1)^n B_n. \tag{1.13}$$

Let $f(y) = (x + y)^{n+1}$. By (1.6), we have the following:

$$\int_{\mathbb{Z}_p} (x + y + 1)^{n+1} d\mu(y) - \int_{\mathbb{Z}_p} (x + y)^{n+1} d\mu(y) = (n + 1)x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.14}$$

By (1.8) and (1.14), we have the following:

$$B_{n+1}(x + 1) - B_{n+1}(x) = (n + 1)x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.15}$$

Thus, by (1.11) and (1.15), we have the following identity.

$$x^n = \frac{1}{n+1} \sum_{l=0}^{n} \binom{n+1}{l} B_l(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.16}$$

As is well known, the Euler polynomials $E_n(x)$ are defined by the generating function as follows:

$$G(t, x) = \frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.17}$$

with the usual convention of replacing $E(x)^n$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ is referred to as the $n$th Euler number. That is, the generating function of Euler numbers is given by

$$G(t) = G(t, 0) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = e^{Et}, \tag{1.18}$$

with the usual convention of replacing $E^n$ by $E_n$, (cf. [1–23]).

From (1.18), we see that the recurrence formula for the Euler numbers is

$$(E + 1)^n + E_n = 2\delta_{0,n}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.19}$$

By (1.17) and (1.18), we easily get the following:

$$E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.20}$$

Let $C(\mathbb{Z}_p)$ be the space of continuous $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x)(-1)^x, \tag{1.21}$$

(cf. [9]). Then it is easy to see that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.22}$$

where $f_1(x) = f(x + 1)$.

By (1.22), we have the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} . \tag{1.23}$$

From (1.23), we can derive the Witt's formula for the $n$-th Euler polynomial as follows:

$$\int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.24}$$

By (1.17), we have the following:

$$E_n(1 - x) = (-1)^n E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.25}$$

Thus, from (1.19), (1.20), and (1.25), we have the following:

$$E_n(1) = -E_n + 2\delta_{0,n} = (-1)^n E_n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.26}$$

By (1.20), we have the following:

$$E_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.27}$$

Especially, for $x = 1$ and $y = 1$,

$$E_n(2) = \sum_{k=0}^{n} \binom{n}{k} E_k(1) = \sum_{k=0}^{n} \binom{n}{k} (-E_n + 2\delta_{0,k}), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.28}$$

Therefore, from (1.25), (1.26), and (1.28), we can derive the following relations. For $n \in \mathbb{Z}_{\geq 0}$,

$$(-1)^n E_n(-1) = E_n(2) = 2 - E_n(1) = 2 + E_n - 2\delta_{0,n} = 2 - (-1)^n E_n. \tag{1.29}$$

Let $f(y) = (x + y)^n$. By (1.22), we have the following:

$$\int_{\mathbb{Z}_p} (x + y + 1)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = 2x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.30}$$

By (1.24) and (1.30), we have the following:

$$E_n(x + 1) + E_n(x) = 2x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.31}$$

Thus, by (1.27) and (1.31), we get the following identity.

$$x^n = \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x) + E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.32}$$

The Bernstein polynomials are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad \text{for } k, n \in \mathbb{Z}_{\geq 0}, \tag{1.33}$$

with $0 \leq k \leq n$ (cf. [14]).

By the definition of $B_{k,n}(x)$, we note that

$$B_{k,n}(x) = B_{n-k,n}(1 - x). \tag{1.34}$$

In this paper, we derive some new and interesting identities involving Bernoulli and Euler numbers from well-known polynomial identities. Here, we note that our results are "complementary" to those in [6], in the sense that we take a fermionic $p$-adic integral where a bosonic $p$-adic integral is taken and vice versa, and we use the identity involving Euler polynomials in (1.32) where that involving Bernoulli polynomials in (1.16) is used and vice versa. Finally, we report that there have been a lot of research activities on this direction of research, namely, on derivation of identities involving Bernoulli and Euler numbers and polynomials by exploiting bosonic and fermionic $p$-adic integrals (cf. [6–8]).

