

Research Article

Euler Basis, Identities, and Their Applications

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Let $V_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ be the $(n+1)$ -dimensional vector space over \mathbb{Q} . We show that $\{E_0(x), E_1(x), \dots, E_n(x)\}$ is a good basis for the space V_n , for our purpose of arithmetical and combinatorial applications. Thus, if $p(x) \in \mathbb{Q}[x]$ is of degree n , then $p(x) = \sum_{l=0}^n b_l E_l(x)$ for some uniquely determined $b_l \in \mathbb{Q}$. In this paper we develop method for computing b_l from the information of $p(x)$.

1. Introduction

The Euler polynomials, $E_n(x)$, are given by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.1)$$

(see [1–20]) with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n th Euler numbers. The Bernoulli numbers are also defined by

$$\frac{t}{e^t - 1} = e^{Bt} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (1.2)$$

(see [1–20]) with the usual convention about replacing B^n by B_n . As is well known, the Bernoulli polynomials are given by

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^n \binom{n}{l} B_{n-l} x^l, \quad (1.3)$$

(see [9–15]) From (1.1), (1.2), and (1.3), we note that

$$B_n(1) - B_n = \delta_{1,n}, \quad E_n(1) + E_n = 2\delta_{0,n}, \quad (1.4)$$

where $\delta_{k,n}$ is the kronecker symbol.

Let $m, n \in \mathbb{Z}_+$ with $m + n \geq 2$. The formula

$$B_m(x)B_n(x) = \sum_r \left\{ \binom{m}{2r}n + \binom{n}{2r}m \right\} \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1} \frac{m!n!B_{m+n}}{(m+n)!}, \quad (1.5)$$

is proved in [4–6]. Let $V_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ be the $(n+1)$ -dimensional vector space over \mathbb{Q} . Probably, $\{1, x, \dots, x^n\}$ is the most natural basis for this space. But $\{E_0(x), E_1(x), \dots, E_n(x)\}$ is also a good basis for the space V_n , for our purpose of arithmetical and combinatorial applications. Thus, if $p(x) \in \mathbb{Q}[x]$ is of degree n , then

$$p(x) = \sum_{l=0}^n b_l E_l(x), \quad (1.6)$$

for some uniquely determined $b_l \in \mathbb{Q}$. Further,

$$b_k = \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \quad (k = 0, 1, 2, \dots, n), \quad (1.7)$$

where $p^{(k)}(x) = d^k p(x)/dx^k$. In this paper we develop methods for computing b_l from the information of $p(x)$. Apply these results to arithmetically and combinatorially interesting identities involving $E_0(x), E_1(x), \dots, E_n(x), B_0(x), \dots, B_n(x)$. Finally, we give some applications of those obtained identities.

2. Euler Basis, Identities, and Their Applications

Let us take $p(x)$ the polynomial of degree n as follows:

$$p(x) = \sum_{k=0}^n B_k(x)B_{n-k}(x). \quad (2.1)$$

From (2.1), we have

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!} \sum_{l=k}^n B_{l-k}(x)B_{n-l}(x). \quad (2.2)$$

By (1.7) and (2.2), we get

$$\begin{aligned} b_k &= \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \\ &= \frac{1}{2} \binom{n+1}{k} \sum_{l=k}^n \{(B_{l-k} + \delta_{1,l-k})(B_{n-l} + \delta_{1,n-l}) + B_{l-k}B_{n-l}\}, \end{aligned} \quad (2.3)$$

Thus, we have

$$b_k = \binom{n+1}{k} \left(\sum_{l=k}^n B_{l-k}B_{n-l} + B_{n-k-1} \right), \quad (0 \leq k \leq n-3), \quad (2.4)$$

$$b_{n-2} = \frac{7}{72} n(n^2 - 1), \quad b_n = n+1, \quad b_{n-1} = 0. \quad (2.5)$$

By (1.6), (2.1), (2.3), and (2.4), we get

$$\begin{aligned} &\sum_{k=0}^n B_k(x) B_{n-k}(x) \\ &= \sum_{k=0}^{n-3} \binom{n+1}{k} \left(\sum_{l=k}^n B_{l-k}B_{n-l} + B_{n-k-1} \right) E_k(x) + \frac{7}{72} n(n^2 - 1) E_{n-2}(x) + (n+1) E_n(x). \end{aligned} \quad (2.6)$$

