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Research Article

On the Homology Theory of Operator Algebras

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We investigate the cyclic homology and free resolution effect of a commutative unital Banach algebra. Using the free resolution operator, we define the relative cyclic homology of commutative Banach algebras. Lemmas and theorems of this investigation are studied and proved. Finally, the relation between cyclic homology and relative cyclic homology of Banach algebra is deduced.

1. Introduction

Many years ago, cyclic homology has been introduced by Connes and Tsygan and defined on suitable categories of algebras, as the homology of a natural chain complex and the target of a natural Chern character from topological (or algebraic) K-Theory.

In order to extend the classical theory of the Chern character to the noncommutative setting, Connes [1] and Tsygan [2] have developed the cyclic homology of associative algebras. Recently, there has been increasing interest in general algebraic structures than associative algebras, characterized by the presence of several algebraic operations. Such structures appear, for example, in homotopy theory [3, 4] and topological field theory [5].

Brylinski and Nistor [6] have extended Conne's computation of the cyclic cohomology groups of smooth algebras arising from foliations with separated graphs and explained some results of Atiyah and Segal on orbifold Euler characteristic in the setting of cyclic homology. Kazhdan [7] studied Hochschild and cyclic homology of finite type algebras using abelian stratifications of their primitive ideal spectrum.

Victor Nistor [8] has studied associative p-summable quasi homomorphism's and p-summable extensions elements in a bivariant cyclic cohomology group defined by Connes, and showed that this generalizes his character on K-homology; furthermore, he studied the properties of this character and showed that it is compatible with analytic index.

Results of Connes [1] have led much research interest into the computation of cyclic (co)homology groups in recent years (see, [4, 6, 9–13]).

A promising approach to the calculation of cyclic cohomology groups is to break it down by making use of extensions of Banach algebras; this is a standard device in the study of various properties of Banach algebras.

The Banach cyclic (co)homology of Banach algebra has been studied by Christensen and Sinclair [3], Helemskii [4, 9], among others. The dihedral cohomology in Banach category and its relation with the cyclic cohomology, the triviality, and nontriviality of dihedral cohomology groups of some classes of operator algebras have been studied [14].

Suppose that A and B be commutative unital Banach algebra with involution (in short B^*-algebra). And let $C_n(A)$ denote the $(n+1)$ fold projective tensor power of A. The elements of this Banach space n-dimensional will be called chains. Let $t_n : C_n(A) \to C_n(A)$, $n = 1, 2, \ldots$ denote the operator uniquely defined by

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}. \tag{1.1}$$

And let $CC_n(A)$ denote the quotient space of $C_n(A)$ modulo the closure of the linear span of elements of the form $x - t_n x$ $(n = 1, 2, \ldots)$. Note that $\operatorname{Im}(id_{C_n(A)} - t_n)$ is closed in $C_n(A)$ and so $CC_n(A) = C_n(A)/\operatorname{Im}(id_{C_n(A)} - t_n)$ and also $CC_0(A) = C_0(A) = A$.

Define the complex $C(A) = (C*(A), \delta*)$, where $C_n(A) = A \otimes \cdots \otimes A$ and, $b^* : C_n(A) \to C_{n-1}(A)$ is the boundary operator;

$$b_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i-1} \otimes \cdots \otimes a_{n-1}. \tag{1.2}$$

We can easily verify that $b_{n+1} b_n = 0$ and hence $\operatorname{Ker} b_n \supset \operatorname{Im} b_{n-1}$. The group $H_n(A) = H(C(A)) = (\operatorname{Ker} b_n)/(\operatorname{Im} b_{n-1})$ is called the simplicial (Hochschild) homology of Banach algebra A.

Note that: $\operatorname{Ker} b_n$ is always closed, but, in general $\operatorname{Im} b_{n-1}$ is not closed.

Considering a unital Banach algebra A, one acts on the complex $C(A)$, by the cyclic group of order $(n+1)$ by means of the operator $t_n : C_n(A) \to C_n(A)$ which we denote.

The quotient complex $CC_n(A) = C_n(A)/\operatorname{Im}(1 - t_n)$ is a subcomplex of the complex $C_n(A)$. Following [15], the cyclic homology of a Banach algebra A is the homology of the complex $CC_n(A)$.