## 2. Identities Involving Bernoulli Numbers

Taking the bosonic $p$-adic integral on both sides of (1.16), we have the following:

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu(x) &= \int_{\mathbb{Z}_p} \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k(x) d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \int_{\mathbb{Z}_p} B_k(x) d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \sum_{j=0}^{k} \binom{k}{j} B_{k-j} \int_{\mathbb{Z}_p} x^j d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \sum_{j=0}^{k} \binom{k}{j} B_{k-j} B_j.
\end{aligned}
\tag{2.1}
$$

Therefore, we obtain the following theorem.

**Theorem 2.1.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then on has the following:*

$$
B_m = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m+1}{k} \binom{k}{j} B_{k-j} B_j.
\tag{2.2}
$$

Let us apply (1.9) to the bosonic $p$-adic integral of (1.16).

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \int_{\mathbb{Z}_p} B_k(x) d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} (-1)^k \int_{\mathbb{Z}_p} B_k(1-x) d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} (-1)^k \sum_{j=0}^{k} \binom{k}{j} B_{k-j} \int_{\mathbb{Z}_p} (1-x)^j d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} (-1)^k \sum_{j=0}^{k} \binom{k}{j} B_{k-j} (-1)^j B_j(-1).
\end{aligned}
\tag{2.3}
$$

Then, we can express (2.3) in three different ways.

By (1.13), (2.3) can be written as

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k}\binom{k}{j} B_{k-j}(j + B_j + \delta_{1,j}) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \left( kB_{k-1}(1) + kB_{k-1} + \sum_{j=0}^{k}\binom{k}{j} B_{k-j}B_j \right) \\
&= -\sum_{k=0}^{m-1} \binom{m}{k}\left( B_k + (-1)^k B_k \right) + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k}\binom{k}{j} B_{k-j}B_j \\
&= -\sum_{k=0}^{m-1} \binom{m}{k}(B_k + B_k + \delta_{1,k}) + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k}\binom{k}{j} B_{k-j}B_j \\
&= -2(B_m(1) - B_m) - (m - \delta_{1,m}) + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k}\binom{k}{j} B_{k-j}B_j \\
&= -\delta_{1,m} - m + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k}\binom{k}{j} B_{k-j}B_j.
\end{aligned}
\tag{2.4}
$$

Thus, we have the following theorem.

**Theorem 2.2.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
B_m = -\delta_{1,m} - m + \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m+1}{k}\binom{k}{j}(-1)^k B_{k-j}B_j.
\tag{2.5}
$$

**Corollary 2.3.** *Let $m$ be an integer $\geq 2$. Then one has the following:*

$$
B_m + m = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m+1}{k}\binom{k}{j}(-1)^k B_{k-j}B_j.
\tag{2.6}
$$

Especially, for an odd integer $m$ with $m \geq 3$, we obtain the following corollary.

**Corollary 2.4.** *Let $m$ be an odd integer with $m \geq 3$. Then one has the following:*

$$
m(m+1) = \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m+1}{k}\binom{k}{j}(-1)^k B_{k-j}B_j.
\tag{2.7}
$$

By (1.13), (2.3) can be written as

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k} \binom{k}{j} B_{k-j}\left(j + (-1)^j B_j\right) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \left( kB_{k-1}(1) + \sum_{j=0}^{k} \binom{k}{j}(-1)^j B_{k-j}B_j \right) \\
&= -\sum_{k=0}^{m-1} \binom{m}{k} B_k + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k} \binom{k}{j}(-1)^j B_{k-j}B_j \\
&= -B_m(1) + B_m + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k} \binom{k}{j}(-1)^j B_{k-j}B_j.
\end{aligned}
\tag{2.8}
$$

By (1.10), (2.8) can be written as

$$
\int_{\mathbb{Z}_p} x^m d\mu(x) = (-1)^{m+1} B_m + B_m + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k} \binom{k}{j}(-1)^j B_{k-j}B_j.
\tag{2.9}
$$

So, we get the following theorem.

**Theorem 2.5.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
B_m = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m+1}{k}\binom{k}{j}(-1)^{m+k+j} B_{k-j}B_j.
\tag{2.10}
$$

By (1.10), (2.8) can also be written as

$$
\int_{\mathbb{Z}_p} x^m d\mu(x) = -\delta_{1,m} + \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k}(-1)^k \sum_{j=0}^{k} \binom{k}{j}(-1)^j B_{k-j}B_j.
\tag{2.11}
$$

Thus, we have the following theorem.

