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Research Article

End-Point Results for Multivalued Mappings in Partially Ordered Metric Spaces

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The purpose of this paper is to prove end-point theorems for multivalued mappings satisfying comparatively a more general contractive condition in ordered complete metric spaces. Afterwards, we extend the results of previous sections and prove common end-point results for a pair of $\mathfrak{T}$-weakly isotone increasing multivalued mappings in the underlying spaces. Finally, we present common end point for a pair of $\mathfrak{T}$-weakly isotone increasing multivalued mappings satisfying weakly contractive condition.

1. Introduction and Preliminaries

Fixed-point theory for multivalued mappings was originally initiated by Von Neumann in the study of game theory. Fixed-point theorem for multivalued mappings is quite useful in control theory and has been frequently used in solving the problem of economics and game theory.

The theory of multivalued nonexpansive mappings is comparatively complicated as compare to the corresponding theory of single-valued nonexpansive mappings. It is therefore natural to expect that the theory of noncontinuous nonself-multivalued mappings would be much more complicated.

The study of fixed-points for multivalued contraction mappings was equally an active topic as single-valued mappings. The development of geometric fixed-point theory for multivalued was initiated with the work of Nadler Jr. [1] in the year 1969. He used the concept of Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case, as following.

Theorem 1.1. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ is a mapping from $\mathcal{X}$ into $CB(\mathcal{X})$ such that for all $x, y \in \mathcal{X}$,*

$$\mathcal{H}(\mathcal{T}x, \mathcal{T}y) \leq \lambda d(x, y), \tag{1.1}$$

where $0 \leq \lambda < 1$. Then $\mathcal{T}$ has a fixed-point.

Since then, this discipline has been further developed, and many profound concepts and results have been established; for example, the work of Border [2], Ćirić [3], Corley [4], Itoh and Takahashi [5], Mizoguchi and Takahashi [6], Petruşel and Luca [7], Rhoades [8], Tarafdar and Yuan [9], and references cited therein.

Let $(\mathcal{X}, d)$ be a metric space. We denote the class of nonempty and bounded subsets of $\mathcal{X}$ by $B(\mathcal{X})$. For $\mathcal{A}, \mathcal{B} \in B(\mathcal{X})$, functions $D(\mathcal{A}, \mathcal{B})$, and $\delta(\mathcal{A}, \mathcal{B})$ are defined as follows:

$$\begin{aligned} D(\mathcal{A}, \mathcal{B}) &= \inf\{d(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}, \\ \delta(\mathcal{A}, \mathcal{B}) &= \sup\{d(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}. \end{aligned} \tag{1.2}$$

If $\mathcal{A} = \{a\}$, then we write $D(\mathcal{A}, \mathcal{B}) = D(a, \mathcal{B})$ and $\delta(\mathcal{A}, \mathcal{B}) = \delta(a, \mathcal{B})$. Also in addition, if $\mathcal{B} = \{b\}$, then $D(\mathcal{A}, \mathcal{B}) = d(a, b)$ and $\delta(\mathcal{A}, \mathcal{B}) = d(a, b)$. Obviously, $D(\mathcal{A}, \mathcal{B}) \leq \delta(\mathcal{A}, \mathcal{B})$.

For all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in B(\mathcal{X})$, the definition of $\delta(\mathcal{A}, \mathcal{B})$ yields the following:

$$\begin{gathered} \delta(\mathcal{A}, \mathcal{B}) = \delta(\mathcal{B}, \mathcal{A}), \\ \delta(\mathcal{A}, \mathcal{B}) \leq \delta(\mathcal{A}, \mathcal{C}) + \delta(\mathcal{C}, \mathcal{B}), \\ \delta(\mathcal{A}, \mathcal{B}) = 0 \quad \text{iff } \mathcal{A} = \mathcal{B} = \{a\}, \\ \delta(\mathcal{A}, \mathcal{A}) = \operatorname{diam} \mathcal{A}. \end{gathered} \tag{1.3}$$

A point $x \in \mathcal{X}$ is called a fixed-point of a multivalued mapping $\mathcal{T} : \mathcal{X} \to B(\mathcal{X})$ if $x \in \mathcal{T}x$. If there exists a point $x \in \mathcal{X}$ such that $\mathcal{T}x = \{x\}$, then x is called an end-point of $\mathcal{T}$ [10].

Definition 1.2. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an ordered metric space if and only if:

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Definition 1.3. Let $(\mathcal{X}, \preceq)$ be a partial ordered set. Then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.4 (see [11]). Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of a partially ordered set $(\mathcal{X}, \preceq)$. The relation between $\mathcal{A}$ and $\mathcal{B}$ is denoted and defined as follows:

$$\mathcal{A} \prec_1 \mathcal{B}, \quad \text{if for every } a \in \mathcal{A} \text{ there exists } b \in \mathcal{B} \text{ such that } a \preceq b. \tag{1.4}$$

Definition 1.5 (see [12]). A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

On the other hand, fixed-point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [13, Theorem 2.1] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [14] extended the result of Ran and Reurings for nondecreasing mappings and applied it to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed-point theorems in ordered metric spaces. For detail see [14–28] and references cited therein. Beg and Butt [11, 17, 29] worked on set-valued mappings and proved common fixed-point for mapping satisfying implicit relation in partially ordered metric space. Recently, Choudhury and Metiya [30] proved fixed-point theorems for multivalued mappings in the framework of a partially ordered metric space.

