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Research Article

The Multiple Gamma-Functions and the Log-Gamma Integrals

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In this paper, which is a companion paper to [W], starting from the Euler integral which appears in a generalization of Jensen's formula, we shall give a closed form for the integral of $\log\Gamma(1\pm t)$. This enables us to locate the genesis of two new functions $A_{1/a}$ and $C_{1/a}$ considered by Srivastava and Choi. We consider the closely related function *A(a)* and the Hurwitz zeta function, which render the task easier than working with the $A_{1/a}$ functions themselves. We shall also give a direct proof of Theorem 4.1, which is a consequence of [CKK, Corollary 1.1], though.

1. Introduction

If $f(z)$ is analytic in a domain D containing the circle $C : |z| = r$ and has no zero on the circle, then the Gauss mean value theorem

$$\log\left|f(0)\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| \, d\theta \tag{1.1}$$

is true. In [1, page 207] the case is considered where $f(z)$ has a zero $re^{i\theta_0}$ on the circle, and (1.1) turns out that the Euler integral

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log 2 \tag{1.2}$$

which is essential in proving a generalization of Jensen's formula [1, pages 207-208].

Let G denote the Catalan constant defined by the absolutely convergent series

$$G = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = L(2, \chi_4), \tag{1.3}$$

where χ_4 is the nonprincipal Dirichlet character mod 4.

As a next step from (1.2) the relation

$$\int_0^{\pi/4} \log \sin t \, dt = -\frac{\pi}{4} \log 2 - \frac{1}{2} G \tag{1.4}$$

holds true. In this connection, in [2] we obtained some results on G viewing it as an intrinsic value to the Barnes G-function. The Barnes G-function (which is Γ_2^{-1} in the class of multiple gamma functions) is defined as the solution to the difference equation (cf. (2.3))

$$\log G(z+1) - \log G(z) = \log \Gamma(z) \tag{1.5}$$

with the initial condition

$$\log G(1) = 0 \tag{1.6}$$

and the asymptotic formula to be satisfied

$$\begin{aligned} \log G(z+N+2) = {} & \frac{N+1+z}{2} \log 2\pi \\ & + \frac{1}{2}\left(N^2 + 2N + 1 + B_2 + z^2 + 2(N+1)z\right) \log N \\ & - \frac{3}{4} N^2 - N - Nz - \log A + \frac{1}{12} + O\left(N^{-1}\right), \end{aligned} \tag{1.7}$$

$N \to \infty$, where $\Gamma(s)$ indicates the Euler gamma function (cf., e.g., [3]).

Invoking the reciprocity relation for the gamma function

$$\Gamma(s) \sin \pi s = \frac{\pi}{\Gamma(1-s)}, \tag{1.8}$$

it is natural to consider the integrals of $\log \Gamma(\alpha + t)$ or of multiple gamma functions Γ_r (cf., e.g., [4, 5]). Barnes' theorem [6, page 283] reads

$$\begin{aligned} \int_0^a \log \Gamma(\alpha + t) dt = {} & -\log \frac{G(\alpha + a)}{G(\alpha)} - (1-\alpha) \log \frac{\Gamma(\alpha + a)}{\Gamma(\alpha)} \\ & + a \log \Gamma(\alpha + a) - \frac{1}{2} a^2 + \frac{1}{2}(\log 2\pi + 1 - 2\alpha) a \end{aligned} \tag{1.9}$$

valid for nonintegral values of a.

In this paper, motivated by the above, we proceed in another direction to developing some generalizations of the above integrals considered by Srivastava and Choi [7]. For q-analogues of the results, compare the recent book of the same authors [8]. Our main result is Theorem 2.1 which gives a closed form for $\int_0^a \log\Gamma(1-t)\,\mathrm{d}t$ and locates its genesis. A slight modification of Theorem 2.1 gives the counterpart of Barnes' formula (1.9) which reads.

