Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 473582, 12 pages
doi:10.1155/2012/473582

Research Article

An Upper Bound of the Bezout Number for Piecewise Algebraic Curves over a Rectangular Partition

Feng-Gong Lang and Xiao-Ping Xu

School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266100, China

Correspondence should be addressed to Feng-Gong Lang, langfg@yahoo.cn

Received 24 March 2012; Accepted 10 June 2012

Academic Editor: Raül Curto

A piecewise algebraic curve is a curve defined by the zero set of a bivariate spline function. Given two bivariate spline spaces $S_m^r(\Delta)$ and $S_n^t(\Delta)$ over a domain D with a partition Δ, the Bezout number BN($m,r;n,t;\Delta$) is defined as the maximum finite number of the common intersection points of two arbitrary piecewise algebraic curves $f(x,y)=0$ and $g(x,y)=0$, where $f(x,y)\in S_m^r(\Delta)$ and $g(x,y)\in S_n^t(\Delta)$. In this paper, an upper bound of the Bezout number for piecewise algebraic curves over a rectangular partition is obtained.

1. Introduction

Let $D\subset R^2$ be a bounded domain, and let $\mathbf{P}_k$ be the collection of real bivariate polynomials with total degree not greater than k. Divide D by using finite number of irreducible algebraic curves we get a partition denoted by Δ. The subdomains $D_1,D_2,\dots,D_N$ are called the cells. The line segments that form the boundary of each cell are called the edges. Intersection points of the edges are called the vertices. The vertices in the inner of the domain are called interior vertices, otherwise are called boundary vertices. For a vertex v, its so-called star $st(v)$ means the union of all cells in Δ sharing v as a common vertex, and its degree $d(v)$ is defined as the number of the edges sharing v as a common endpoint. If $d(v)$ is odd, we call v an odd vertex. For integers k and μ with $k>\mu\ge 0$, the bivariate spline space with degree k and smoothness μ over D with respect to Δ is defined as follows [1, 2]:

$$S_k^{\mu}(\Delta):=\left\{s\in C^{\mu}(D)\mid s_{|D_i}\in\mathbf{P}_k,\ i=1,\dots,N\right\}. \tag{1.1}$$

For a spline $s \in S_k^{\mu}(\Delta)$, the zero set

$$Z(s) := \{(x, y) \in D \mid s(x, y) = 0\} \tag{1.2}$$

is called a piecewise algebraic curve [1, 2]. Obviously, the piecewise algebraic curve is a generalization of the usual algebraic curve [3, 4].

In fact, the definition of piecewise algebraic curve is originally introduced by Wang in the study of bivariate spline interpolation. He pointed out that the given interpolation knots are properly posed if and only if they are not lying on a same nonzero piecewise algebraic curve [1, 2]. Hence, to solve a bivariate spline interpolation problem, it is necessary to deal with the properties of piecewise algebraic curve. Moreover, piecewise algebraic curve is also helpful for us to study the usual algebraic curve. Besides, piecewise algebraic curve also relates to the remarkable Four-Color conjecture [5–7]. In fact, the Four-Color conjecture holds if and only if, for any triangulation, there exist three linear piecewise algebraic curves such that the union of them equals the union of all central lines of all triangles in the triangulation. We know that any triangulation is 2-vertex signed [6, 7], which means the vertices of the triangulation can be marked by -1 or 1 such that the vertices of every triangle in the triangulation are marked by different numbers. So, we remark that the Four-Color conjecture holds if and only if there exist three nonequivalent mark methods. Based on these observations, in a word, piecewise algebraic curve is a new and important topic of computational geometry and algebraic geometry. It is of important theoretical and practical significance in many fields such as bivariate spline interpolation and computer-aided geometric design (CAGD). Hence, it is necessary to continue to study it.