Given commutative unital Banach algebras A and B, let $f : A \to B$ be algebras homomorphism. We define a free resolution of a Banach algebra B over the homomorphism $f : A \xrightarrow{i} R \xrightarrow{\pi} B$, where i is an inclusion and π is a quasi-isomorphism, and use this fact to define the relative cyclic homology

$$HC_*\left(A \xrightarrow{f} B\right) = H_*\left(\frac{R}{(A + [R, R] + (1 - t_n)}\right), \tag{1.3}$$

where $[R, R]$ is the commutant of Banach algebra R.

Definition 1.1. A graded Banach algebra is a Banach algebra that has a graded normed algebra as a dense subalgebra.

We discuss the existence of the free Banach algebra resolution. Let $V = \sum_{n=0}^{\infty} V_n$ be a graded vector space over ring $F(F = R$ or $C)$. Suppose that R is a differential graded F-algebra and let $R\langle V\rangle = R^{*}T_k(V)$ be the free product of Banach algebras, where $T_k(V) = \sum_{j\geq 0} V^{\otimes j}$ is the tensor Banach algebra over F. The product in $R\langle V\rangle$ is given by

$$\begin{aligned}
&(r_1e_1,\ldots,r_ne_nr_{n+1})\cdot(\widehat{r}_1\widehat{e}_1,\ldots,\widehat{r}_k\widehat{e}_k\widehat{r}_{k+1})\\
&\quad=(r_1e_1,\ldots,r_ne_n(r_{n+1}\widehat{r}_1)\widehat{e}_1,\ldots,\widehat{r}_k\widehat{e}_k\widehat{r}_{k+1}),\quad r_i,\widehat{r}_j\in R,\ e_i\widehat{e}_j\in T_k(V).
\end{aligned} \tag{1.4}$$

Definition 1.2. Let $f : R_1 \to R_2$ be a homomorphism of differential graded F-algebras. An algebra R_2 is a free Banach algebra over the homomorphism$f$ if there exists an isomorphism $\alpha : R_1\langle V\rangle \approx R_2$, where E is a differential graded vector space with the following commutative diagram:

$$\begin{tikzcd}
R_1 \arrow[r, "f"] \arrow[dr, "i"'] & R_2 \arrow[d, "\alpha"] \\
& R_1\langle E\rangle
\end{tikzcd} \tag{1.5}$$

where i is an inclusion map.

Lemma 1.3. *Let $f : A \to B$ be a homomorphism of F-algebras. Then there exists a differential graded Banach algebra $R = \sum_{i=0} R_i$ with the following properties.*

(i) *π is surjection, and the following diagram is commutative:*

$$\begin{tikzcd}
& R \arrow[d, "\pi"] \\
A \arrow[ur, "i"] \arrow[r, "f"'] & B
\end{tikzcd} \tag{1.6}$$

where i is the inclusion map.

Clearly, there is an isomorphism $j : R \to A$ such that $j \circ i = 1_A$.

(ii) *π is quasi-isomorphism, that is, $\pi_* : H_*(A) \to H_*(B) = B$, where B is a differential graded Banach algebra,*

$$(B)_i = \begin{cases} B, & i = 0 \\ 0, & i > 0, \quad \text{and the differential } \partial^B = 0. \end{cases} \tag{1.7}$$

(iii) *The differential graded Banach algebra R is free over the homomorphism $i : A \to R$.*

Proof. We proof this theorem by two steps.

(1) We construct a commutative diagram of a Banach algebras

$$\begin{tikzcd} & R^{(0)} \arrow[d, "\pi_0"] \\ A \arrow[ur, "i_0"] \arrow[r, "f"'] & B \end{tikzcd} \tag{1.8}$$

where $R^{(0)}$ is free over the homomorphism $i_0 : A \to R^{(0)}$, π_0 an surjection. Define $A\langle (t_i)\rangle = E(t_i)$, where $E(t_i)$ is an involutive vector space generated by $\{t_i\}$, or generated by the family $\{t_i t_i^*\}$. The automorphism $* : E(t_i) \to E(t_i)$ is given as follows $^*(t_i) = (t_i^*)^*(t_i^*) = t_i$.

We choose a system $\{\Re_i^{(0)}\}$ of generators in a Banach algebra B. This family is assumed to be closed under an involutive on B.