Corollary 1.1. *Except for integral values of a, one has*

$$\begin{aligned}\int_0^a \log\Gamma(\alpha-t)\mathrm{d}t &= \log\frac{G(\alpha-a)}{G(\alpha)} + (1-\alpha)\log\frac{\Gamma(\alpha-a)}{\Gamma(\alpha)} \\ &\quad + a\log\Gamma(\alpha-a) + \frac{1}{2}a^2 + \frac{1}{2}(\log 2\pi + 1 - 2\alpha)a.\end{aligned} \tag{1.10}$$

Srivastava and Choi introduced two functions $\log A_{1/a}$ and $\log C_{1/a}$ by (2.9) and (2.9) with formal replacement of $1/a$ by $-1/a$, respectively. They state $C_{1/a} = A_{-1/a}$, which is rather ambiguous as to how we interpret the meaning because (2.9) is defined for $a > 0$ [7, page 347, l.11]. They use this $C_{1/a}$ function to express the integral $\int_0^a \log\Gamma(1-t)\mathrm{d}t$, without giving proof. This being the case, it may be of interest to locate the integral of $\log\Gamma(1-t)$ [7, (13), page 349], thereby $\log C_{1/a}$ [7, page 347].

For this purpose we use a more fundamental function $A(a)$ than $A_{1/a}$ defined by

$$\log A(a) = -\zeta'(-1,a) + \frac{1}{12}, \tag{1.11}$$

where $\zeta(s,a)$ is the Hurwitz zeta-function

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \quad \operatorname{Re} s = \sigma > 1 \tag{1.12}$$

in the first instance. For its theory, compare, for instance, [3], [9, Chapter 3].

We shall prove the following corollary which gives the right interpretation of the function $C_{1/a}$.

Corollary 1.2. *For $0 < a < 1$,*

$$\log C_{1/a} = \log A(1-a) - \frac{1}{4}a^2, \tag{1.13}$$

or

$$\log C_{1/a} = \log A_{1-1/a} + \frac{1}{4}(1-a)^2 + (1-a)\log(1-a) - \frac{1}{4}a^2. \tag{1.14}$$

2. Barnes Formula

There is a generalization of (1.4) as well as (1.2) in the form [7, equation (28), page 31]:

$$\int_0^a \log \sin \pi t \, dt = a \log \frac{\sin \pi a}{2\pi} + \log \frac{G(1+a)}{G(1-a)}, \quad a \notin \mathbb{Z}. \tag{2.1}$$

Equation (2.1) is Barnes' formula [6, page 279] which is equivalent to Kinkelin's 1860 result [10] [7, equation (26), page 30]:

$$\int_0^z \pi t \cot \pi t \, dt = \log \frac{G(1-z)}{G(1+z)} + z \log 2\pi. \tag{2.2}$$

Since (1.5) is equivalent to

$$G(z+1) = G(z)\Gamma(z), \tag{2.3}$$

it follows that

$$\int_0^a \log \sin \pi t \, dt = a \log \frac{\sin \pi a}{2\pi} + \log \frac{G(a)}{G(1-a)} + \log \Gamma(a). \tag{2.4}$$

Putting $a = 1/2$, we obtain

$$\pi^{-1} \int_0^{\pi/2} \log \sin x \, dx = \int_0^{1/2} \log \sin \pi t \, dt = -\frac{1}{2} \log 2\pi + \log \Gamma\left(\frac{1}{2}\right) = -\frac{1}{2} \log 2, \tag{2.5}$$

which is (1.2).

The counterpart of (2.1) follows from the reciprocity relation (1.8), known as Alexeievsky's Theorem [7, equation (42), page 32].

$$\int_0^a \log \Gamma(1+t) \, dt = \frac{1}{2}(\log 2\pi - 1)a - \frac{a^2}{2} + a \log \Gamma(a+1) - \log G(a+1), \tag{2.6}$$

which in turn is a special case of (1.9).

Indeed, in [7, page 207], only (1.9) and the integral of $\log G(t+\alpha)$ are in closed form and the integral of $\log \Gamma_3(t+\alpha)$ is not. A general formula is given by Barnes [4] with constants to be worked out. We shall state a concrete form for this integral in Section 3, using the relation [7, equation (455), page 210] between $\log \Gamma_3(t+\alpha)$ and the integral of ψ and appealing to a closed form for the latter in [11].