In this paper, we mainly focus on the Bezout number for piecewise algebraic curves. It is well known that the Bezout theorem for usual algebraic curves is very important in algebraic geometry [3, 4]. Its weak form says that two algebraic curves with degree m and n, respectively, will have infinite number of common intersection points if they have more intersection points (including multiple points) than the product mn of their degrees, that is, the so-called Bezout number. Similarly, for two given bivariate spline spaces $S_m^r(\Delta)$ and $S_n^t(\Delta)$, the following number [2]

$$\mathrm{BN}(m, r; n, t; \Delta) := \max_{f,g}\{\#(f, g) < \infty \mid f(x, y) \in S_m^r(\Delta), g(x, y) \in S_n^t(\Delta)\} \tag{1.3}$$

is called the Bezout number, where $\#(f, g)$ denotes the number of the common intersection points (including multiple points) of $f(x, y) = 0$ and $g(x, y) = 0$. It also implies that two piecewise algebraic curves determined by two splines in $S_m^r(\Delta)$ and $S_n^t(\Delta)$ respectively will have infinite number of common intersection points if they have more intersection points than $\mathrm{BN}(m, r; n, t; \Delta)$. Needless to say, it is crucial for us to obtain $\mathrm{BN}(m, r; n, t; \Delta)$. Obviously, if $r \leq t$, then we have

$$\mathrm{BN}(m, r; n, t; \Delta) \leq BN(m, r; n, r; \Delta) \leq Nmn. \tag{1.4}$$

However, we remark that it is very hard to obtain exact $\mathrm{BN}(m, r; n, t; \Delta)$. On the one hand, piecewise algebraic curve itself is difficult; on the other hand, the Bezout number $\mathrm{BN}(m, r; n, t; \Delta)$ is also complicated; we know it not only relies on the degrees m, n and the smoothness orders r, t, but also the dimensions of $S_m^r(\Delta)$ and $S_n^t(\Delta)$ and the geometric

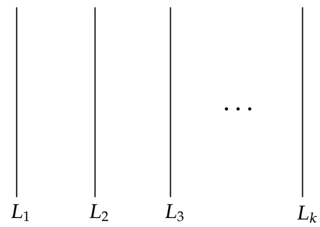

Figure 1: Δ_k.

characteristics of Δ as well [6, 7]. Hence, we can only get an upper bound for $\mathrm{BN}(m, r; n, t; \Delta)$ sometimes.

Throughout the relative literatures, we find that there have been some results on the Bezout number for piecewise algebraic curves over triangulations [6–9], while this paper, as a continue paper to fill a gap, is mainly devoted to an upper bound of the Bezout number for two piecewise algebraic curves over a rectangular partition. Our method is different from the methods in [6–9] and is new and effective. The remainder is organized as follows. In Section 2, for the sake of integrity, some prevenient results on the Bezout number for piecewise algebraic curves over parallel lines partition and triangulation are introduced; Section 3 is the main section, in this section; assume Δ to be a rectangular partition, an upper bound of $\mathrm{BN}(m, r; n, r; \Delta)$ is well estimated; finally, this paper is concluded in Section 4 with a conjecture.

2. Preliminary

Denoted by Δ_k the partition containing only k parallel lines (Figure 1), we have the followeing theorem.

Theorem 2.1 (see [10]). *One has*

$$\mathrm{BN}(m, r; n, t; \Delta_k) \leq (k+1)mn - \min\{r, t\}k. \tag{2.1}$$

For a triangulation Δ, the Bezout number for two continuous piecewise algebraic curves satisfying the following theorem.

Theorem 2.2 (see [6]). *Cosnider*

$$\mathrm{BN}(m, 0; n, 0; \Delta) \leq Nmn - \left[\frac{V_{\mathrm{odd}} + 2}{3}\right], \tag{2.2}$$

where N and V_{odd} is the number of the triangles and the odd interior vertices in Δ, respectively, and $[x]$ means the maximum integer not greater than x.

By using resultants and polar coordinates, Zhao studied the Bezout number for two C^1 piecewise algebraic curves over a non-obtuse-angled star $st(v)$ (Figures 2 and 3).

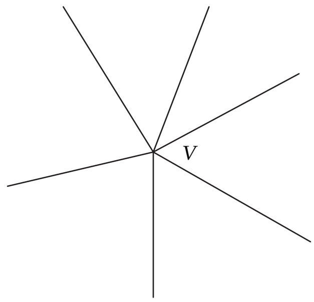

Figure 2: An interior star.

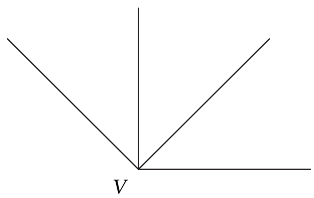

Figure 3: A boundary star.