Now, let $R^{(0)} = A\langle t_i^{(0)}\rangle$, where $t_i^{(0)}$ is equivalent to the generator $\{\Re^{(0)}\}$ in a Banach algebra B, and suppose that $\beta_i^{(0)} = t_i^{(0)}$ or $(t_i^{(0)})^*$. We define π_0 using the universal property of $R^{(0)}$. Let π_0 be the unique involutive Banach algebras $R^{(0)} \to B$ homomorphism, which restricts f on A and sends $t_i^{(0)}$ to $\Re_i^{(0)}$.

Since $i_0 : A \to A\langle t_i^{(0)}\rangle$ is an inclusion map, $i_0(a) = a$, i_0 is a Banach algebras homomorphism, and $\pi_0\, i_0(a) = \pi_0(a) = f(a)$. Hence, diagram (1.8) is commutative and π_0 is surjective.

Let $j_0 : R^{(0)} \to A$ be the unique algebras homomorphism restricting to the identity on A and mapping $t_i^{(0)}$ to zero. $R^{(0)}$ is a differential graded K-Banach algebra (see[13]);

$$\left(R^{(0)}\right)_i = \begin{cases} R^{(0)}, & i = 0 \\ 0, & i > 0, \quad \text{and the differential } \partial^{R^{(0)}}\ B_i^{(0)} = 0. \end{cases} \tag{1.9}$$

The algebra $R^{(0)}$ is free over the homomorphism $i_0 : A \to R^{(0)}$ since $R^{(0)} = A\langle t_i^{(0)}\rangle$.

(2) We construct the second commutative diagram

$$\begin{tikzcd} & R^{(1)} \arrow[d, "\pi_1"] \\ A \arrow[ur, "i_1"] \arrow[r, "f"'] & B \end{tikzcd} \tag{1.10}$$

where $R^{(1)}$ is free over the homomorphism $i_1 : A \to R^{(1)}$ and π_1 surjection. Choose a system $\{\Re_j^{(1)}\}$ of generators of $\operatorname{Ker}\pi_0$, which is closed under involution. Let $t_i^{(1)}$ be indeterminate which are bijection with the $\Re_j^{(1)}$. Define $R^{(1)} = A\langle t_i^{(0)}, t_j^{(1)}\rangle$, where $t_i^{(0)}$ is as defined above. Suppose that $\beta_j^{(1)}$ denotes $t_j^{(1)}$ or $(t_j^{(1)})^*$. The homomorphism π_1 is defined as to be the unique algebras $R^{(1)} \rightarrow B$ restricting to π_0 on $R^{(0)}$ and sending $t_j^{(1)}$ to zero. As can be seen, the homomorphism π_1 can be defined as π_0 and that π_1 is surjective since $\pi_1(\beta_i^{(0)}) = \Re_i$, $\pi_1(\beta_j^{(1)}) = 0_i$. The homomorphism $i_1 : A \rightarrow A\langle t_i^{(0)}, t_j^{(1)}\rangle$ is inclusion. The diagram (1.10) is commutative since $(\pi_1 i_1)(a) = \pi_1(a) = f(a)$.

The homomorphism j_1 is defined to be the unique homomorphism: $R^{(1)} \rightarrow A$ of involutive Banach algebras restricting to identity on A and mapping $t_i^{(1)}$ to zero. The algebra $R^{(1)} = A\langle t_i^{(0)}, t_j^{(1)}\rangle$ is free over i.

Finally, we have a differential graded Banach algebra

$$R^{(1)} = \left(R^{(1)}\right)_0 \oplus \left(R^{(1)}\right)_1 \oplus, \dots, \deg \beta_i^{(1)} = 0, \deg \beta_j^{(1)} = 1. \tag{1.11}$$

The differential $\partial_i^{R^{(1)}}$ is the unique derivation on $R^{(1)}$ satisfying the graded Leibntiz rule and commuting with the involution which restricts to zero on $R^{(1)}$ and sends $t_i^{(1)}$ to $\Re_j^{(1)}$. So $\partial_0^{R^{(1)}} \beta_i^{(0)} = 0$, $\partial_1^{R^{(1)}} \beta_j^{(0)} = \Re_j^{(1)} \in \operatorname{Ker} \pi_0$, $i > 1$.