Formula (2.6) is stated in the following form [7, equation (12), page 349]:

$$\int_0^a \log \Gamma(1+t) \, dt = \frac{1}{2}(\log 2\pi - 1)a - \frac{3}{4}a^2 + \log A - \log A_{1/a}, \tag{2.7}$$

where $\log A$ is the Glaisher-Kinkelin constant defined by [7, equation (2), page 25]

$$\log A = \lim_{N\to\infty}\left(\sum_{n=1}^{N} n\log n - \frac{1}{2}\left(N^2+N+B_2\right)\log N + \frac{1}{4}N^2\right), \tag{2.8}$$

and $\log A_{1/a}$ is defined by [7, equation (9), page 347]

$$\begin{aligned}\log A_{1/a} = \lim_{N\to\infty}\Bigg(&\sum_{n=1}^{N}(n+a)\log(n+a)\\ &-\frac{1}{2}\left(N^2+(2a+1)N+a^2+a+B_2\right)\log(N+a)+\frac{1}{4}N^2+\frac{a}{2}N\Bigg),\end{aligned} \tag{2.9}$$

for $a > 0$.

Comparing (2.6) and (2.7), we immediately obtain

$$\begin{aligned}\log A_{1/a} &= \log G(a+1) - a\log\Gamma(a+1) + \log A - \frac{a^2}{4}\\ &= \log G(a) + (1-a)\log\Gamma(a) + \log a - \frac{a^2}{4} - a\log a,\end{aligned} \tag{2.10}$$

on using the difference relation $\Gamma(a+1) = a\Gamma(a)$.

Thus, in a sense we have located the genesis of the function $\log A_{1/a}$. although they prove (2.7) by an elementary method [7, page 348].

Indeed, $A_{1/a}$ and $A(a)$ are almost the same:

$$\log A_{1/a} = \log A(a) - \frac{1}{4}a^2 - a\log a, \tag{2.11}$$

a proof being given below. However, $\log A(a)$ is more directly connected with $\zeta'(-1,a)$ for which we have rich resources of information as given in [9, Chapter 3].

We prove the following theorem which gives a closed form for $\int_0^a \log\Gamma(1-t)\,dt$, thereby giving the genesis of the constant $C_{1/a}$.

Theorem 2.1. *For $a \notin \mathbb{Z}$, one has*

$$\int_0^a \log\Gamma(1-t)\,dt = \log G(1-a) + a\log\Gamma(1-a) + \frac{1}{2}a^2 + \frac{1}{2}(\log 2\pi - 1)a. \tag{2.12}$$

If $0 < a < 1$, then

$$\int_0^a \log\Gamma(1-t)\,dt = \log A(1-a) - \log A + \frac{1}{2}a^2 + \frac{1}{2}(\log 2\pi - 1)a. \tag{2.13}$$

Proof. We evaluate the integral

$$I = \int_0^a \log\Gamma(1+t)\sin\pi t\,\mathrm{d}t \tag{2.14}$$

in two ways. First,

$$I = a\log\pi + a\log a - a - \int_0^a \log\Gamma(1-t)\mathrm{d}t. \tag{2.15}$$

On the other hand, noting that I is the sum of (2.1) and (2.7), we deduce that

$$\begin{aligned} I = {} & a\log\frac{\sin\pi a}{2\pi} + \log G(a+1) + \log A - \log G(1-a) \\ & + \frac{1}{2}(\log 2\pi - 1)a - \frac{3}{4}a^2 - \log A_{1/a}. \end{aligned} \tag{2.16}$$

Substituting (1.5), we obtain

$$\begin{aligned} I = {} & a\log\frac{\sin\pi a}{2\pi} + a\log\Gamma(a) + \log A(a) - \log A_{1/a} \\ & - \log G(1-a) + \frac{1}{2}(\log 2\pi - 1)a - \frac{3}{4}a^2. \end{aligned} \tag{2.17}$$

The first two terms on the right of (2.17) become

$$a\log\frac{\Gamma(a)\sin\pi a}{2\pi} = a\log\frac{1}{2}\Gamma(1-a) = -a\left(\log 2 + \log\Gamma(1-a)\right), \tag{2.18}$$

while the 3rd and the 4th terms give, in view of (2.11), $(1/4)a^2 + a\log a$.

Hence, altogether

$$I = -a\log 2 - a\log\Gamma(1-a) - \log G(1-a) + a\log a - \frac{1}{2}a^2 + \frac{1}{2}(\log 2\pi - 1)a. \tag{2.19}$$

Comparing (2.15) and (2.19) proves (2.12), completing the proof. □

Comparing (2.13) and [7, equation (13), page 349]

$$\int_0^a \log\Gamma(1-t)\mathrm{d}t = \log A(1-a) - \log A + \frac{3}{4}a^2 + \frac{1}{2}(\log 2\pi - 1)a, \tag{2.20}$$

we prove Corollary 1.2.

