Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 424189, 7 pages
doi:10.1155/2012/424189

Research Article

Symmetry Fermionic p-Adic q-Integral on $\mathbb{Z}_p$ for Eulerian Polynomials

Daeyeoul Kim[1] and Min-Soo Kim[2]

[1] *National Institute for Mathematical Sciences, Yuseong-daero 1689-gil, Yuseong-gu, Daejeon 305-811, Republic of Korea*
[2] *Division of Cultural Education, Kyungnam University, Changwon 631-701, Republic of Korea*

Correspondence should be addressed to Min-Soo Kim, mskim@kyungnam.ac.kr

Received 18 June 2012; Accepted 14 August 2012

Academic Editor: Cheon Ryoo

Kim et al. (2012) introduced an interesting p-adic analogue of the Eulerian polynomials. They studied some identities on the Eulerian polynomials in connection with the Genocchi, Euler, and tangent numbers. In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, defined by Kim (2008), we show a symmetric relation between the q-extension of the alternating sum of integer powers and the Eulerian polynomials.

1. Introduction

The Eulerian polynomials $A_n(t), n = 0, 1, \ldots$, which can be defined by the generating function

$$\frac{1-t}{e^{(t-1)x}-t} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, \tag{1.1}$$

have numerous important applications in number theory, combinatorics, and numerical analysis, among other areas. From (1.1), we note that

$$(A(t) + (t-1))^n - tA_n(t) = (1-t)\delta_{0,n}, \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [1]). Thus far, few recurrences for the Eulerian polynomials other than (1.2) have been reported in the literature. Other recurrences are of importance as they might reveal new aspects and properties of the Eulerian polynomials, and they can help simplify the proofs of known properties. For more important properties, see, for instance, [1] or [2].

Let p be a fixed odd prime number. Let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ be the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ be the p-adic valuation on $\mathbb{Q}$, where $|p|_p = p^{-1}$. The extended valuation on $\mathbb{C}_p$ is denoted by the same symbol $|\cdot|_p$. Let q be an indeterminate, where $|1-q|_p < 1$. Then, the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.3}$$

For a uniformly (or strictly) differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$ (see [1, 3–6]), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.4}$$

Then, it is easy to see that

$$\frac{1}{q} I_{-1/q}(f_1) + I_{-1/q}(f) = [2]_{1/q} f(0), \tag{1.5}$$

where $f_1(x) = f(x+1)$.

By using the same method as that described in [1], and applying (1.5) to f, where

$$f(x) = q^{(1-\omega)x} e^{-x(1+q)\omega t} \tag{1.6}$$

for $\omega \in \mathbb{Z}_{>0}$, we consider the generalized Eulerian polynomials on $\mathbb{Z}_p$ by using the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{(1-\omega)x} e^{-x(1+q)\omega t} d\mu_{-1/q}(x) &= \frac{1+q}{q^{1-\omega} e^{-(1+q)\omega t} + q} \\ &= \sum_{n=0}^{\infty} A_n(-q,\omega) \frac{t^n}{n!}. \end{aligned} \tag{1.7}$$

By expanding the Taylor series on the left-hand side of (1.7) and comparing the coefficients of the terms $t^n/n!$, we get

$$\int_{\mathbb{Z}_p} q^{(1-\omega)x} x^n d\mu_{-1/q}(x) = \frac{(-1)^n}{\omega^n (1+q)^n} A_n(-q,\omega). \tag{1.8}$$

We note that, by substituting $\omega = 1$ into (1.8),

$$A_n(-q,1) = A_n(-q) = (-1)^n(1+q)^n \int_{\mathbb{Z}_p} x^n d\mu_{-1/q}(x) \tag{1.9}$$

is the Witt's formula for the Eulerian polynomials in [1, Theorem 1]. Recently, Kim et al. [1] investigated new properties of the Eulerian polynomials $A_n(-q)$ at $q = 1$ associated with the Genocchi, Euler, and tangent numbers.

Let $T_{k,1/q}(n)$ denote the q-extension of the alternating sum of integer powers, namely,

$$T_{k,1/q}(n) = \sum_{i=0}^{n} (-1)^i i^k q^{-i} = 0^k q^0 - 1^k q^{-1} + \cdots + (-1)^n n^k q^{-n}, \tag{1.10}$$

where $0^0 = 1$. If $q \to 1$, $T_{k,q}(n) \to T_k(n) = \sum_{i=0}^{n}(-1)^i i^k$ is the alternating sum of integer powers (see [4]). In particular, we have

$$T_{k,1/q}(0) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases} \tag{1.11}$$

Let ω_1, ω_2 be any positive odd integers. Our main result of symmetry between the q-extension of the alternating sum of integer powers and the Eulerian polynomials is given in the following theorem, which is symmetric in ω_1 and ω_2.