Similarly, we can consider the commutative diagram

$$\begin{tikzcd} & R^{(2)} \arrow[d, "\pi_2"] \\ A \arrow[ur, "i_2"] \arrow[r, "f"'] & B \end{tikzcd} \tag{1.12}$$

where $R^{(2)} = A\langle t_i^{(0)}, t_j^{(1)}, t_k^{(2)}\rangle$ is a differential graded Banach algebra

$$\begin{gathered} R^{(2)} = \left(R^{(2)}\right)_0 \oplus \left(R^{(2)}\right)_1 \oplus \left(R^{(2)}\right)_2 \oplus, \dots, \deg \beta_i^{(0)} = 0, \\ \deg \beta_j^{(0)} = 1, \qquad \deg \beta_j^{(0)} = 2. \end{gathered} \tag{1.13}$$

The differential Banach algebra $R^{(2)}$ is also defined by using a universal property and, hence,

$$\partial_0^{R^{(2)}} \beta_i^{(0)} = 0, \quad \partial_1^{R^{(2)}} \beta_j^{(0)} = \Re_j^{(1)}, \quad \partial_1^{R^{(2)}} \beta_j^{(2)} = \Im_k^{(2)} = 0, \quad i > 2. \tag{1.14}$$

Consequently, we can construct an involutive Banach algebra $R^{(i)}$, $i \geq 0$ with the following commutative diagram:

$$\begin{tikzcd} & R^{(0)} \arrow[r, "p_0"] \arrow[d, "\pi_0"] & R^{(1)} \arrow[r, "p_1"] \arrow[d, "\pi_1"] & \cdots R^{(n-1)} \arrow[r, "p_{n-1}"] \arrow[d, "\pi_{n-1}"] & R^{(n)} \arrow[r, "p_n"] \arrow[d, "\pi_n"] & {} \\ A \arrow[ur, "i_0"] \arrow[r] & B \arrow[r, equal] & B \arrow[r, equal] & \cdots B \arrow[r, equal] & B \arrow[r, equal] & {} \end{tikzcd} \tag{1.15}$$

where π_i is surjection, $i \geq 0$, $i_n = P_{n-1} \circ \cdots \circ P_0 \circ i_o$ is an inclusion map from A to $R^{(n)}$, P_i is also an inclusion map from

$$P_i : A\left\langle t_{m_o}^{(0)}, t_{m_1}^{(1)}, \ldots, t_{m_i}^{(i)} \right\rangle \text{ to } A\left\langle t_{m_o}^{(0)}, t_{m_1}^{(1)}, \ldots, t_{m_i}^{(i)}, t_{m_i+1}^{(i+1)} \right\rangle. \tag{1.16}$$

Define $i_n = q_{n-1} \circ \cdots \circ q_0 \circ j_o$, where q_n is the projection of P_i.

The diagram (1.15) is commutative since $i_{n+1}(\beta_i^{(n)}) = \pi_n(\beta_i^{(n)}) = 0$, $n \geq o$.

Define $R = \lim R_n$, $\pi = \lim \pi_n$, $i = \lim i_n$, $j = \lim j_n$. Then the differential Banach graded algebra R satisfies the items of Lemma 1.3 since:

(1) $\pi = \lim \pi_n$ is surjection, the diagram

$$\begin{tikzcd} & R^{(2)} \arrow[d, "\pi_2"] \\ A \arrow[ur, "i_2"] \arrow[r, "f"'] & B \end{tikzcd} \tag{1.17}$$

is commutative since $i(a) = a$, $\pi(a) = f(a)$

(2) π is quasi-isomorphism of differential graded algebras

$$\begin{tikzcd} (R)_0 \arrow[d, "\pi_1"] & (R)_1 \arrow[l, "\partial_0^R"'] \arrow[d, "\pi_2"] & \cdots \arrow[l, "\partial_1^R"'] & (R)_n \arrow[l, "\partial_n^R"'] \arrow[d, "\pi_n"] & \cdots \arrow[l, "\partial_{n+1}^R"'] \\ B & 0 \arrow[l] & \cdots \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \end{tikzcd} \tag{1.18}$$

where $\partial_i^R = \lim \partial_i^R$, $(R)_0 = \ker(\pi)_0 = B$, $\operatorname{Im} \partial^R = \ker \partial_0^R$, that is,

$$H_0(R) = H_i(R) = 0. \tag{1.19}$$

(3) the differential graded Banach algebras R is free over the homomorphism $i : A \to R$, since $R = E$, E is a vector space generated by the system:

$$\left\{ t_{i_0}^{(0)}, t_{i_1}^{(1)}, \ldots, t_{i_n}^{(n)} \right\}. \tag{1.20}$$

□

Definition 1.4. The differential graded Banach algebra which satisfies the conditions (i), (ii), and (iii) of Lemma 1.3 is called a free resolution of Banach algebra B over f.