The results of this paper are divided in three sections. In the first section we establish the existence of end-points for a multivalued mapping under a more general contractive condition in partially ordered metric spaces. The consequences of the main theorem are also given. The second section is devoted for common end-point results for a pair of weakly isotone increasing multivalued mappings. In the third section, we present common end-point results for a pair of weakly isotone increasing multivalued mappings satisfying weakly contractive condition.

2. End-Point Theorems for a Multivalued Mapping

In this section, we prove end-point theorems for a multivalued mapping in ordered complete metric space.

Theorem 2.1. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \to B(\mathcal{X})$ be such that the following conditions are satisfied:*

(i) *there exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_1 \mathcal{T}x_0$,*

(ii) *for $x, \ y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \prec_1 \mathcal{T}y$,*

(iii)

$$\psi(\delta(\mathcal{T}x, \mathcal{T}y)) \leq \alpha\psi(M(x, y)) + L \min\{D(x, \mathcal{T}x), D(y, \mathcal{T}y), D(x, \mathcal{T}y), D(y, \mathcal{T}x)\}, \tag{2.1}$$

for all comparable $x, y \in \mathcal{X}$, where $L \geq 0$, $0 < \alpha < 1$ and ψ is an altering distance function and

$$M(x,y) = \max\left\{d(x,y), D(x,\mathcal{T}x), D(y,\mathcal{T}y), \frac{D(x,\mathcal{T}y) + D(y,\mathcal{T}x)}{2}\right\}. \tag{2.2}$$

If the property

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \quad \text{then } x_n \prec z \forall n \tag{2.3}$$

holds, then $\mathcal{T}$ has a end-point.

Proof. By the assumption (i), there exists $x_1 \in \mathcal{T}x_0$ such that $x_0 \preceq x_1$. By the assumption (ii), $\mathcal{T}x_0 \prec_1 \mathcal{T}x_1$. Then there exists $x_2 \in \mathcal{T}x_1$ such that $x_1 \preceq x_2$. Continuing this process we construct a monotone increasing sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{n+1} \in \mathcal{T}x_n$, for all $n \geq 0$. Thus we have

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{2.4}$$

If $x_{n_0} \in \mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \neq x_{n+1}$ for all $n \geq 0$.

Using the monotone property of ψ and the condition (iii), we have for all $n \geq 0$,

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(\delta(\mathcal{T}x_n, \mathcal{T}x_{n+1})) \\
&\leq \alpha\psi\Bigg(\max\bigg\{d(x_n, x_{n+1}), D(x_n, \mathcal{T}x_n), D(x_{n+1}, \mathcal{T}x_{n+1}), \\
&\qquad \frac{D(x_n, \mathcal{T}x_{n+1}) + D(x_{n+1}, \mathcal{T}x_n)}{2}\bigg\}\Bigg) \\
&\quad + L\min\{D(x_n, \mathcal{T}x_n), D(x_{n+1}, \mathcal{T}x_{n+1}), D(x_n, \mathcal{T}x_{n+1}), D(x_{n+1}, \mathcal{T}x_n)\} \\
&\leq \alpha\psi\Bigg(\max\bigg\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\bigg\}\Bigg) \\
&\quad + L\min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\}.
\end{aligned} \tag{2.5}$$

Since $d(x_n, x_{n+2})/2 \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$\psi(d(x_{n+1}, x_{n+2})) \leq \alpha\psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{2.6}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, for some positive integer n.

Then from (2.6), we have

$$\psi(d(x_{n+1},x_{n+2})) \le \alpha\psi(d(x_{n+1},x_{n+2})), \tag{2.7}$$

it implies that $d(x_{n+1},x_{n+2}) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that $x_n \ne x_{n+1}$, for each n.

Therefore, $d(x_{n+1},\ x_{n+2}) < d(x_n,\ x_{n+1})$, for all $n \ge 0$ and $\{d(x_n,\ x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \ge 0$ such that

$$d(x_n,x_{n+1}) \longrightarrow r \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

Taking the limit as $n \to \infty$ in (2.6) and using the continuity of ψ, we have $\psi(r) \le \alpha\psi(r)$, which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.9}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)},\ x_{n(k)}) \ge \epsilon$.

Assuming that $n(k)$ is the smallest such positive integer, we get $n(k) > m(k) > k$,

$$d\left(x_{m(k)},x_{n(k)}\right) \ge \epsilon, \qquad d\left(x_{m(k)},x_{n(k)-1}\right) < \epsilon. \tag{2.10}$$

Now,

$$\epsilon \le d\left(x_{m(k)},x_{n(k)}\right) \le d\left(x_{m(k)},x_{n(k)-1}\right) + d\left(x_{n(k)-1},x_{n(k)}\right), \tag{2.11}$$

that is,

$$\epsilon \le d\left(x_{m(k)},x_{n(k)}\right) < \epsilon + d\left(x_{n(k)-1},x_{n(k)}\right). \tag{2.12}$$