Theorem 1.1. *Let ω_1, ω_2 be any positive odd integers and $n \geq 0$. Then, one has*

$$\begin{aligned} &\sum_{i=0}^{n} \binom{n}{i} A_i(-q,\omega_1) T_{n-i,q^{-\omega_2}}(\omega_1 - 1)\omega_2^{n-i}(-1-q)^{n-i} \\ &\quad = \sum_{i=0}^{n} \binom{n}{i} A_i(-q,\omega_2) T_{n-i,q^{-\omega_1}}(\omega_2 - 1)\omega_1^{n-i}(-1-q)^{n-i}. \end{aligned} \tag{1.12}$$

Observe that Theorem 1.1 can be obtained by the same method as that described in [4]. If $q = 1$, Theorem 1.1 reduces to the form stated in the remark in [4, page 1275].

Using (1.11), if we take $\omega_2 = 1$ in Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let ω_1 be any positive odd integer and $n \geq 0$. Then, one has*

$$A_n(-q) = \sum_{i=0}^{n} \binom{n}{i} A_i(-q,\omega_1) T_{n-i,q^{-1}}(\omega_1 - 1)(-1-q)^{n-i}. \tag{1.13}$$

2. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will need the following two identities (see (2.4) and (2.5)) related to the Eulerian polynomials and the q-extension of the alternating sum of integer powers.

Let ω_1, ω_2 be any positive odd integers. From (1.7), we obtain

$$\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_1)x} e^{-x(1+q)\omega_1 t} d\mu_{-1/q}(x)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} = \frac{1+\left(q^{-\omega_1} e^{-(1+q)\omega_1 t}\right)^{\omega_2}}{1+q^{-\omega_1} e^{-(1+q)\omega_1 t}}. \tag{2.1}$$

This has an interesting p-adic analytic interpretation, which we shall discuss below (see Remark 2.1). It is easy to see that the right-hand side of (2.1) can be written as

$$\begin{aligned} \frac{1+\left(q^{-\omega_1} e^{-(1+q)\omega_1 t}\right)^{\omega_2}}{1+q^{-\omega_1} e^{-(1+q)\omega_1 t}} &= \sum_{i=0}^{\omega_2-1} (-1)^i q^{-\omega_1 i} e^{-(1+q)\omega_1 t i} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\omega_2-1} (-1)^i i^k \left(q^{\omega_1}\right)^{-i} \omega_1^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

In (1.10), let $q = q^{\omega_1}$. The left-hand, right-hand side, by definition, becomes

$$\frac{1+\left(q^{-\omega_1} e^{-(1+q)\omega_1 t}\right)^{\omega_2}}{1+q^{-\omega_1} e^{-(1+q)\omega_1 t}} = \sum_{k=0}^{\infty} \left(T_{k,q^{-\omega_1}}(\omega_2 - 1)\omega_1^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \tag{2.3}$$

A comparison of (2.1) and (2.3) yields the identity

$$\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_1)x} e^{-x(1+q)\omega_1 t} d\mu_{-1/q}(x)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} = \sum_{k=0}^{\infty} \left(T_{k,q^{-\omega_1}}(\omega_2 - 1)\omega_1^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \tag{2.4}$$

By slightly modifying the derivation of (2.4), we can obtain the following identity:

$$\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_2)x} e^{-x(1+q)\omega_2 t} d\mu_{-1/q}(x)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} = \sum_{k=0}^{\infty} \left(T_{k,q^{-\omega_2}}(\omega_1 - 1)\omega_2^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \tag{2.5}$$

Remark 2.1. The derivations of identities are based on the fermionic p-adic q-integral expression of the generating function for the Eulerian polynomials in (1.7) and the quotient of integrals in (2.4), (2.5) that can be expressed as the exponential generating function for the q-extension of the alternating sum of integer powers.

Observe that similar identities related to the Eulerian polynomials and the q-extension of the alternating sum of integer powers in (2.4) and (2.5) can be found, for instance, in [3, (1.8)], [4, (21)], and [6, Theorem 4].

Proof of Theorem 1.1. Let ω_1, ω_2 be any positive odd integers. Using the iterated fermionic p-adic q-integral on $\mathbb{Z}_p$ and (1.7), we have

$$\begin{aligned}
&\frac{\int\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1+(1-\omega_2)x_2} e^{-(1+q)(\omega_1 x_1+\omega_2 x_2)t} d\mu_{-1/q}(x_1) d\mu_{-1/q}(x_2)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} \\
&\quad = [2]_{1/q} \frac{q^{-\omega_1\omega_2} e^{-(1+q)\omega_1\omega_2 t} + 1}{\left(q^{-\omega_1} e^{-(1+q)\omega_1 t} + 1\right)\left(q^{-\omega_2} e^{-(1+q)\omega_2 t} + 1\right)}.
\end{aligned} \tag{2.6}$$

Now, we put

$$I^* = \frac{\int\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1+(1-\omega_2)x_2} e^{-(1+q)(\omega_1 x_1+\omega_2 x_2)t} d\mu_{-1/q}(x_1) d\mu_{-1/q}(x_2)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)}. \tag{2.7}$$