2. The Relative Cyclic Homology of Banach Algebra

In this part, we define the relative cyclic homology of commutative unital Banach algebra and study its properties. Let f be a homomorphism of Banach algebras A and B over a field K (K is a real or a complex number set). Let R_f^B be a free resolution of Banach algebra B over f and, for $r_1, r_2 \in R_f^B$, let $[r_1, r_2] = r_1, r_2 - (-1)^{|r_1||r_2|} r_2 r_1$ where $|ri| = \deg\, r_i$, $i = 1, 2$.

Let $C = [R_f^B, R_f^B]$ be the linear space generated by $[r_1, r_2]$, $r_1, r_2 \in R_f^B$.

We construct the complex $C = [R_f^B, R_f^B]$. Clearly, from the definition of R_f^B, that $\mathrm{Im}\,(1 - t_n)$ is a subcomplex of R_f^B. We have

$$\begin{aligned}
\partial[r_1 r_2] &= r_1 r_2 - (-1)^{|lr_1||r_2|} r_2 r_1 \\
&= \partial r_1 r_2 + (-1)^{|r_2|} r_1 \partial r_2 - (-1)^{|lr_1||r_2|} \left(\partial r_2 r_1 - (-1)^{|r_2|} r_2 \partial r_1\right) \\
&= \partial r_1 r_2 - (-1)^{|r_2|(|r_1|+1)} r_2 \partial r_1 + (-1)^{|r_1|} \left(r_1 \partial r_2 - (-1)^{|r_1|(|r_2|+1)} \partial r_2 r_1\right) \\
&= [\partial r_1 r_2] + (-1)^{|r_1|} [r_1, \partial r_2], \qquad |\partial r_i| = |r_1| - 1, \quad i = 1, 2.
\end{aligned} \tag{2.1}$$

Then $[R_f^B, R_f^B]$ is subcomplex in R_f^B. Therefore, the chain complex of K-module $[R_f^B, R_f^B]$ is a subcomplex of R_f^B.

Definition 2.1. Let $f : A \to B$ be F-algebras (char $F = 0$) homomorphism, and R_f^B be a free resolution of a Banach algebra B over f. Then the relative cyclic homology is defined as follows:

$$HC_*\left(A \xrightarrow{f} B\right) = H_*\left(\frac{R_f^B}{(A + [R, R] + \mathrm{Im}(1 - t_n))}\right). \tag{2.2}$$

Definition 2.2. The F-algebra $A\langle t\rangle$ generated by the elements $a_0 t a_1 t, \ldots, t a_n$, $n \geq 0$ can be considered as differential graded Banach algebras by requiring that the morphism $A \to A\langle t\rangle$ is a morphism of differential graded algebras (A is viewed as a differential graded Banach algebra concentrated in degree 0) and the $\deg t = 1$, $\partial t = 0$ and $t^* = t$.

Lemma 2.3. *A Banach algebra $A\langle t\rangle$ is splitable. One has a free algebra resolution of Banach algebra $B = 0$ over the homomorphism $A \to 0$.*

Proof. Using [6] from the chain complex

$$A \xleftarrow{\partial} AtA \xleftarrow{\partial} AtAt \xleftarrow{\partial} \cdots \xleftarrow{\partial} At \cdots tA \xleftarrow{\partial} \cdots, \tag{2.3}$$

where $At, \ldots, tA$ (n-times) is a K-module and the boundary operator ∂ is given by

$$\begin{aligned}\partial(a_0ta_1t,\ldots,ta_{n-1}ta_n) &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t,\ldots,ta_i(\partial t)a_{i+1}t,\ldots,ta_n\\ &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t,\ldots,t(a_ia_{i+1})t,\ldots,ta_n.\end{aligned}\tag{2.4}$$

Note that the differential ∂ in $A\langle t\rangle$ is equivalent to the operator $\delta_n' : C_n(A) \to C_{n-1}(A)$, defined by

$$\delta_n'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n.\tag{2.5}$$

Following [11], the complex $(C_n(A), \delta_n')$ is splitable and so is the complex $A\langle t\rangle$, that is, $H_*(A\langle t\rangle) = 0$. Therefore, Banach algebra $A\langle t\rangle$ is free resolution of the Banach algebra $B = 0$ over the homomorphism $A \to 0$. □