Theorem 2.3 (see [9]). *If v is an interior vertex, then*

$$\mathrm{BN}(m,1;n,1;\mathrm{st}(v)) \leq d(v)mn - (d(v)-1), \tag{2.3}$$

and if v is a boundary vertex, then

$$\mathrm{BN}(m,1;n,1;\mathrm{st}(v)) \leq (d(v)-1)mn - (d(v)-2). \tag{2.4}$$

Based on Theorem 2.3, by a combinatorial optimization method, a result for a general nonobtuse triangulation is obtained.

Theorem 2.4 (see [7]). *For any given nonobtuse triangulation Δ,*

$$\mathrm{BN}(m,1;n,1;\Delta) \leq Nmn - \frac{2E_I - V_I - V_B}{3}, \tag{2.5}$$

where N, E_I, V_I, and V_B is the number of the triangles, the interior edges, the interior vertices, and the boundary vertices in Δ.

We know that the Bezout number for piecewise algebraic curves over stars is the key issue for the Bezout number for piecewise algebraic curves over a general partition. In order to get $\mathrm{BN}(m,r;n,r;\Delta)$, Wang and Xu generalized the smoothness order of Theorem 2.3 from 1 to r.

Theorem 2.5 (see [7]). *For any nonobtuse star $st(v)$, if v is an interior vertex, then*

$$\mathrm{BN}(m,r;n,r;\mathrm{st}(v)) \leq d(v)mn - \max\left\{d(v)-4, \left[\frac{d(v)+1}{2}\right]\right\}r, \tag{2.6}$$

and if v is a boundary vertex, then

$$\mathrm{BN}(m,r;n,r;\mathrm{st}(v)) \leq (d(v)-1)mn - \max\left\{d(v)-2, \left[\frac{d(v)-1}{2}\right]\right\}r. \tag{2.7}$$

Here, we note that if v is a boundary vertex, then $d(v) \geq 2$, so $d(v)-2 \geq [(d(v)-1)/2]$ holds unconditionally; hence we have

$$\mathrm{BN}(m,r;n,r;\mathrm{st}(v)) \leq (d(v)-1)mn - (d(v)-2)r. \tag{2.8}$$

Recently, by using the theory of resultants, polar coordinates and periodic trigonometric spline, Theorem 2.5 is improved greatly when v is an interior vertex.

Theorem 2.6 (see [5, 8]). *For any nonobtuse star $st(v)$, if v is an interior vertex, then*

$$\mathrm{BN}(m,r;n,r;\mathrm{st}(v)) \leq 2\left[\frac{d(v)(mn-r)}{2}\right], \tag{2.9}$$

and if v is a boundary vertex, then

$$\mathrm{BN}(m,r;n,r;\mathrm{st}(v)) \leq (d(v)-1)(mn-r)+r. \tag{2.10}$$

Here, we also remark that if v is a boundary vertex, then the second formula is likewise equivalent to

$$\mathrm{BN}(m,r;n,r;\mathrm{st}(v)) \leq (d(v)-1)mn - (d(v)-2)r. \tag{2.11}$$

By Theorem 2.6, we get a better upper bound of $\mathrm{BN}(m,r;n,r;\Delta)$ as follows.

Theorem 2.7. *For any nonobtuse triangulation Δ, one has*

$$BN(m,r;n,r;\Delta) \leq \begin{cases} Nmn - \dfrac{2}{3}E_I r, & \text{if } mn-r \text{ is even}, \\ Nmn - \dfrac{2}{3}E_I r - \dfrac{1}{3}V_{\mathrm{odd}}, & \text{if } mn-r \text{ is odd}, \end{cases} \tag{2.12}$$

where V_{odd} stands for the number of the odd interior vertices in Δ.

Proof. Suppose that the number of the common intersection points of two piecewise algebraic curves $f(x,y)=0$ and $g(x,y)=0$ is finite and equals $\mathrm{BN}(m,r;n,r;\Delta)$, where $f \in S_m^r(\Delta)$ and