**Theorem 2.6.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
B_m = -\delta_{1,m} + \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \binom{m+1}{k}\binom{k}{j}(-1)^{k+j} B_{k-j}B_j.
\tag{2.12}
$$

## 3. Identities Involving Euler Numbers

Taking the fermionic $p$-adic integral on both sides of (1.32), we have the following:

$$
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \left( E_m(x) + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} E_k(x) \right) d\mu_{-1}(x)
$$

$$
= \sum_{l=0}^{m} \binom{m}{l} E_{m-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} E_{k-j} \int_{\mathbb{Z}_p} x^j d\mu_{-1}(x) \qquad (3.1)
$$

$$
= \sum_{l=0}^{m} \binom{m}{l} E_{m-l} E_l + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} E_{k-j} E_j.
$$

So, we obtain the following theorem.

**Theorem 3.1.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
E_m = \sum_{l=0}^{m} \binom{m}{l} E_{m-l} E_l + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k} \binom{k}{j} E_{k-j} E_j. \qquad (3.2)
$$

Let us apply (1.25) to the fermionic $p$-adic integral of (1.32).

$$
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) = (-1)^m \int_{\mathbb{Z}_p} E_m(1-x) d\mu_{-1}(x) + \frac{1}{2} \sum_{k=0}^{m-1} \binom{m}{k} (-1)^k \int_{\mathbb{Z}_p} E_k(1-x) d\mu_{-1}(x)
$$

$$
= (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k} (-1)^k E_k(-1) \qquad (3.3)
$$

$$
+ \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k} \binom{k}{j} (-1)^k E_{k-j} (-1)^j E_j(-1).
$$

Then, we can express (3.3) in two different ways.

By (1.29), (3.3) can be written as

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(2 + E_k - 2\delta_{0,k}) \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}(2 + E_j - 2\delta_{0,j}) \\
&= 2E_m + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k} E_k + 2(-1)^{m+1} E_m + \sum_{k=0}^{m-1} \binom{m}{k} E_k \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j} E_j + \sum_{k=0}^{m-1} \binom{m}{k}(-1)^{k+1} E_k \\
&= 2E_m + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k} E_k + 2(-1)^{m+1} E_m + E_m(1) - E_m \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j} E_j + (-1)^{m+1}(E_m(-1) - E_m) \\
&= -2 + 2\delta_{0,m} + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k} E_k + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j} E_j.
\end{aligned}
\tag{3.4}
$$

Thus, we get the following theorem.

**Theorem 3.2.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
E_m = -2 + 2\delta_{0,m} + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k} E_k + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j} E_j.
\tag{3.5}
$$

**Corollary 3.3.** *Let $m \in \mathbb{Z}_{>0}$. Then one has the following:*

$$
E_m + 2 = (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k} E_k + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j} E_j.
\tag{3.6}
$$

By (1.29), (3.3) can be written as

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}\left(2 - (-1)^k E_k\right) \\
&\quad + \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}\left(2 - (-1)^j E_j\right) \\
&= 2E_m + (-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k}(-1)^k E_{m-k} E_k \\
&\quad + \sum_{k=0}^{m-1} \binom{m}{k} E_k - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k(-1)^j E_{k-j} E_j \\
&= 2E_m + (-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k}(-1)^k E_{m-k} E_k \\
&\quad + E_m(1) - E_m - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k(-1)^j E_{k-j} E_j \\
&= 2\delta_{0,m} + (-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k}(-1)^k E_{m-k} E_k \\
&\quad - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k(-1)^j E_{k-j} E_j.
\end{aligned}
\tag{3.7}
$$

So, we have the following theorem.