Hence the relation between $C_{1/a}$ and $A_{1/a}$ is (1.14), that is, one between $C_{1/a}$ and $A_{1-1/a}$ rather than $C_{1/a} = A_{-1/a}$ as Srivastava and Choi state.

At this point we shall dwell on the underlying integral representation for (the derivative of) the Hurwitz zeta-function, which makes the argument rather simple and lucid as in [12] and gives some consequences.

Proof of (2.11). Consider that

$$\zeta'(s,a)-\frac{1}{12}=-\frac{1}{2}a^2\log a-\frac{1}{4}a^2-\frac{1}{2}a\log a \\ -\frac{B_2}{2}\log a-\frac{1}{3!}\int_0^\infty \overline{B}_3(t)(t+a)^{-2}\mathrm{d}t \tag{2.21}$$

[9, (3.15), page 59], where the last integral may be also expressed as

$$-\frac{1}{2!}\int_0^\infty \overline{B}_2(t)(t+a)^{-1}\mathrm{d}t, \tag{2.22}$$

and where $\overline{B}_k(t)$ is the kth periodic Bernoulli polynomial. Then

$$-\zeta'(-1,a)=\sum_{0\le n\le x}(n+a)\log(n+a)-\frac{1}{2}(x+a)^2\log(x+a) \\ +\frac{1}{4}(x+a)^2+\overline{B}_1(x)(x+a)-\frac{1}{2}\overline{B}_2(x)(x+a)+O\left(x^{-1}\log x\right); \tag{2.23}$$

whence in particular, we have the generic formula for $\zeta'(-1,a)$ and consequently for $\log A(a)$ through (1.11):

$$\log A(a)=\lim_{N\to\infty}\left(\sum_{n=0}^{\infty}(n+a)\log(n+a)-\frac{1}{2}\log(N+a) \\ \times\left((N+a)^2+N+a+B_2\right)+\frac{1}{4}(N+a)^2\right). \tag{2.24}$$

This may be slightly modified in the form

$$\log A(a)=\lim_{N\to\infty}\left(\sum_{n=0}^{N}(n+a)\log(n+a) \\ -\frac{1}{2}\left(N^2+(2a+1)N+a^2+a+B_2\right)\log(N+a)+\frac{1}{4}N^2+\frac{1}{2}aN\right) \\ +\frac{1}{4}a^2+a\log a. \tag{2.25}$$

Comparing (2.9) and (2.25), we verify (2.11). □

The merit of using $A(a)$ is that by way of $\zeta'(-1,a)$, we have a closed form for it:

$$\log A(a) = \frac{1}{2}a^2\log a - \frac{1}{4}a^2 + \frac{1}{2}a\log a + \frac{B_2}{2}\log a + \frac{1}{2!}\int_0^\infty \overline{B}_2(t)(t+a)^{-1}dt. \tag{2.26}$$

In the same way, via another important relation [7, equation (23), page 94],

$$\log G(a) = -\left(\zeta'(-1,a) - \frac{1}{12}\right) - \log A - (1-a)\log\Gamma(a). \tag{2.27}$$

Equation (2.21) gives a closed form for $\log G(a)$, too. We also have from (1.11) and (2.27)

$$\begin{aligned}\log A(a) &= \log G(a) + (1-a)\log\Gamma(a) + \log A \\ &= \log G(a+1) - a\log\Gamma(a) + \log A.\end{aligned} \tag{2.28}$$

There are some known expressions not so handy as given by (2.27). For example, [7, page 25] and [7, equation (440), page 206], one of which reads

$$\frac{G'}{G}(1+z) = \sum_{n=1}^{\infty}\left(\frac{n}{z+n} - 1 + \frac{z}{n}\right) + \frac{1}{2}(\log 2\pi - 1) - (1+\gamma)z, \tag{2.29}$$

with γ designating the Euler constant. Equation (2.29) is a basis of (2.2) (cf. proof of [2, Lemma 1]).