Taking the limit as $k \to \infty$ in the above inequality and using (2.9), we have

$$\lim_{k\to\infty} d\left(x_{m(k)},x_{n(k)}\right) = \epsilon. \tag{2.13}$$

Again,

$$\begin{aligned} d\left(x_{m(k)},x_{n(k)}\right) &\le d\left(x_{m(k)},x_{m(k)+1}\right) + d\left(x_{m(k)+1},x_{n(k)+1}\right) + d\left(x_{n(k)+1},x_{n(k)}\right), \\ d\left(x_{m(k)+1},x_{n(k)+1}\right) &\le d\left(x_{m(k)+1},x_{m(k)}\right) + d\left(x_{m(k)},x_{n(k)}\right) + d\left(x_{n(k)},x_{n(k)+1}\right). \end{aligned} \tag{2.14}$$

Taking the limit as $k \to \infty$ in the above inequalities and using (2.9) and (2.13), we have

$$\lim_{k\to\infty} d\left(x_{m(k)+1},x_{n(k)+1}\right) = \epsilon. \tag{2.15}$$

Again,

$$\begin{aligned}
d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right), \\
d\left(x_{m(k)}, x_{n(k)+1}\right) &\leq d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right).
\end{aligned} \tag{2.16}$$

Letting $k \to \infty$ in the above inequalities and using (2.9) and (2.13), we have

$$\lim_{k \to \infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \epsilon. \tag{2.17}$$

Similarly, we have that

$$\lim_{k \to \infty} d\left(x_{n(k)}, x_{m(k)+1}\right) = \epsilon. \tag{2.18}$$

For each positive integer k, $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}
\psi\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) &\leq \psi\left(\delta\left(\mathcal{T}x_{m(k)}, \mathcal{T}x_{n(k)}\right)\right) \\
&\leq \alpha\psi\Bigg(\max\Bigg\{ d\left(x_{m(k)}, x_{n(k)}\right), D\left(x_{m(k)}, \mathcal{T}x_{m(k)}\right), D\left(x_{n(k)}, \mathcal{T}x_{n(k)}\right), \\
&\qquad \frac{D\left(x_{m(k)}, \mathcal{T}x_{n(k)}\right) + D\left(x_{n(k)}, \mathcal{T}x_{m(k)}\right)}{2} \Bigg\}\Bigg) \\
&\quad + L\min\left\{ D\left(x_{m(k)}, \mathcal{T}x_{m(k)}\right), D\left(x_{n(k)}, \mathcal{T}x_{n(k)}\right), D\left(x_{m(k)}, \mathcal{T}x_{n(k)}\right), \right. \\
&\qquad \left. D\left(x_{n(k)}, \mathcal{T}x_{m(k)}\right) \right\} \\
&\leq \alpha\psi\Bigg(\max\Bigg\{ d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right), \\
&\qquad \frac{d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)}, x_{m(k)+1}\right)}{2} \Bigg\}\Bigg) \\
&\quad + L\min\left\{ d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right), d\left(x_{m(k)}, x_{n(k)+1}\right), \right. \\
&\qquad \left. d\left(x_{n(k)}, x_{m(k)+1}\right) \right\}.
\end{aligned} \tag{2.19}$$

Letting $k \to \infty$ in above inequality, using (2.9), (2.13), (2.15), (2.17), and (2.18) and the continuity of ψ, we have

$$\psi(\epsilon) \leq \alpha\psi(\epsilon), \tag{2.20}$$

which is a contradiction by virtue of a property of ψ.

Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that

$$x_n \longrightarrow z \quad \text{as } n \longrightarrow \infty. \tag{2.21}$$

By the assumption (2.3), $x_n \preceq z$, for all n.

Then by the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}
\psi(\delta(x_{n+1}, \mathcal{T}z)) &\le \psi(\delta(\mathcal{T}x_n, \mathcal{T}z)) \\
&\le \alpha\psi\left(\max\left\{d(x_n, z), D(x_n, \mathcal{T}x_n), D(z, \mathcal{T}z), \frac{D(x_n, \mathcal{T}z) + D(z, \mathcal{T}x_n)}{2}\right\}\right) \\
&\quad + L\min\{D(x_n, \mathcal{T}x_n), D(z, \mathcal{T}z), D(x_n, \mathcal{T}z), D(z, \mathcal{T}x_n)\} \\
&\le \alpha\psi\left(\max\left\{d(x_n, z), d(x_n, x_{n+1}), D(z, Tz), \frac{D(x_n, Tz) + d(z, x_{n+1})}{2}\right\}\right) \\
&\quad + L\min\{d(x_n, x_{n+1}), D(z, \mathcal{T}z), D(x_n, \mathcal{T}z), d(z, x_{n+1})\}.
\end{aligned} \tag{2.22}$$

Taking the limit as $n \to \infty$ in the above inequality, using (2.9) and (2.21) and the continuity of ψ, we have

$$\psi(\delta(z, \mathcal{T}z)) \le \alpha\psi(D(z, \mathcal{T}z)) \le \alpha\psi(\delta(z, \mathcal{T}z)), \tag{2.23}$$

which implies that $\delta(z, \mathcal{T}z) = 0$, or that $\{z\} = \mathcal{T}z$. Moreover, z is a end-point of $\mathcal{T}$. □

Taking ψ an identity function in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \to B(\mathcal{X})$ be such that the following conditions are satisfied:*

(i) *there exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_1 \mathcal{T}x_0$,*

(ii) *for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \prec_1 \mathcal{T}y$,*

(iii)

$$\delta(\mathcal{T}x, \mathcal{T}y) \le \alpha M(x, y) + L\min\{D(x, \mathcal{T}x), D(y, \mathcal{T}y), D(x, \mathcal{T}y), D(y, \mathcal{T}x)\}, \tag{2.24}$$

for all comparable $x, y \in \mathcal{X}$, where $L \ge 0$, $0 < \alpha < 1$ and

$$M(x, y) = \max\left\{d(x, y), D(x, \mathcal{T}x), D(y, \mathcal{T}y), \frac{D(x, \mathcal{T}y) + D(y, \mathcal{T}x)}{2}\right\}. \tag{2.25}$$

If the property

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \quad \text{then } x_n \preceq z \ \forall n \tag{2.26}$$

holds, then $\mathcal{T}$ has a end-point.