$g \in S_n^r(\Delta)$. For a vertex v in Δ, let $k(v)$ be the number of the common intersection points of $f(x,y) = 0$ and $g(x,y) = 0$ in $st(v)$. Summing $k(v)$ for each vertex, so every common intersection point is counted triply. Hence, we get

$$\sum_v k(v) = 3\mathrm{BN}(m,r;n,r;\Delta). \tag{2.13}$$

By Theorem 2.6, we have

$$k(v) \le \mathrm{BN}(m,r;n,r;\ \mathrm{st}(v)) \le \delta(v), \tag{2.14}$$

where

$$\delta(v) = \begin{cases} \begin{cases} d(v)(mn-r), & \text{if } d(v)(mn-r) \text{ is even} \\ d(v)(mn-r)-1, & \text{if } d(v)(mn-r) \text{ is odd} \end{cases} & \text{if } v \text{ is an interior vertex} \\ (d(v)-1)mn - (d(v)-2)r, & \text{if } v \text{ is } a \text{ boundary vertex.} \end{cases} \tag{2.15}$$

By the Euler formulae $E_I = V_B + 3V_I - 3,\ N = V_B + 2V_I - 2$ and the equations $\sum_v d(v) = 2(E_I + E_B),\ E_B = V_B$, if $mn - r$ is even, we get

$$\begin{aligned} \mathrm{BN}(m,r;n,r;\Delta) &= \frac{1}{3}\sum_v k(v) \le \frac{1}{3}\sum_v \delta(v) \\ &= \frac{1}{3}\left\{ \sum_{\text{interior } v} d(v)(mn-r) + \sum_{\text{boundary } v} ((d(v)-1)mn - (d(v)-2)r) \right\} \\ &= \frac{1}{3}\left(\sum_v d(v) - V_B\right)mn - \frac{1}{3}\left(\sum_v d(v) - 2V_B\right)r \\ &= Nmn - \frac{2}{3}E_I r. \end{aligned} \tag{2.16}$$

Similarly, if $mn - r$ is odd, we have

$$\mathrm{BN}(m,r;n,r;\Delta) \le Nmn - \frac{2}{3}E_I r - \frac{1}{3}V_{\text{odd}}. \tag{2.17}$$

□

In the next section, we will apply Theorems 2.1 and 2.6 to a rectangular partition. A good upper bound of the Bezout number for piecewise algebraic curves over a rectangular partition is derived, which fills a gap in the study of the Bezout number for piecewise algebraic curves over any partition.

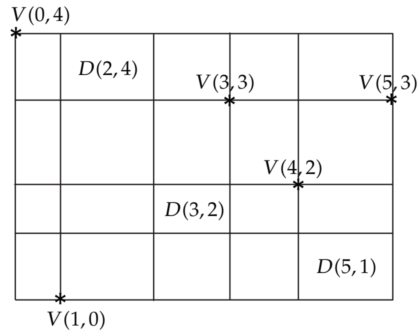

Figure 4: A rectangular partition $\Delta_{a\times b}$, $a = 5$, $b = 4$.

3. Results

Without loss of generality, let integers $a \geq b \geq 1$. For a rectangular domain D, subdivide it into $N := ab$ subrectangular cells by $(a-1)$ vertical lines and $(b-1)$ horizontal lines, and denote by $\Delta_{a\times b}$ the partition; see Figure 4 for an example. Let V_0, V_1^0, and V_1^1 be the collection of the interior vertices $(d(v) = 4)$, the boundary vertices that lying in the interior of the boundary edges $(d(v) = 3)$, and the else four corner vertices $(d(v) = 2)$, respectively. The cardinality of V_0, V_1^0, and V_1^1 is $(a-1)(b-1)$, $2(a+b-2)$, and 4, respectively. If a cell lies in the ith column (from left to right) and in the jth row (from bottom to top), we denote it by $D(i,j)$, $i = 1,2,\dots,a$; $j = 1,2,\dots,b$. For an interior vertex $v \in V_0$, if it is the intersection point of the pth vertical interior line and the qth horizontal interior line, then we denote it by $v(p,q)$, $p = 1,2,\dots,a-1$; $q = 1,2,\dots,b-1$. The else boundary vertices can also be denoted similarly.

Assume that the number of the common intersection points of two piecewise algebraic curves $f(x,y) = 0$ and $g(x,y) = 0$ is finite and equals $\mathrm{BN}(m,r;n,r;\Delta_{a\times b})$, where $f \in S_m^r(\Delta_{a\times b})$ and $g \in S_n^r(\Delta_{a\times b})$. If $b = 1$, then $\Delta_{a\times 1}$ is a partition Δ_{a-1} containing only $a-1$ parallel lines essentially (Figure 1); by Theorem 2.1, we have

$$\mathrm{BN}(m,r;n,r;\Delta_{a\times 1}) \leq amn - (a-1)r. \tag{3.1}$$

In the rest of this section, assume $a \geq b \geq 2$.