Remark 2.2. The Glaisher-Kinkelin constant A is connected with $A(1)$ and A_1 as follows:

$$\log A = \log A(1) = \log A_1 + \frac{1}{4}. \tag{2.30}$$

This can also be seen from Vardi's formula [7, (31), page 97]:

$$\log A = -\zeta'(-1) + \frac{1}{12}, \tag{2.31}$$

which is (1.11) with $a = 1$.

We may also give another direct proof of Corollary 1.2.

Proof of Corollary 1.2 (another proof). $\log C_{1/a}$ is the limit of the expression

$$\begin{aligned}S_N = \sum_{k=1}^{N}(k-1+\alpha)\log(k-1+\alpha) - \left(\frac{1}{2}N^2 + \left(\alpha - \frac{1}{2}\right)N + \frac{1}{2}B_2(\alpha)\right) \\ \times\log(N-1+\alpha) + \frac{1}{4}N^2 + \frac{N}{2}(\alpha-1),\end{aligned} \tag{2.32}$$

where $\alpha = 1 - a$. Let $N = M + 1$. Then

$$S_N = \sum_{k=0}^{M}(k+\alpha)\log(k+\alpha) - \left(\frac{1}{2}(M+1)^2 + \left(\alpha - \frac{1}{2}\right)(M+1) + \frac{1}{2}B_2(\alpha)\right) \times \log(M+\alpha) + \frac{1}{4}(M+1)^2 + \frac{M+1}{2}\alpha - \frac{M+1}{2}. \tag{2.33}$$

Hence, simplifying, we find that

$$S_N = \sum_{k=1}^{M}(k+\alpha)\log(k+\alpha) - \left(\frac{1}{2}M^2 + \left(\alpha + \frac{1}{2}\right)M + \frac{1}{2}\left(\alpha^2 + \alpha + B_2\right)\right) \times \log(M+\alpha) + \frac{1}{4}M^2 + \frac{1}{2}\alpha M + \alpha\log\alpha - \frac{(\alpha-1)^2}{4} + \frac{1}{4}\alpha^2. \tag{2.34}$$

Hence

$$\log C_{1/a} = \log A_\alpha + \alpha\log\alpha - \frac{(\alpha-1)^2}{4} + \frac{1}{4}\alpha^2, \tag{2.35}$$

which is (1.14). This completes the proof. □

As an immediate consequence of Corollary 1.2, we prove (2.36) as can be found in [7, pages 350–351].

$$A_{1/a} = \left(\frac{\pi a}{\sin \pi a}\right)^{-a}\frac{G(1+a)}{G(1-a)}C_{1/a}, \quad 0 < a < 1. \tag{2.36}$$

Proof of (2.36). From (2.28), (1.5), and (1.8), we obtain

$$\log A(a) - \log A(1-a) = \log\frac{G(1+a)}{G(1-a)} - a\log\frac{\pi}{\sin \pi a}. \tag{2.37}$$

On the other hand, by (2.11) and (1.13), we see that the left-hand side of (2.37) is

$$\log\frac{A_{1/a}}{C_{1/a}} + a\log a, \tag{2.38}$$

whence we conclude that

$$\log\frac{A_{1/a}}{C_{1/a}} = \log\frac{G(1+a)}{G(1-a)} - a\log\frac{\pi a}{\sin \pi a}. \tag{2.39}$$

On exponentiating, (2.37) leads to (2.36). □

3. Polygamma Function of Negative Order

In this section we introduce the function $\widetilde{A}_k(q)$ [13]:

$$\widetilde{A}_k(q) = k\zeta'(1-k,q), \tag{3.1}$$

which is closely related to the polygamma function of negative order and states some simple applications. We recall some properties of $\widetilde{A}_k(q)$:

$$\begin{aligned} \widetilde{A}_2(q+1) &= \widetilde{A}_2(q) + 2q\log q, \\ \widetilde{A}_2\left(\frac{1}{2}\right) &= -\zeta'(-1) - \frac{1}{12}\log 2, \\ \widetilde{A}_2\left(\frac{1}{4}\right) &= -\frac{1}{4}\zeta'(-1) + \frac{G}{2\pi}, \end{aligned} \tag{3.2}$$

$$\widetilde{A}_2\left(\frac{3}{4}\right) = -\frac{1}{2}\zeta'(-1) - \widetilde{A}_2\left(\frac{1}{4}\right). \tag{3.3}$$