The following corollary is a special case of Theorem 2.1 when $\mathcal{T}$ is a single-valued mapping.

Corollary 2.3. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be such that the following conditions are satisfied:*

(i) *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq \mathcal{T}x_0$,*

(ii) *for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$,*

(iii)

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) \leq \alpha\psi(M(x, y)) + L\min\{d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}, \tag{2.27}$$

for all comparable $x, y \in \mathcal{X}$, where $L \geq 0$, $0 < \alpha < 1$ and ψ is an altering distance function and

$$M(x, y) = \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2}\right\}. \tag{2.28}$$

If the property (2.3) holds, then $\mathcal{T}$ has a fixed-point.

In the following theorem we replace condition (2.3) of the above corollary by requiring $\mathcal{T}$ to be continuous.

Theorem 2.4. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous mapping such that the following conditions are satisfied:*

(i) *there exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_1 \mathcal{T}x_0$,*

(ii) *for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \prec_1 \mathcal{T}y$,*

(iii)

$$\psi(d(\mathcal{T}x, \mathcal{T}y)) \leq \alpha\psi(M(x, y)) + L\min\{d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}, \tag{2.29}$$

for all comparable $x, y \in \mathcal{X}$, where $L \geq 0$, $0 < \alpha < 1$ and ψ is an altering distance function and

$$M(x, y) = \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2}\right\}. \tag{2.30}$$

Then $\mathcal{T}$ has a end-point.

Proof. If we assume $\mathcal{T}$ as a multivalued mapping in which $\mathcal{T}x$ is a singleton set for every $x \in \mathcal{X}$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. Follows the line of proof of Theorem 2.1, we have that $\{x_n\}$ is a Cauchy sequence and

$$\lim_{n \to \infty} x_n = z. \tag{2.31}$$

Then, the continuity of $\mathcal{T}$ implies that

$$z = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \mathcal{T} x_n = \mathcal{T} z \tag{2.32}$$

and this proves that z is a end-point of $\mathcal{T}$. □

3. Common End-Point Theorems for a Pair of Multivalued Mappings

In this section, we prove common end-point theorems for a pair of $\mathcal{T}$-weakly isotone increasing multivalued mappings.

To complete the result, we need notion of $\mathcal{T}$-weakly isotone increasing for multivalued mappings given by Vetro [31, Definition 4.2].

Definition 3.1. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to B(\mathcal{X})$ be two maps. The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if $\mathcal{S}x \preceq_2 \mathcal{T}y \preceq_2 \mathcal{S}z$ for all any $x \in \mathcal{X}$, $y \in \mathcal{S}x$ and $z \in \mathcal{T}y$.

Note that, in particular, for single-valued mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$, mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if [31, Definition 2.2] if for each $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x \preceq \mathcal{S}\mathcal{T}\mathcal{S}x$.

Theorem 3.2. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to B(\mathcal{X})$ be such that*

$$\psi(\delta(\mathcal{T}x, \mathcal{S}y)) \le \alpha\psi(M(x,y)) + L\min\{D(x,\mathcal{T}x), D(y,\mathcal{S}y), D(x,\mathcal{S}y), D(y,\mathcal{T}x)\}, \tag{3.1}$$

for all comparable $x, y \in \mathcal{X}$, where $L \ge 0$, $0 < \alpha < 1$ and ψ is an altering distance function and

$$M(x,y) = \max\left\{d(x,y), D(x,\mathcal{T}x), D(y,\mathcal{S}y), \frac{D(x,\mathcal{S}y) + D(y,\mathcal{T}x)}{2}\right\}. \tag{3.2}$$

Also suppose that $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing and there exists an $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_2 \mathcal{S}x_0$. If the property

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \quad \text{then } x_n \preceq z \ \forall n \tag{3.3}$$

holds, then $\mathcal{S}$ and $\mathcal{T}$ have a common end-point.

Proof. Define a sequence $\{x_n\} \subset \mathcal{X}$ and prove that the limit point of that sequence is a unique common end-point for $\mathcal{T}$ and $\mathcal{S}$. For a given $x_0 \in \mathcal{X}$ and nonnegative integer n let

$$x_0 = x, \quad x_{2n+1} \in \mathcal{S}x_{2n}, \ x_{2n+2} \in \mathcal{T}x_{2n+1} \quad \text{for } n \ge 0. \tag{3.4}$$

If $x_{n_0} \in \mathcal{S}x_{n_0}$ or $x_{n_0} \in \mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \ne x_{n+1}$ for all n.