3.1. *The First Method*

For a vertex v in $\Delta_{a\times b}$, let $k(v)$ be the number of the common intersection points of $f(x,y) = 0$ and $g(x,y) = 0$ in $st(v)$. Summing $k(v)$ for each vertex, so that every common intersection point is counted fourfold. Hence, we get

$$\sum_{v} k(v) = 4\mathrm{BN}(m,r;n,r;\Delta_{a\times b}). \tag{3.2}$$

By Theorem 2.6, we have

$$k(v) \leq \mathrm{BN}(m,r;n,r;\ \mathrm{st}(v)) \leq \delta(v), \tag{3.3}$$

where

$$\delta(v) = \begin{cases} 4mn - 4r, & \text{if } v \in V_0,\ d(v) = 4 \\ 2mn - r, & \text{if } v \in V_1^0,\ d(v) = 3 \\ mn, & \text{if } v \in V_1^1,\ d(v) = 2. \end{cases} \tag{3.4}$$

We have

$$\begin{aligned} \mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \frac{1}{4}\sum_{v} k(v) \leq \frac{1}{4}\sum_{v}\delta(v) \\ &= \frac{1}{4}\left(\sum_{v\in V_0}\delta(v) + \sum_{v\in V_1^0}\delta(v) + \sum_{v\in V_1^1}\delta(v)\right) \\ &= \frac{1}{4}\{(4mn-4r)(a-1)(b-1) + (2mn-r)(2(a+b-2)) + 4mn\} \\ &= Nmn - r\left(N - \frac{a+b}{2}\right). \end{aligned} \tag{3.5}$$

Let $\mathrm{BN}_1 = Nmn - r(N - ((a+b)/2))$.

3.2. *The Second Method*

Let $k(i,j)$ be the number of the common intersection points of $f(x,y) = 0$ and $g(x,y) = 0$ in the cell $D(i,j)$ $(i = 1,2,\ldots,a; j = 1,2,\ldots,b)$ summing $k(i,j)$ for all cells, we get

$$\sum_{i,j} k(i,j) = \mathrm{BN}(m,r;n,r;\Delta_{a\times b}). \tag{3.6}$$

For the cells in the jth row, by Theorem 2.1, we have

$$\sum_{i} k(i,j) \leq \mathrm{BN}(m,r;n,r;\Delta_{a-1}) \leq amn - (a-1)r. \tag{3.7}$$

Hence,

$$\begin{aligned} \mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \sum_{i,j} k(i,j) = \sum_{j}\left(\sum_{i} k(i,j)\right) \\ &\leq (amn - (a-1)r)b = Nmn - r(N-b). \end{aligned} \tag{3.8}$$

Similarly,

$$\begin{aligned}\mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \sum_{i,j} k(i,j) = \sum_i \left(\sum_j k(i,j)\right) \\ &\le (bmn-(b-1)r)a = Nmn - r(N-a).\end{aligned} \tag{3.9}$$

Let $\mathrm{BN}_2 = Nmn - r(N-b)$, $\mathrm{BN}_3 = Nmn - r(N-a)$. Considering $a \ge b$, we have $\mathrm{BN}_2 \le \mathrm{BN}_1 \le \mathrm{BN}_3$. This tells us a better summation method results in a better upper bound.

3.3. The Third Method

In this subsection, we will derive a rather better upper bound of $\mathrm{BN}(m,r;n,r;\Delta_{a\times b})$ than BN_1, BN_2, and BN_3 by using a mixed method based on Theorems 2.1 and 2.6. We prefer this summation method.

(1) If a and b are both even, then $\Delta_{a\times b} = \cup_{p=1}^{a/2}\cup_{q=1}^{b/2} st(v(2p-1,2q-1))$; so

$$\begin{aligned}\mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \sum_{p=1}^{a/2}\sum_{q=1}^{b/2} k(v(2p-1,2q-1)) \le \sum_{p=1}^{a/2}\sum_{q=1}^{b/2} \delta(v(2p-1,2q-1)) \\ &= (4mn-4r)\times\frac{a}{2}\times\frac{b}{2} = Nmn - Nr.\end{aligned} \tag{3.10}$$