Equation (3.3) is [2, equation (2.31)], which is used in proving [2, Theorem 2] and can be read off from the distribution property [9, equation (3.72), page 76] as follows:

$$\sum_{a=1}^{4}\zeta\left(s,\frac{a}{4}\right) = 4^s\zeta(s). \tag{3.4}$$

Differentiation gives

$$\sum_{n=1}^{4}\zeta'\left(s,\frac{a}{4}\right) = 4^s\left((\log 4)\zeta(s) + \zeta'(s)\right). \tag{3.5}$$

Putting $s=-1$, we obtain

$$\zeta'(-1) + \zeta'\left(-1,\frac{1}{2}\right) + \zeta'\left(-1,\frac{1}{4}\right) + \zeta'\left(-1,\frac{3}{4}\right) = 4^{-1}\left((\log 4)\zeta(-1) + \zeta'(-1)\right), \tag{3.6}$$

which we solve in $\zeta'(-1,3/4)$:

$$\begin{aligned} \zeta'\left(-1,\frac{3}{4}\right) = {} & \frac{1}{4}\left((2\log 2)\zeta(-1) + \zeta'(-1)\right) \\ & - \zeta'(-1) - \frac{1}{2}\widetilde{A}_2\left(\frac{1}{2}\right) - \zeta'\left(-1,\frac{1}{4}\right). \end{aligned} \tag{3.7}$$

Substituting (3.2) and $\zeta(-1) = -B_2/2 = -1/12$ and simplifying, we conclude that

$$\zeta'\left(-1, \frac{3}{4}\right) = -\frac{1}{4}\zeta'(-1) - \zeta'\left(-1, \frac{1}{4}\right) \tag{3.8}$$

and that

$$\widetilde{A}_2\left(\frac{3}{4}\right) = 2\zeta'\left(-1, \frac{3}{4}\right) = -\frac{1}{2}\zeta'(-1) - 2\zeta'\left(-1, \frac{1}{4}\right), \tag{3.9}$$

whence (3.3).

Using these, we deduce from (2.37) the following.

Example 3.1.

$$\log A_{1/4} = \frac{5}{64} + \frac{1}{2}\log 2 - \frac{1}{8}\log A - \frac{G}{2\pi}. \tag{3.10}$$

Proof. By (1.11) and (3.1), for $q > 0$,

$$\log A(q) = -\frac{1}{2}\widetilde{A}_2(q) + \frac{1}{12}. \tag{3.11}$$

Since $\log A(1/4) - \log A(3/4) = -1/2(\widetilde{A}_2(1/4) - \widetilde{A}_2(3/4))$, it follows from (3.3) that the left-hand side of (2.37) is

$$-\widetilde{A}_2\left(\frac{1}{4}\right) - \frac{1}{4}\zeta'(-1), \tag{3.12}$$

which is

$$2\log A\left(\frac{1}{4}\right) - \frac{1}{6} + \frac{1}{4}\left(\log A - \frac{1}{12}\right) \tag{3.13}$$

where we used (2.31).

The right-hand side of (2.37), $\log(G(5/4)/G(3/4)) - 1/4\log(\pi/\sin(\pi/4))$, becomes $-(G/2\pi)$, in view of known values of G [7, page 30].

Hence, altogether, (2.37) with $a = 1/4$ reads

$$-\frac{G}{2\pi} = 2\log A\left(\frac{1}{4}\right) + \frac{1}{4}\log A - \frac{3}{16}. \tag{3.14}$$

Invoking (2.11), this becomes (3.10). □

We note that (3.14) gives a proof of the third equality in (3.2). Both (2.36) and (3.10) are contained in [14, 1999a] and are given as exercises in [7].

4. The Triple Gamma Function

For general material, we refer to [7, page 42]. As can been seen on [7, page 207], the important integral $\int_0^z \log\Gamma_3(t+a)\mathrm{d}t$ is not in closed form. Recently, Chakraborty-Kanemitsu-Kuzumaki [5, Corollary 1.1] have given a general expressions for all the integrals in $\log\Gamma_r$, by appealing to Barnes' original results.