Lemma 2.4. *The complex $A\langle t\rangle/[A, A\langle t\rangle]$ is a standard simplicial (Hochschild) complex.*

Proof. Consider the factor complex $A\langle t\rangle/[A, A\langle t\rangle]$. It is generated by the elements $a_0ta_1t,\ldots,ta_{n-1}t$, since

$$a_0ta_1t,\ldots,ta_{n-1}ta_n = a_na_0 \times ta_1t,\ldots,ta_{n-1}t(\operatorname{mod}[A, A\langle t\rangle]).\tag{2.6}$$

The action of the differential ∂ on the complex $A\langle t\rangle/[A, A\langle t\rangle]$ is given by

$$\begin{aligned}\partial(a_0ta_1t,\ldots,ta_{n-1}ta_n) = {}&\sum_{i=0}^{n-1}(-1)^i a_0ta_1t,\ldots,t(a_iaa_{i+1})t,\ldots,ta_n\\ &+ (-1)^n a_na_0ta_1t,\ldots,a_{n-1}t.\end{aligned}\tag{2.7}$$

Consider the complex

$$A \xleftarrow{id} A \xleftarrow{\delta} A^{\otimes 2} \xleftarrow{\delta} \cdots \xleftarrow{\delta} A^{\otimes n} \xleftarrow{\delta} \cdots,\tag{2.8}$$

where δ is the differential in the standard Hochschild complex (see [7, 16]). Since the space $(A\langle t\rangle/[A, A\langle t\rangle])_{n+1}$ identifies with the space

$$A^{\otimes n+1} : a_0ta_1,\ldots,ta_nt \longrightarrow a_0 \otimes \cdot a_1 \otimes \cdots \otimes a_n\tag{2.9}$$

and the differential in $A\langle t\rangle/[A, A\langle t\rangle]$ identifies with the differential in the standard Hochschild complex, then the complex $A\langle t\rangle/[A, A\langle t\rangle]$ is the Hochschild complex. □

Theorem 2.5. *Let A be Banach algebra with unity and involution. Then $HC_i(A \to B) = HC_{i-1}(A)$, where $HC_i(A)$ is the cyclic homology of F-algebras (char $F = 0$).*

Proof. Consider the factor complex: $A\langle t\rangle / [A\langle t\rangle, A\langle t\rangle] + \mathrm{Im}(1 - r_n)$, such that

$$a_0 t a_1 t, \ldots, t a_{n-1} t = (-1)^n a_n t a_0 t \ldots a_{n-1} t, \tag{2.10}$$

where deg $a_0 t a_1 t, \ldots, t a_{n-1} t = n$, deg $a_0 t a_1 t, \ldots, t a_n t = n + 1$.

The cyclic homology of $A\langle t\rangle$ is the homology of the complex $A\langle t\rangle / [A\langle t\rangle, A\langle t\rangle] + \mathrm{Im}(1 - t_n)$. By factoring $A\langle t\rangle$, first by the subcomplex $A \leftarrow 0 \leftarrow 0 \leftarrow \cdots$ and second by the subcomplex $[A\langle t\rangle, A\langle t\rangle] + \mathrm{Im}(1 - t_n)]$, we get a homomorphism $CC_*(A \to 0) \to CC_{*_{-1}}(A)$, which induces an isomorphism of the cyclic one homology groups $HC_*(A \to 0) \to HC_{*_{-1}}(A)$. ☐

Theorem 2.6. *Let $f : A \to B$ be a homomorphism of a Banach algebras over a field K(char $K = 0$). Then the relative cyclic homology $HC_i(A \xrightarrow{f} B)$ does not depend on the choice of the resolution.*

Proof. The homomorphism f induces homomorphism of chain complexes

$$f_* : CC_*(A) \longrightarrow CC_*(B), \tag{2.11}$$

where $CC_*(A)$ is a cyclic complex. Consider the diagram

$$\begin{tikzcd} & R_f^B \arrow[d, "\pi"] \\ A \arrow[ur, "i"] \arrow[r, "f"'] & B \end{tikzcd} \tag{2.12}$$

where R_f^B is defined above, and i is an inclusion map. The idea of proof is to show that the cone of the map i is quasi-isomorphic to an arbitrary category (see [8, 17]), to the complex $R_f^B / [R_f^B, R_f^B] + \mathrm{Im}(1 - t_n)$, Since

$$H_i\left(R_f^B\right) = \begin{cases} B, & i = 0 \\ 0, & i > 0. \end{cases} \tag{2.13}$$