**Theorem 3.4.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
E_m = 2\delta_{0,m} + (-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k}(-1)^k E_{m-k} E_k - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^{k+j} E_{k-j} E_j.
\tag{3.8}
$$

**Corollary 3.5.** *Let $m \in \mathbb{Z}_{>1}$. Then one has the following:*

$$
E_m = (-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k}(-1)^k E_{m-k} E_k - \frac{1}{2} \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^{k+j} E_{k-j} E_j.
\tag{3.9}
$$

## 4. Identities Involving Bernoulli and Euler Numbers

By (1.16) and (1.32), we have the following:

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^{m+n} d\mu(x) &= \int_{\mathbb{Z}_p} \left( \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k(x) \right) \left( E_n(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x) \right) d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \int_{\mathbb{Z}_p} B_k(x) E_n(x) d\mu(x) \\
&\quad + \frac{1}{2(m+1)} \sum_{k=0}^{m} \binom{m+1}{k} \sum_{l=0}^{n-1} \binom{n}{l} \int_{\mathbb{Z}_p} B_k(x) E_l(x) d\mu(x) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \sum_{l=0}^{n} \binom{m+1}{k} \binom{k}{j} \binom{n}{l} B_{k-j} E_{n-l} B_{j+l} \\
&\quad + \frac{1}{2(m+1)} \sum_{k=0}^{m} \sum_{l=0}^{n-1} \sum_{j=0}^{k} \sum_{i=0}^{l} \binom{m+1}{k} \binom{n}{l} \binom{k}{j} \binom{l}{i} B_{k-j} E_{l-i} B_{j+i}.
\end{aligned}
\tag{4.1}
$$

Therefore, we get the following theorem.

**Theorem 4.1.** *Let $m, n \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
\begin{aligned}
B_{m+n} &= \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} \sum_{l=0}^{n} \binom{m+1}{k} \binom{k}{j} \binom{n}{l} B_{k-j} E_{n-l} B_{j+l} \\
&\quad + \frac{1}{2(m+1)} \sum_{k=0}^{m} \sum_{l=0}^{n-1} \sum_{j=0}^{k} \sum_{i=0}^{l} \binom{m+1}{k} \binom{n}{l} \binom{k}{j} \binom{l}{i} B_{k-j} E_{l-i} B_{j+i}.
\end{aligned}
\tag{4.2}
$$

By (1.16) and (1.33), we have the following:

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m B_{k,n}(x) d\mu(x) &= \int_{\mathbb{Z}_p} \frac{1}{m+1} \sum_{l=0}^{m} \binom{m+1}{l} B_l(x) B_{k,n}(x) d\mu(x) \\
&= \frac{1}{m+1} \binom{n}{k} \sum_{l=0}^{m} \sum_{i=0}^{l} \binom{m+1}{l} \binom{l}{i} B_{l-i} \int_{\mathbb{Z}_p} x^{i+k} (1-x)^{n-k} d\mu(x) \\
&= \frac{1}{m+1} \binom{n}{k} \sum_{l=0}^{m} \sum_{i=0}^{l} \sum_{j=0}^{n-k} \binom{m+1}{l} \binom{l}{i} \binom{n-k}{j} (-1)^j B_{l-i} \int_{\mathbb{Z}_p} x^{i+k+j} d\mu(x) \\
&= \frac{1}{m+1} \binom{n}{k} \sum_{l=0}^{m} \sum_{i=0}^{l} \sum_{j=0}^{n-k} \binom{m+1}{l} \binom{l}{i} \binom{n-k}{j} (-1)^j B_{l-i} B_{i+k+j}.
\end{aligned}
\tag{4.3}
$$

By (1.33), we have the following:

$$
\int_{\mathbb{Z}_p} x^m B_{k,n}(x) d\mu(x) = \binom{n}{k} \int_{\mathbb{Z}_p} x^{m+k}(1-x)^{n-k} d\mu(x)
$$

$$
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j}(-1)^j \int_{\mathbb{Z}_p} x^{m+k+j} d\mu(x) \tag{4.4}
$$

$$
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j}(-1)^j B_{m+k+j}.
$$

By (4.3) and (4.4), we obtain the following theorem.