Since $\{x_0\} \preceq_2 \mathcal{S}x_0$, $x_1 \in \mathcal{S}x_0$ can be chosen so that $x_0 \preceq x_1$. Since $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing, it is $\mathcal{S}x_0 \preceq_2 \mathcal{T}x_1$; in particular, $x_2 \in \mathcal{T}x_1$ can be chosen so that $x_1 \preceq x_2$. Now, $\mathcal{T}x_1 \preceq_2 \mathcal{S}x_2$ (since $x_2 \in \mathcal{T}x_1$); in particular, $x_3 \in \mathcal{S}x_2$ can be chosen so that $x_2 \preceq x_3$.

Continuing this process we construct a monotone increasing sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots . \tag{3.5}$$

If $x_{n_0} \in \mathcal{S}x_{n_0}$ or $x_{n_0} \in \mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \neq x_{n+1}$ for all n.

Suppose that n is an odd number. Substituting $x = x_n$ and $y = x_{n+1}$ in (3.1) and using properties of function ψ, we have for all $n \geq 0$,

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(\delta(\mathcal{T}x_n, \mathcal{S}x_{n+1})) \\
&\leq \alpha\psi\Bigg(\max\Bigg\{ d(x_n, x_{n+1}), D(x_n, \mathcal{T}x_n), D(x_{n+1}, \mathcal{S}x_{n+1}), \\
&\qquad \frac{D(x_n, \mathcal{S}x_{n+1}) + D(x_{n+1}, \mathcal{T}x_n)}{2} \Bigg\}\Bigg) \\
&\quad + L\min\{D(x_n, \mathcal{T}x_n), D(x_{n+1}, \mathcal{S}x_{n+1}), D(x_n, \mathcal{S}x_{n+1}), D(x_{n+1}, \mathcal{T}x_n)\} \\
&\leq \alpha\psi\Bigg(\max\Bigg\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2} \Bigg\}\Bigg) \\
&\quad + L\min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\}.
\end{aligned} \tag{3.6}$$

Since $d(x_n,\ x_{n+2})/2 \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$\psi(d(x_{n+1}, x_{n+2})) \leq \alpha\psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{3.7}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, for some positive integer n.

Then from (3.7), we have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \alpha\psi(d(x_{n+1}, x_{n+2})), \tag{3.8}$$

it implies that $d(x_{n+1},\ x_{n+2}) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each n and so we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{3.9}$$

In the similar fashion, we can also show inequalities (3.9) when n is an even number. Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geq 0$ such that

$$d(x_n, x_{n+1}) \longrightarrow r \quad \text{as } n \longrightarrow \infty. \tag{3.10}$$

Taking the limit as $n \to \infty$ in (3.7) and using the continuity of ψ, we have $\psi(r) \leq \alpha\psi(r)$, which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.11}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$.

Assuming that $n(k)$ is the smallest such positive integer, we get $n(k) > m(k) > k$,

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon, \qquad d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon. \tag{3.12}$$

Now,

$$\epsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right), \tag{3.13}$$

that is,

$$\epsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) < \epsilon + d\left(x_{n(k)-1}, x_{n(k)}\right). \tag{3.14}$$

Taking the limit as $k \to \infty$ in the above inequality and using (3.11), we have

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon. \tag{3.15}$$

Again,

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right), \\ d\left(x_{m(k)+1}, x_{n(k)+1}\right) &\leq d\left(x_{m(k)+1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right). \end{aligned} \tag{3.16}$$

Taking the limit as $k \to \infty$ in the above inequalities and using (3.11) and (3.15), we have

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \epsilon. \tag{3.17}$$

Again,

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right), \\ d\left(x_{m(k)}, x_{n(k)+1}\right) &\leq d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right). \end{aligned} \tag{3.18}$$

Letting $k \longrightarrow \infty$ in the above inequalities and using (2.9) and (3.15), we have

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \epsilon. \tag{3.19}$$

Similarly, we have that

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)+1}\right) = \epsilon. \tag{3.20}$$

For each positive integer k, $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and the condition (3.1), we have

$$\begin{aligned}
\psi\left(d\left(x_{m(k)+1}, x_{n(k)+2}\right)\right) &\leq \psi\left(\delta\left(\mathcal{T}x_{m(k)}, \mathcal{S}x_{n(k)+1}\right)\right) \\
&\leq \alpha\psi\Bigg(\max\Bigg\{d\left(x_{m(k)}, x_{n(k)+1}\right), D\left(x_{m(k)}, \mathcal{T}x_{m(k)}\right), D\left(x_{n(k)+1}, \mathcal{S}x_{n(k)+1}\right), \\
&\qquad \frac{D\left(x_{m(k)}, \mathcal{S}x_{n(k)+1}\right) + D\left(x_{n(k)+1}, \mathcal{T}x_{m(k)}\right)}{2}\Bigg\}\Bigg) \\
&\quad + L\min\left\{D\left(x_{m(k)}, \mathcal{T}x_{m(k)}\right), D\left(x_{n(k)+1}, \mathcal{S}x_{n(k)+1}\right), D\left(x_{m(k)}, \mathcal{S}x_{n(k)+1}\right),\right. \\
&\qquad \left. D\left(x_{n(k)+1}, \mathcal{T}x_{m(k)}\right)\right\} \\
&= \alpha\psi\Bigg(\max\Bigg\{d\left(x_{m(k)}, x_{n(k)+1}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)+1}, x_{n(k)+1}\right), \\
&\qquad \frac{d\left(x_{m(k)}, x_{n(k)+2}\right) + d\left(x_{n(k)+1}, x_{m(k)+1}\right)}{2}\Bigg\}\Bigg) \\
&\quad + L\min\left\{d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)+1}, x_{n(k)+2}\right), d\left(x_{m(k)}, x_{n(k)+2}\right),\right. \\
&\qquad \left. d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right\}.
\end{aligned} \tag{3.21}$$

Letting $k \longrightarrow \infty$ in above inequality, using (3.11), (3.15), (3.17), (3.19), and (3.20) and using the continuity of ψ, we have

$$\psi(\epsilon) \leq \alpha\psi(\epsilon), \tag{3.22}$$

which is a contradiction by virtue of a property of ψ.

Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that

$$x_n \longrightarrow z \quad \text{as } n \longrightarrow \infty. \tag{3.23}$$

By the assumption (3.3), $x_n \preceq z$, for all n.

Then by the monotone property of ψ and the condition (3.1), we have

$$\begin{aligned}
\psi(\delta(x_{n+1},\mathcal{S}z)) &\le \psi(\delta(\mathcal{T}x_n,\mathcal{S}z))\\
&\le \alpha\psi\left(\max\left\{d(x_n,z),D(x_n,\mathcal{T}x_n),D(z,\mathcal{S}z),\frac{D(x_n,\mathcal{S}z)+D(z,\mathcal{T}x_n)}{2}\right\}\right)\\
&\quad + L\min\{D(x_n,\mathcal{T}x_n),D(z,\mathcal{S}z),D(x_n,\mathcal{S}z),D(z,\mathcal{T}x_n)\}\\
&\le \alpha\psi\left(\max\left\{d(x_n,z),d(x_n,x_{n+1}),D(z,\mathcal{S}z),\frac{D(x_n,\mathcal{S}z)+d(z,x_{n+1})}{2}\right\}\right)\\
&\quad + L\min\{d(x_n,x_{n+1}),D(z,\mathcal{S}z),D(x_n,\mathcal{S}z),d(z,x_{n+1})\}.
\end{aligned} \tag{3.24}$$

Taking the limit as $n\to\infty$ in the above inequality, using (3.11) and (3.23) and the continuity of ψ, we have

$$\psi(\delta(z,\mathcal{S}z)) \le \alpha\psi(D(z,\mathcal{S}z)) \le \alpha\psi(\delta(z,\mathcal{S}z)), \tag{3.25}$$

it implies that $\delta(z,\mathcal{S}z)=0$, or that $\{z\}=\mathcal{S}z$. Similarly $\{z\}=\mathcal{T}z$. Moreover, z is a common end-point of $\mathcal{T}$ and $\mathcal{S}$. $\square$

Putting $\mathcal{S}=\mathcal{T}$ in Theorem 3.2, we immediately obtain the following result.

Corollary 3.3. *Let $(\mathcal{X},d,\preceq)$ be an ordered complete metric space. Let $\mathcal{T}:\mathcal{X}\to B(\mathcal{X})$ be such that*

$$\psi(\delta(\mathcal{T}x,\mathcal{T}y)) \le \alpha\psi(M(x,y)) + L\min\{D(x,\mathcal{T}x),D(y,\mathcal{T}y),D(x,\mathcal{T}y),D(y,\mathcal{T}x)\}, \tag{3.26}$$

for all comparable $x,y\in\mathcal{X}$, where $L\ge 0$, $0<\alpha<1$ and ψ is an altering distance function and

$$M(x,y)=\max\left\{d(x,y),D(x,\mathcal{T}x),D(y,\mathcal{T}y),\frac{D(x,\mathcal{T}y)+D(y,\mathcal{T}x)}{2}\right\}. \tag{3.27}$$

Also suppose that $\mathcal{T}x\preceq_1\mathcal{T}(\mathcal{T}x)$ for all $x\in\mathcal{X}$ and there is $x_0\in\mathcal{X}$ such that $\{x_0\}\prec_1\mathcal{T}x_0$. If the property

$$\{x_n\}\subset\mathcal{X} \text{ is a nondecreasing sequence with } x_n\longrightarrow z \text{ in } \mathcal{X}, \quad \text{then } x_n\prec z\ \forall n \tag{3.28}$$

holds, then $\mathcal{T}$ has a end-point.

In Theorem 3.2, if $\mathcal{T},\mathcal{S}$ are single valued mappings, then we have the following result.

Theorem 3.4. *Let $(\mathcal{X},d,\preceq)$ be an ordered complete metric space. Let $\mathcal{T},\mathcal{S}:\mathcal{X}\to\mathcal{X}$ be such that*

$$\psi(d(\mathcal{T}x,\mathcal{S}y)) \le \alpha\psi(M(x,y)) + L\min\{d(x,\mathcal{T}x),d(y,\mathcal{S}y),d(x,\mathcal{S}y),d(y,\mathcal{T}x)\}, \tag{3.29}$$

for all comparable $x, y \in \mathcal{X}$, where $L \geq 0$, $0 < \alpha < 1$ and φ is an altering distance function and

$$M(x,y) = \max\left\{ d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{S}y), \frac{d(x,\mathcal{S}y) + d(y,\mathcal{T}x)}{2} \right\}. \tag{3.30}$$

Also suppose that $\mathcal{S}$ and $\mathcal{T}$ are weakly isotone increasing. If

$$\mathcal{S} \text{ is continuous} \tag{3.31}$$

or

$$\mathcal{T} \text{ is continuous} \tag{3.32}$$

or

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \quad \text{then } x_n \preceq z \ \forall n \tag{3.33}$$

holds, then $\mathcal{S}$ and $\mathcal{T}$ have a common end-point.