Then the isomorphism $\pi_* : CC_*(R_f^B) \to CC_*(B)$ induces an isomorphism of the homology of these complexes. Since $i_* : CC_*(A) \to CC_*(R_f^B)$ is an inclusion, then

$$HC_i\left(A \xrightarrow{f} B\right) \longrightarrow HC_i\left(A \xrightarrow{g \circ f} C\right) \longrightarrow HC_i\left(A \xrightarrow{g} C\right) \longrightarrow HC_{i-1}\left(A \xrightarrow{f} B\right) \longrightarrow \cdots \tag{2.14}$$

$M(i_*) \approx CC_*(R_f^B)/CC_*(A)$, where $M(i_*)$ is a cone of i (see [12, 15, 18]).

Note that, the symbol $\approx$ denotes a quasi-isomorphism. It is clear, from the above discussion, that the following diagram is commutative

$$\begin{tikzcd} & CC_*(R_f^B) \arrow[d, "\pi_*"] \\ CC_*(A) \arrow[ur, dash, "i_*"] \arrow[r, "f_*"'] & CC_*(B) \end{tikzcd} \tag{2.15}$$

and hence $M(f_*) \approx CC_*(R_f^B)/CC_*(A)$.

Following [11], we have $CC_*(R_f^B)/CC_*(A) \approx R_f^B/A + [R_f^B, R_f^B]$, where CC_* is the Connes cyclic complex, and by using the spectral sequence $E_{ij}^2 = {}^{\varepsilon}H_*(Z/2, H_*(R_f^B)) = {}^{\varepsilon}HC_{i+j}(R_f^B)$, we have

$$\frac{CC_*\left(R_f^B\right)}{CC_*(A) \approx R_f^B/A + \left[R_f^B, R_f^B\right] + \mathrm{Im}(1 - t_n)}. \tag{2.16}$$

So, $M(f_*) \approx R_f^B/A + [R_f^B, R_f^B] + \mathrm{Im}(1 - t_n)$. Then $HC_i(A \xrightarrow{f} B)$ does not depend on the choice of R_f^B. $\square$

Theorem 2.7. *Let A, B, and D be involutive Banach algebra. Then the following sequence $A \xrightarrow{f} B \xrightarrow{g} D$ induces the long exact sequence of relative cyclic homology;*

$$HC_i\left(A \xrightarrow{f} B\right) \longrightarrow HC_i\left(A \xrightarrow{g\circ f} D\right) \longrightarrow HC_i\left(B \xrightarrow{g} D\right) \longrightarrow HC_{i-1}\left(A \xrightarrow{f} B\right) \longrightarrow \cdots . \tag{2.17}$$

Proof. In Theorem 2.6, it has been proved that any homomorphism $f : A \to B$ of involutive algebra in an arbitrary category is equivalent to an inclusion $i : A \to R_f^B$. Then, for a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of involutive Banach algebra, we have the following complex:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & R_f^B \arrow[r, equal] \arrow[d, equal] & B \arrow[r, "i'"] \arrow[dr, "g"'] & R_g^D \arrow[d, equal] \\ & B & & D \end{tikzcd} \tag{2.18}$$

Consider the following sequence of mapping cones

$$o \longrightarrow M(i_*) \longrightarrow M(i'_*) \longrightarrow M(i_* \circ i'_*) \longrightarrow o. \tag{2.19}$$

In general, the sequence (2.17) is not exact. The composition of two morphisms will be zero. However, the cone over the morphism $M(i_*) \to M(i'_*)$ is canonically homotopy equivalent to $M(i_* \circ i'_*)$. So, we get the following exact sequence of the relative cyclic homology

$$HC_i\Big(A \xrightarrow{f} B\Big) \longrightarrow HC_i\Big(A \xrightarrow{g\circ f} C\Big) \longrightarrow HC_i\Big(B \xrightarrow{g} C\Big) \longrightarrow HC_{i-1}\Big(A \xrightarrow{f} B\Big) \to \cdots. \quad (2.20)$$