**Theorem 4.2.** *Let $m, n, k \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
\sum_{j=0}^{n-k} \binom{n-k}{j}(-1)^j B_{m+k+j} = \frac{1}{m+1} \sum_{l=0}^{m} \sum_{i=0}^{l} \sum_{j=0}^{n-k} \binom{m+1}{l}\binom{l}{i}\binom{n-k}{j}(-1)^j B_{l-i} B_{i+k+j}. \tag{4.5}
$$

*Especially, one has the following:*

$$
(m+1)B_{m+n} = \sum_{l=0}^{m} \sum_{i=0}^{l} \binom{m+1}{l}\binom{l}{i} B_{l-i} B_{i+n}. \tag{4.6}
$$

By (4.2) and (4.6), we have the following theorem. Note that (4.8) in the following was obtained in [6].

**Theorem 4.3.** *Let $m, n \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
B_{m+n} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} B_{m+l} + \frac{1}{2} \sum_{l=0}^{n-1} \sum_{i=0}^{l} \binom{n}{l}\binom{l}{i} E_{l-i} B_{m+i}. \tag{4.7}
$$

In particular, we have the following:

$$
B_n = \sum_{l=0}^{n} \binom{n}{l} E_{n-1} B_l + \frac{1}{2} \sum_{l=0}^{n-1} \sum_{i=0}^{l} \binom{n}{l}\binom{l}{i} E_{l-i} B_i. \tag{4.8}
$$

## References

[1] A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.

[2] D. Ding and J. Yang, "Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 1, pp. 7–21, 2010.

[3] G. Kim, B. Kim, and J. Choi, "The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 17, no. 2, pp. 137–145, 2008.

 [4] L. Jang, "A note on Kummer congruence for the Bernoulli numbers of higher order," *Proceedings of the Jangjeon Mathematical Society*, vol. 5, no. 2, pp. 141–146, 2002.

 [5] L. C. Jang and H. K. Pak, "Non-Archimedean integration associated with $q$-Bernoulli numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 5, no. 2, pp. 125–129, 2002.

 [6] D. S. Kim, T. Kim, D. V. Dolgy, J. Choi, and Y. H. Kim, "Some identities on Bernoulli and Euler numbers," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 486158, 12 pages, 2012.

 [7] D. S. Kim, T. Kim, S.-H. Lee, D. V. Dolgy, and S.-H. Rim, "Some new identities on the Bernoulli and Euler numbers," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 856132, 11 pages, 2011.

 [8] D. S. Kim, A. Bayad, T. Kim, and S.-H. Rim, "Identities on the Bernoulli and Euler numbers and polynomials," *Ars Combinatoria*. In press.

 [9] T. Kim, "Euler numbers and polynomials associated with zeta functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 581582, 11 pages, 2008.

[10] T. Kim, "Note on the Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 17, no. 2, pp. 131–136, 2008.

[11] T. Kim, "Symmetry of power sum polynomials and multivariate fermionic $p$-adic invariant integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.

[12] T. Kim, "Symmetry $p$-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials," *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.

[13] T. Kim, "$q$-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[14] T. Kim, "A note on $q$-Bernstein polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 1, pp. 73–82, 2011.

[15] A. Kudo, "A congruence of generalized Bernoulli number for the character of the first kind," *Advanced Studies in Contemporary Mathematics*, vol. 2, pp. 1–8, 2000.

[16] Q.-M. Luo and F. Qi, "Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 7, no. 1, pp. 11–18, 2003.

[17] Q.-M. Luo, "Some recursion formulae and relations for Bernoulli numbers and Euler numbers of higher order," *Advanced Studies in Contemporary Mathematics*, vol. 10, no. 1, pp. 63–70, 2005.

[18] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on $q$-Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.

[19] Y.-H. Kim and K.-W. Hwang, "Symmetry of power sum and twisted Bernoulli polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 127–133, 2009.

[20] S.-H. Rim, J.-H. Jin, E.-J. Moon, and S.-J. Lee, "Some identities on the $q$-Genocchi polynomials of higher-order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 860280, 14 pages, 2010.

[21] C. S. Ryoo, "On the generalized Barnes type multiple $q$-Euler polynomials twisted by ramified roots of unity," *Proceedings of the Jangjeon Mathematical Society*, vol. 13, no. 2, pp. 255–263, 2010.

[22] C. S. Ryoo, "Some relations between twisted $q$-Euler numbers and Bernstein polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 2, pp. 217–223, 2011.

[23] Y. Simsek, "Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–278, 2008.