Proof. If we assume $\mathcal{T}$ and $\mathcal{S}$ as a multivalued mapping in which $\mathcal{T}x$ and $\mathcal{S}x$ are a singleton set for every $x \in \mathcal{X}$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 3.4. Follows the line of proof of Theorem 3.4, we have that $\{x_n\}$ is a Cauchy sequence and

$$\lim_{n \to \infty} x_n = z. \tag{3.34}$$

Then, if $\mathcal{T}$ is continuous, we have

$$z = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \mathcal{T}x_n = \mathcal{T}z \tag{3.35}$$

and this proves that z is a end-point of $\mathcal{T}$ and so z is a end-point of $\mathcal{S}$. Similarly, if $\mathcal{S}$ is continuous, we have the result. Thus it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common end-point. □

4. Common End-Point Theorems for a Pair of Multivalued Mappings Satisfying Weakly Contractive Condition

In this section, we prove common end-point theorems for a pair of weakly isotone increasing multivalued mappings under weakly contractive condition.

To complete the result, we need notion of weakly contractive condition given by Rhoades [32].

Definition 4.1 (Weakly Contractive Mapping). Let $\mathcal{X}$ be a metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called weakly contractive if and only if

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x,y) - \varphi(d(x,y)), \quad \forall x, y \in \mathcal{X}, \tag{4.1}$$

where φ is an altering distance function.

Theorem 4.2. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to B(\mathcal{X})$ be such that*

$$\begin{aligned} \psi(\delta(\mathcal{T}x, \mathcal{S}y)) \leq \psi\left(\max\left\{d(x,y), D(x,\mathcal{T}x), D(y,\mathcal{S}y), \frac{D(x,\mathcal{S}y) + D(y,\mathcal{T}x)}{2}\right\}\right) \\ - \phi(\max\{d(x,y), \delta(y,\mathcal{S}y)\}), \end{aligned} \tag{4.2}$$

for all comparable $x, y \in \mathcal{X}$, where $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are an altering distance functions.

Also suppose that $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing and there exists an $x_0 \in \mathcal{X}$ such that $\{x_0\} \prec_2 \mathcal{S}x_0$. If the property

$$\{x_n\} \subset \mathcal{X} \text{ is a nondecreasing sequence with } x_n \longrightarrow z \text{ in } \mathcal{X}, \quad \text{then } x_n \preceq z \ \forall n \tag{4.3}$$

holds, then $\mathcal{S}$ and $\mathcal{T}$ have a common end-point.

Proof. Define a sequence $\{x_n\} \subset \mathcal{X}$ and prove that the limit point of that sequence is a unique common end-point for $\mathcal{T}$ and $\mathcal{S}$. For a given $x_0 \in \mathcal{X}$ and nonnegative integer n let

$$x_0 = x, \quad x_{2n+1} \in \mathcal{S}x_{2n}, x_{2n+2} \in \mathcal{T}x_{2n+1} \quad \text{for } n \geq 0. \tag{4.4}$$

Since $\{x_0\} \preceq_2 \mathcal{S}x_0$, $x_1 \in \mathcal{S}x_0$ can be chosen so that $x_0 \preceq x_1$. Since $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing, it is $\mathcal{S}x_0 \preceq_2 \mathcal{T}x_1$; in particular, $x_2 \in \mathcal{T}x_1$ can be chosen so that $x_1 \preceq x_2$. Now, $\mathcal{T}x_1 \preceq_2 \mathcal{S}x_2$ (since $x_2 \in \mathcal{T}x_1$); in particular, $x_3 \in \mathcal{S}x_2$ can be chosen so that $x_2 \preceq x_3$.

Continuing this process, we conclude that $\{x_n\}$ can be an increasing sequence in $\mathcal{X}$:

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{4.5}$$

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a common end-point of $\mathcal{T}$ and $\mathcal{S}$. Hence we will assume that $x_n \neq x_{n+1}$, for all $n \geq 0$.

Suppose that n is an odd number. Substituting $x = x_n$ and $y = x_{n+1}$ in (2.6) and using properties of functions ψ and ϕ, we have for all $n \geq 0$,

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &\leq \psi(\delta(\mathcal{T}x_n, \mathcal{S}x_{n+1})) \\ &\leq \psi\Bigg(\max\Bigg\{d(x_n, x_{n+1}), D(x_n, \mathcal{T}x_n), D(x_{n+1}, \mathcal{S}x_{n+1}), \\ &\qquad \frac{D(x_n, \mathcal{S}x_{n+1}) + D(x_{n+1}, \mathcal{T}x_n)}{2}\Bigg\}\Bigg) \\ &\quad - \phi(\max\{d(x_n, x_{n+1}), \delta(x_{n+1}, \mathcal{S}x_{n+1})\}) \\ &\leq \psi\Bigg(\max\Bigg\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\qquad \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\Bigg\}\Bigg) \\ &\quad - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \end{aligned} \tag{4.6}$$

Since $d(x_n, x_{n+2})/2 \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$, it follows that