(2) If a is odd and b is even, then $\Delta_{a\times b} = \cup_{p=1}^{((a-1)/2)}\cup_{q=1}^{b/2} st(v(2p-1,2q-1)) \cup_{j=1}^{b} D(a,j)$; so we have

$$\begin{aligned}\mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \sum_{p=1}^{((a-1)/2)}\sum_{q=1}^{b/2} k(v(2p-1,2q-1)) + \sum_{j=1}^{b} k(a,j) \\ &\le \sum_{p=1}^{((a-1)/2)}\sum_{q=1}^{b/2} \delta(v(2p-1,2q-1)) + BN(m,r;n,r;\Delta_{b-1}) \\ &= (4mn-4r)\times\frac{a-1}{2}\times\frac{b}{2} + (bmn-(b-1)r) \\ &= Nmn - (N-1)r.\end{aligned} \tag{3.11}$$

(3) If a is even and b is odd, then $\Delta_{a\times b} = \cup_{p=1}^{a/2} \cup_{q=1}^{((b-1)/2)} st(v(2p-1,2q-1)) \cup_{i=1}^{a} D(i,b)$; so

$$\begin{aligned}
\mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \sum_{p=1}^{a/2} \sum_{q=1}^{((b-1)/2)} k\big(v(2p-1,2q-1)\big) + \sum_{i=1}^{a} k(i,b) \\
&\le \sum_{p=1}^{a/2} \sum_{q=1}^{((b-1)/2)} \delta\big(v(2p-1,2q-1)\big) + \mathrm{BN}(m,r;n,r;\Delta_{a-1}) \\
&= (4mn-4r) \times \frac{a}{2} \times \frac{b-1}{2} + (amn-(a-1)r) \\
&= Nmn-(N-1)r.
\end{aligned} \tag{3.12}$$

(4) If a and b are both odd, then $\Delta_{a\times b} = \cup_{p=1}^{((a-1)/2)} \cup_{q=1}^{((b-1)/2)} st(v(2p-1,2q-1)) \cup_{j=1}^{b} D(a,j) \cup_{i=1}^{a-1} D(i,b)$, hence

$$\begin{aligned}
\mathrm{BN}(m,r;n,r;\Delta_{a\times b}) &= \sum_{p=1}^{((a-1)/2)} \sum_{q=1}^{((b-1)/2)} k\big(v(2p-1,2q-1)\big) + \sum_{j=1}^{b} k\big(a,j\big) + \sum_{i=1}^{a-1} k(i,b) \\
&\le \sum_{p=1}^{((a-1)/2)} \sum_{q=1}^{((b-1)/2)} \delta\big(v(2p-1,2q-1)\big) + BN(m,r;n,r;\Delta_{b-1}) \\
&\quad + \mathrm{BN}(m,r;n,r;\Delta_{a-2}) \\
&= (4mn-4r) \times \frac{a-1}{2} \times \frac{b-1}{2} + (bmn-(b-1)r) + ((a-1)mn-(a-2)r) \\
&= Nmn-(N-2)r.
\end{aligned} \tag{3.13}$$

Theorem 3.1. *Let $\Delta_{a\times b}$ be a rectangular partition, then*

$$BN(m,r;n,r;\Delta_{a\times b}) \le \begin{cases} Nmn-Nr, & \text{if } a \text{ and } b \text{ are both even} \\ Nmn-(N-1)r, & \text{if } (a-b) \text{ is odd} \\ Nmn-(N-2)r, & \text{if } a \text{ and } b \text{ are both odd} \end{cases}. \tag{3.14}$$

Let BN denotes the upper bound given in Theorem 3.1. We remark that BN is better than BN_1, BN_2, and BN_3. Since $a \ge b \ge 2$ and $\mathrm{BN}_2 \le \mathrm{BN}_1 \le \mathrm{BN}_3$, we only give the comparisons between BN and $\mathrm{BN}_2 = Nmn - r(N-b)$. For fixed m, n, and r, we have the following results; see Table 1. In a word, we have $\mathrm{BN} \le \mathrm{BN}_2$.

4. Conclusions

In this paper, we mainly derive an upper bound for the Bezout number $\mathrm{BN}(m,r;n,r;\Delta_{a\times b})$. The results of Theorem 3.1 are excellent than BN_1, BN_2, and BN_3. It is very useful in the fields

Table 1: Comparisons of BN and BN_2 (where $a \geq b \geq 2$ and $N = ab$).