In this section, we shall give a direct derivation of a closed form by combining [7, (455), page 210] and [11, Corollary 3] (with $\lambda = 3$). The first reads

$$\begin{aligned}
2\int_0^z \log\Gamma_3(t+a)\mathrm{d}t = &-\int_0^z t^3\psi(t+a)\mathrm{d}t + 2z\log\Gamma_3(z+a) \\
&-2(2a-3)\frac{\log\Gamma_3(z+a)}{\log\Gamma_3(a)} + \left(3a^2-9a+7\right)\frac{\log G(z+a)}{\log G(a)} \\
&-(a-1)^3\frac{\log\Gamma(z+a)}{\log\Gamma(a)} + \frac{3}{8}z^4 + \frac{1}{3}(1-\log 2\pi)z^3 \\
&+\left(-\frac{3}{4}a^2 + \frac{7}{4}a - \frac{9}{8} + \frac{1}{4}(2a-3)\log 2\pi + \log A\right)z^2, \\
&+\left(a^2 - \frac{3}{2}a + \frac{1}{4} + \frac{1}{2}(a-2-3a+2)\log 2\pi + 2(3-2a)\log A\right)z,
\end{aligned} \tag{4.1}$$

while the second reads (cf. also [15])

$$\begin{aligned}
\int_0^z t^3\log\psi(t+a)\mathrm{d}t = &-\sum_{r=0}^{3} C_3(r,a)\log\frac{\Gamma_{r+1}(a+z)}{\Gamma_{r+1}(a)} \\
&-\sum_{l=1}^{3}(-1)^l\left(\binom{3}{l}\zeta'(l-3) + \frac{B_{4-l}(a)}{l(4-l)}\right)z^l + \frac{11}{24}z^4,
\end{aligned} \tag{4.2}$$

where $C_3(r,a)$ are defined by

$$C_3(r,a) = (-1)^r r!\sum_{m=r}^{3}\binom{3}{m}(-1)^m S(m,n)(a-1)^{3-m} \tag{4.3}$$

and where $S(m,n)$ are the Stirling numbers of the second kind [7, page 58]. To express the values of $\zeta'(l-3)$, we appeal to [7]

(i) $\zeta'(0) = -(1/2)\log 2\pi$ [7, (20), page 92],

(ii) $\zeta'(-2) = \log B = \zeta(3)/4\pi^2$ [7, pages 99-100] and (2.31). After some elementary but long calculations, we arrive at

$$\begin{aligned}\int_0^z t^3 \log \psi(t+a)\mathrm{d}t = -3! \log \frac{\Gamma_4(a+z)}{\Gamma_4(a)} - 6(a-2) \log \frac{\Gamma_3(a+z)}{\Gamma_3(a)} \\ -\left(3a^2 - 9a + 7\right) \log \frac{\Gamma_2(a+z)}{\Gamma_2(a)} - (a-1)^3 \log \frac{\Gamma(a+z)}{\Gamma(a)} \frac{11}{24} z^4 \\ + \left(-\frac{1}{2}\log 2\pi + \frac{1}{3}B_1(a)\right) z^3 - \left(\frac{1}{4} - 3\log A + \frac{1}{4}B_2(a)\right) z^2 \\ + 3\left(\log B + \frac{1}{3}B_3(a)\right) z.\end{aligned} \tag{4.4}$$

Combining we have the following.

Theorem 4.1 (see [5, Example 2.3]). *Except for the singularities of the multiple gamma function, one has*

$$\begin{aligned}\int_0^z \log \Gamma_3(t+a)\, \mathrm{d}\, t = 3 \log \frac{\Gamma_4(a+z)}{\Gamma_4(a)} + z \log \Gamma_3(z+a) \\ + (a-3) \log \frac{\Gamma_3(a+z)}{\Gamma_3(a)} - \frac{1}{24} z^4 - \frac{1}{6}\left(a - \frac{3}{2} - \frac{1}{2}\log 2\pi\right) z^3 \\ + \frac{1}{8}\left(-2a^2 + 6a - \frac{10}{3} + (2a-3)\log 2\pi - 8\log A\right) z^2 \\ - \frac{1}{2}\left(a^3 - \frac{5}{2}a^2 + 2a - \frac{1}{4} + \frac{1}{2}\left(a^2 - 3a + 2\right)\log 2\pi \right. \\ \left. + 2(2a-3)\log A + 3\log B\right) z.\end{aligned} \tag{4.5}$$

This theorem enables us to put many formulas in [7] in closed form including, for instance, [7, (698), page 245]. Compare [5].

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