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) - \phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{4.7}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, for some positive integer n. Then from (4.7), we have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_{n+1}, x_{n+2})) - \phi(d(x_{n+1}, x_{n+2})), \tag{4.8}$$

that is, $\phi(d(x_{n+1}, x_{n+2})) \leq 0$, which implies that $d(x_{n+1}, x_{n+2}) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that $x_n \neq x_{n+1}$. So we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{4.9}$$

In the similar fashion, we can also show inequalities (4.9) when n is an even number. Therefore, for all $n \geq 0$ and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r. \tag{4.10}$$

In view of the above facts, from (4.7) we have for all $n \geq 0$,

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})). \tag{4.11}$$

Taking the limit as $n \to \infty$ in the above inequality, using (4.10) and the continuities of ϕ and ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r), \tag{4.12}$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{4.13}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then using an argument similar to that given in Theorem 3.2, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ for which

$$\begin{aligned} &\lim_{k \to \infty} d\big(x_{m(k)}, x_{n(k)}\big) = \epsilon, \qquad \lim_{k \to \infty} d\big(x_{m(k)+1}, x_{n(k)+1}\big) = \epsilon, \\ &\lim_{k \to \infty} d\big(x_{m(k)}, x_{n(k)+1}\big) = \epsilon, \qquad \lim_{k \to \infty} d\big(x_{n(k)}, x_{m(k)+1}\big) = \epsilon. \end{aligned} \tag{4.14}$$

For each positive integer k, $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and (4.2), we have

$$\begin{aligned}
\psi\left(d\left(x_{m(k)+1}, x_{n(k)+2}\right)\right) &\leq \psi\left(\delta\left(\mathcal{T}x_{m(k)}, \mathcal{S}x_{n(k)+1}\right)\right) \\
&\leq \psi\Bigg(\max\Bigg\{ d\left(x_{m(k)}, x_{n(k)+1}\right), D\left(x_{m(k)}, \mathcal{T}x_{m(k)}\right), D\left(x_{n(k)+1}, \mathcal{S}x_{n(k)+1}\right), \\
&\qquad \frac{D\left(x_{m(k)}, \mathcal{S}x_{n(k)+1}\right) + D\left(x_{n(k)+1}, \mathcal{T}x_{m(k)}\right)}{2} \Bigg\}\Bigg) \\
&\quad - \phi\left(\max\left\{d\left(x_{m(k)}, x_{n(k)+1}\right), \delta\left(x_{n(k)+1}, \mathcal{S}x_{n(k)+1}\right)\right\}\right), \\
&= \psi\Bigg(\max\Bigg\{ d\left(x_{m(k)}, x_{n(k)+1}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)+1}, x_{n(k)+2}\right), \\
&\qquad \frac{d\left(x_{m(k)}, x_{n(k)+2}\right) + d\left(x_{n(k)+1}, Tx_{m(k)+1}\right)}{2} \Bigg\}\Bigg) \\
&\quad - \phi\left(\max\left\{d\left(x_{m(k)}, x_{n(k)+1}\right), d\left(x_{n(k)+1}, x_{n(k)+2}\right)\right\}\right).
\end{aligned} \tag{4.15}$$

Letting $k \to \infty$ in the above inequality, using (4.14) and the continuities of ψ and ϕ, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon), \tag{4.16}$$

which is a contradiction by virtue of a property of ϕ. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that

$$x_n \longrightarrow z \quad \text{as } n \longrightarrow \infty. \tag{4.17}$$

By the condition (4.3), $x_n \preceq z$, for all n. Then by the monotone property of ψ and (4.2), we have

$$\begin{aligned}
\psi(\delta(x_{n+1}, \mathcal{S}z)) &\leq \psi(\delta(\mathcal{T}x_n, \mathcal{S}z)) \\
&\leq \psi\left(\max\left\{ d(x_n, z), D(x_n, \mathcal{T}x_n), D(z, \mathcal{S}z), \frac{D(x_n, \mathcal{S}z) + D(z, \mathcal{T}x_n)}{2} \right\}\right) \\
&\quad - \phi(\max\{d(x_n, z), \delta(z, \mathcal{S}z)\}) \\
&\leq \psi\left(\max\left\{ d(x_n, z), d(x_n, x_{n+1}), D(z, \mathcal{S}z), \frac{D(x_n, \mathcal{S}z) + d(z, x_{n+1})}{2} \right\}\right) \\
&\quad - \phi(\max\{d(x_n, z), \delta(z, \mathcal{S}z)\}).
\end{aligned} \tag{4.18}$$

Taking the limit as $n \to \infty$ in the above inequality, using (4.13), (4.17) and the continuities of ψ and ϕ, we have

$$\psi(\delta(z, \mathcal{S}z)) \leq \psi(D(z, \mathcal{S}z)) - \phi(\delta(z, \mathcal{S}z)), \tag{4.19}$$

which implies that

$$\psi(\delta(z, \mathcal{S}z)) \leq \psi(\delta(z, \mathcal{S}z)) - \phi(\delta(z, \mathcal{S}z)), \tag{4.20}$$

which is a contradiction unless $\delta(z, \mathcal{S}z) = 0$, or that $\{z\} = \mathcal{S}z$; that is, z is a end-point of $\mathcal{S}$. Similarly $\{z\} = \mathcal{T}z$. Moreover, z is a common end-point of $\mathcal{T}$ and $\mathcal{S}$. □

Similar corollaries can be derived from Theorem 4.2.

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