In the following we give an example of the cyclic homology of tensor algebra by using the free resolution fact. Let a Banach A be F-algebra, (char $F = 0$) and M be A-bimodule. For a chain complex $V_\bullet$ of modules, consider the complex $S^n(A, V_\bullet) = A \otimes_{A\otimes A^{\mathrm{op}}} V_\bullet^{\otimes(k+1)}$. If we act on $S^n(A, V_\bullet)$ by the cyclic group z_{n+1} of order $(n+1)$ by means of automorphisms, we get

$$t_n(v_0 \otimes \cdots \otimes v_n) = (-1)^{\mu} v_n \otimes v_0 \otimes \cdots \otimes v_{n-1}, \quad (2.21)$$

where $\mu = (\deg p_n)(\sum_{i=0}^{n-1} \deg p_i)$.

If $V_\bullet$ is a free resolution of A-bimodule M, then the complex $S^n(A, V_\bullet)$ can be considered as complex $S^n(A, M)$. □

Example 2.8. Let M be A-bimodule, where A is K-Banach algebra, $T_A(M)$ be a tensor Banach algebra and $\mathrm{Tor}_i^A(M, M) = 0,\ i > 0$, then

$$HC_i(T_A(M)) = HC_i(A) \oplus \left(\bigoplus_{n=0}^{\infty} H_i(Z_{n+1}; S^n(A, M))\right). \quad (2.22)$$

Proof. Suppose $V_\bullet$ is a free resolution of A-bimodule M. Then according to the condition $\mathrm{Tor}_i^A(M, M) = 0,\ i > 0$, the space $T_A(V_\bullet)$ is a free resolution of algebra $T_A(M)$ over inclusion $i : A \to T_A(M)$. Using Theorem 2.7 the long exact sequence of relative cyclic homology of the following sequence, $A \xrightarrow{i} T_A(M) \to 0$, we get

$$\begin{aligned} \cdots &\longrightarrow HC_i\Big(A \xrightarrow{i} T_A(M)\Big) \longrightarrow HC_i\Big(A \xrightarrow{0} 0\Big) \\ &\longrightarrow HC_i(T_A(M) \longrightarrow 0) \longrightarrow HC_{i-1}\Big(A \xrightarrow{i} T_A(M)\Big) \xrightarrow{0} \cdots. \end{aligned} \quad (2.23)$$

Since A is a direct sum of $T_A(M)$, we have

$$0 \longrightarrow HC_i\Big(A \xrightarrow{i} T_A(M)\Big) \longrightarrow HC_i(A) \longrightarrow HC_i(T_A(M)) \longrightarrow 0 \quad (2.24)$$

and hence

$$HC_i(T_A(M)) = HC_i(A) \oplus HC_i\Big(A \xrightarrow{i} T_A(M)\Big). \quad (2.25)$$

To prove the theorem we show that

$$HC_i\Big(A \xrightarrow{i} T_A(M)\Big) = \bigoplus_{n=0}^{\infty} H_i(Z_{n+1}; S^n(A, M)). \quad (2.26)$$

Clearly,

$$\frac{T_A(V_\bullet)}{(A+[T_A(V_\bullet),T_A(V_\bullet)]+\operatorname{Im}(1-t_n))}=\frac{\oplus_{n=0}^{\infty}V_\bullet^{\otimes(n+1)}}{([T_A(V_\bullet),T_A(V_\bullet)]+\operatorname{Im}(1-t_n))}. \tag{2.27}$$

Then we have the following isomorphism:

$$\frac{\oplus_{n=0}^{\infty}P_\bullet^{\otimes(n+1)}}{([T_A(V_\bullet),T_A(V_\bullet)]+\operatorname{Im}(1-t_n))}\approx\frac{\oplus_{n=0}^{\infty}A\otimes_{A\otimes A^{\mathrm{op}}}P_\bullet^{\otimes(n+1)}}{\operatorname{Im}(1-t_n)}. \tag{2.28}$$

The homology of the chain complex:

$$\frac{\oplus_{n=0}^{\infty}A\otimes_{A\otimes A^{\mathrm{op}}}V_\bullet^{\otimes(n+1)}}{\operatorname{Im}(1-t_n)} \tag{2.29}$$

is equivalent to

$$\bigoplus_{n=0}^{\infty}H_i(Z_{n+1};S^n(A,M)). \tag{2.30}$$

From (2.25) and (2.26), the proof is completed. □

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