From (1.7) and (2.5), we see that

$$\begin{aligned}
I^* &= \left(\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1} e^{-(1+q)(\omega_1 x_1)t} d\mu_{-1/q}(x_1)\right) \times \left(\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_2)x_2} e^{-(1+q)(\omega_2 x_2)t} d\mu_{-1/q}(x_2)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)}\right) \\
&= \left(\sum_{k=0}^{\infty} A_k(-q, \omega_1) \frac{t^k}{k!}\right) \times \left(\sum_{l=0}^{\infty} \left(T_{l,q^{-\omega_2}}(\omega_1 - 1)\omega_2^l (-1)^l (1+q)^l\right) \frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} A_i(-q, \omega_1) T_{n-i,q^{-\omega_2}}(\omega_1 - 1)\omega_2^{n-i} (1+q)^{n-i}\right) \frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

On the other hand, from (1.7) and (2.4), we have

$$\begin{aligned}
I^* &= \left(\int_{\mathbb{Z}_p} q^{(1-\omega_2)x_2} e^{-(1+q)(\omega_2 x_2)t} d\mu_{-1/q}(x_2)\right) \times \left(\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1} e^{-(1+q)(\omega_1 x_1)t} d\mu_{-1/q}(x_1)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)}\right) \\
&= \left(\sum_{k=0}^{\infty} A_k(-q, \omega_2) \frac{t^k}{k!}\right) \times \left(\sum_{l=0}^{\infty} \left(T_{l,q^{-\omega_1}}(\omega_2 - 1)\omega_1^l (-1)^l (1+q)^l\right) \frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} A_i(-q, \omega_2) T_{n-i,q^{-\omega_1}}(\omega_2 - 1)\omega_1^{n-i} (1+q)^{n-i}\right) \frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

By comparing the coefficients on both sides of (2.8) and (2.9), we obtain the result in Theorem 1.1. □

3. Concluding Remarks

Note that many other interesting symmetric properties for the Euler, Genocchi, and tangent numbers are derivable as corollaries of the results presented herein. For instance, considering [1, (5)],

$$A_n(-1,\omega) = (-2\omega)^n E_n \quad (n \geq 0), \tag{3.1}$$

where E_n denotes the nth Euler number defined by $E_n := E_n(0)$, and the Euler polynomials are defined by the generating function

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{3.2}$$

and on putting $q = 1$ in Theorem 1.1 and Corollary 1.2, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}\omega_1^i E_i T_{n-i}(\omega_1-1)\omega_2^{n-i} = \sum_{i=0}^{n}\binom{n}{i}\omega_2^i E_i T_{n-i}(\omega_2-1)\omega_1^{n-i}, \tag{3.3}$$

$$E_n = \sum_{i=0}^{n}\binom{n}{i}\omega_1^i E_i T_{n-i}(\omega_1-1). \tag{3.4}$$

These formulae are valid for any positive odd integers ω_1, ω_2. The Genocchi numbers G_n may be defined by the generating function

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n\frac{t^n}{n!}, \tag{3.5}$$

which have several combinatorial interpretations in terms of certain surjective maps on finite sets. The well-known identity

$$G_n = 2(1-2^n)B_n \tag{3.6}$$

shows the relation between the Genocchi and the Bernoulli numbers. It follows from (3.6) and the Staudt-Clausen theorem that the Genocchi numbers are integers. It is easy to see that

$$G_n = 2nE_{2n-1} \quad (n \geq 1), \tag{3.7}$$

and from (3.2), (3.5) we deduce that

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}}{k+1}x^{n-k}. \tag{3.8}$$

It is well known that the tangent coefficients (or numbers) T_n, defined by

$$\tan t = \sum_{n=1}^{\infty} (-1)^{n-1} T_{2n} \frac{t^{2n-1}}{(2n-1)!}, \tag{3.9}$$

are closely related to the Bernoulli numbers, that is, (see [1])

$$T_n = 2^n (2^n - 1) \frac{B_n}{n}. \tag{3.10}$$

Ramanujan ([7, page 5]) observed that $2^n(2^n - 1)B_n/n$ and, therefore, the tangent coefficients, are integers for $n \geq 1$. From (3.3), (3.6), (3.7), and (3.10), the obtained symmetric formulae involve the Bernoulli, Genocchi, and tangent numbers (see [1]).

Acknowledgment

This work was supported by the Kyungnam University Foundation grant, 2012.

References

[1] D. S. Kim, T. Kim, W. J. Kim, and D. V. Dolgy, "A note on Eulerian polynomials," *Abstract and Applied Analysis*, vol. 2012, Article ID 269640, 10 pages, 2012.

[2] P. Luschny, "Eulerian polynomials," http://www.luschny.de/math/euler/EulerianPolynomials.html, 2011.

[3] D. S. Kim and K. H. Park, "Identities of symmetry for Euler polynomials arising from quotients of fermionic integrals invariant under S_3," *Journal of Inequalities and Applications*, vol. 2010, Article ID 851521, 16 pages, 2010.

[4] T. Kim, "Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials," *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.

[5] C. S. Ryoo, "A note on q-Bernoulli numbers and polynomials," *Applied Mathematics Letters*, vol. 20, no. 5, pp. 524–531, 2007.

[6] Y. Simsek, "Complete sum of products of (h, q)-extension of Euler polynomials and numbers," *Journal of Difference Equations and Applications*, vol. 16, no. 11, pp. 1331–1348, 2010.

[7] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, Providence, RI, USA, 2000.