	BN	BN_2	$BN_2 - BN$
a and b are both even	$Nmn - Nr$	$Nmn - r(N - b)$	$br \geq 0$
$(a - b)$ is odd	$Nmn - (N - 1)r$	$Nmn - r(N - b)$	$(b - 1)r \geq 0$
a and b are both odd	$Nmn - (N - 2)r$	$Nmn - r(N - b)$	$(b - 2)r \geq 0$

of CAGD. For example, we are frequently needed to get the common intersection points of two piecewise algebraic curves [11, 12]. By Theorem 3.1, a prior estimation of the number of the common intersection points of two piecewise algebraic curves over a rectangular partition will be obtained. In future, we will try best to improve Theorem 3.1 to get the exact number or a lower upper bound for $\mathrm{BN}(m, r; n, r; \Delta_{a\times b})$. In order to attract readers' interest, here, we also give a conjecture on the Bezout number $\mathrm{BN}(m, r; n, r; \Delta_{a\times b})$.

Conjecture 4.1. Consider

$$\begin{aligned}\mathrm{BN}(m, r; n, r; \Delta_{a\times b}) &= mn + [(a-1) + (b-1)](mn - r) + (a-1)(b-1)(mn - 2r) \\ &= Nmn - (2N - a - b)r.\end{aligned} \tag{4.1}$$

If this conjecture holds, it can be applied into the study of the Nöther-type theorem [13, 14] and the Cayley-Bacharach theorem [15] for piecewise algebraic curves over rectangular partition.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities (no. 201113037). The author appreciate the reviewers and editors for their careful reading, valuable suggestions, timely review, and reply.

References

[1] R. H. Wang, "Structure of multivariate splines, and interpolation," *Acta Mathematica Sinica*, vol. 18, no. 2, pp. 91–106, 1975, English translation, vol. 18, pp. 10–39.
[2] R.-H. Wang, *Multivariate Spline Functions and Their Applications*, Science Press, Beijing, China; Kluwer Academic, New York, NY, USA, 1994/2001.
[3] R. Hartshorne, *Algebraic Geometry*, Springer, New York, NY, USA, 1977.
[4] R. J. Walker, *Algebraic Curves*, Dover, New York, NY, USA, 1950.
[5] D. X. Gong, *Some Research on Theory of Piecewise Algebraic Variety and RBF Interpolation [Ph.D. thesis]*, Dalian University of Technology, Dalian, China, 2009.
[6] X. Shi and R. Wang, "The Bezout number for piecewise algebraic curves," *BIT Numerical Mathematics*, vol. 39, no. 2, pp. 339–349, 1999.
[7] R. Wang and Z. Xu, "Estimation of the Bezout number for piecewise algebraic curve," *Science in China A*, vol. 46, no. 5, pp. 710–717, 2003, English translation, vol. 46, no. 5, pp. 710–717.
[8] R.-H. Wang, "Recent researches on multivariate spline and piecewise algebraic variety," *Journal of Computational and Applied Mathematics*, vol. 221, no. 2, pp. 460–471, 2008.
[9] G. H. Zhao, *On Some Problems for Multivariate Splines [Ph.D. thesis]*, Dalian University of Technology, Dalian, China, 1996.
[10] Z. X. Luo, *Researches On Nonlinear Splines [Ph.D. thesis]*, Dalian University of Technology, Dalian, China, 1991.

[11] F.-G. Lang and R.-H. Wang, "Intersection points algorithm for piecewise algebraic curves based on Groebner bases," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 357–366, 2009.
[12] X. Zhang and R. Wang, "Isolating the real roots of the piecewise algebraic variety," *Computers and Mathematics with Applications*, vol. 57, no. 4, pp. 565–570, 2009.
[13] R. Wang and C. Zhu, "Nöther-type theorem of piecewise algebraic curves," *Progress in Natural Science*, vol. 14, no. 4, pp. 309–313, 2004.
[14] C. Zhu and R. Wang, "Nöther-type theorem of piecewise algebraic curves on quasi-cross-cut partition," *Science in China A*, vol. 52, no. 4, pp. 701–708, 2009.
[15] R.-H. Wang and C.-G. Zhu, "Cayley-Bacharach theorem of piecewise algebraic curves," in *Proceedings of the International Symposium on Computational Mathematics and Applications (Dalian, 2002)*, vol. 163, no. 1, pp. 269–276, 2004.