Let us consider the following triple identities:

$$p(x) = \sum_{r+s+t=n} B_r(x) B_s(x) B_t(x) = \sum_{k=0}^n b_k E_k(x), \quad (2.7)$$

where the sum runs over all $r, s, t \in \mathbb{Z}_+$ with $r + s + t = n$. Thus, by (2.7), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3) \sum_{r+s+t=n-k} B_r(x) B_s(x) B_t(x). \quad (2.8)$$

From (1.7) and (2.8), we have

$$\begin{aligned} b_k &= \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \\ &= \frac{\binom{n+2}{k}}{2} \sum_{r+s+t=n-k} \{B_r(1)B_s(1)B_t(1) + B_rB_sB_t\} \\ &= \frac{\binom{n+2}{k}}{2} \left\{ 2 \sum_{r+s+t=n-k} B_rB_sB_t + \sum_{r+s+t=n-k} \delta_{1,r}B_sB_t + \sum_{r+s+t=n-k} B_r\delta_{1,s}B_t \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{r+s+t=n-k} B_r B_s \delta_{1,t} + \sum_{r+s+t=n-k} \delta_{1,r} \delta_{1,s} B_t + \sum_{r+s+t=n-k} \delta_{1,r} B_s \delta_{1,t} \\
& + \sum_{r+s+t=n-k} B_r \delta_{1,s} \delta_{1,t} + \sum_{r+s+t=n-k} \delta_{1,r} \delta_{1,s} \delta_{1,t} \Big\}. \tag{2.9}
\end{aligned}$$

Therefore, by (2.7) and (2.9), we obtain the following theorem.

Theorem 2.1. For $r, s, t \in \mathbb{Z}_+$, and $n \in \mathbb{N}$ with $n \geq 3$, one has

$$\begin{aligned}
& \sum_{r+s+t=n} B_r(x) B_s(x) B_t(x) \\
& = \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ 2 \sum_{r+s+t=n-k} B_r B_s B_t + 3 \sum_{r+s=n-k-1} B_r B_s + 3B_{n-k-2} + \delta_{k,n-3} \right\} E_k(x) \\
& + \binom{n+2}{2} E_n(x). \tag{2.10}
\end{aligned}$$

Let us take the polynomial $p(x)$ as follows:

$$p(x) = \sum_{r+s+t=n} B_r(x) B_s(x) E_t(x). \tag{2.11}$$

Then, by (2.11), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3) \sum_{r+s+t=n-k} B_r(x) B_s(x) E_t(x). \tag{2.12}$$

From (1.6), (1.7), and (2.12), we have

$$\begin{aligned}
b_k & = \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} = \frac{\binom{n+2}{k}}{2} \sum_{r+s+t=n-k} \{ B_r(1) B_s(1) E_t(1) + B_r B_s E_t \} \\
& = \frac{\binom{n+2}{k}}{2} \sum_{r+s+t=n-k} \{ (B_r + \delta_{1,r})(B_s + \delta_{1,s})(-E_t + 2\delta_{0,t}) + B_r B_s E_t \} \\
& = \frac{\binom{n+2}{k}}{2} \left\{ - \sum_{r+s+t=n-k} \delta_{1,r} B_s E_t - \sum_{r+s+t=n-k} B_r \delta_{1,s} E_t + 2 \sum_{r+s+t=n-k} B_r B_s \delta_{0,t} \right. \\
& \quad \left. - \sum_{r+s+t=n-k} \delta_{1,r} \delta_{1,s} E_t + 2 \sum_{r+s+t=n-k} \delta_{1,r} B_s \delta_{0,t} + 2 \sum_{r+s+t=n-k} B_r \delta_{1,s} \delta_{0,t} \right. \\
& \quad \left. + 2 \sum_{r+s+t=n-k} \delta_{1,r} \delta_{1,s} \delta_{0,t} \right\}. \tag{2.13}
\end{aligned}$$

Note that

$$\begin{aligned} b_{n-1} &= \binom{n+2}{n-1} \left\{ -\sum_{s+t=0} B_s E_t - \sum_{r+t=0} B_r E_t + 2 \sum_{r+s=1} B_r B_s - 0 + 2B_0 + 2B_0 + 2 \cdot 0 \right\} \\ &= \frac{1}{2} \binom{n+2}{n-1} \{-1 - 1 + 2(B_1 + B_1) + 2 + 2\} = 0. \end{aligned} \quad (2.14)$$

Therefore, we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\begin{aligned} &\sum_{r+s+t=n} B_r(x) B_s(x) E_t(x) \\ &= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ 2 \sum_{r+s=n-k} B_r B_s - 2 \sum_{r+t=n-k-1} B_r E_t - E_{n-k-2} + 4B_{n-k-1} + 2\delta_{k,n-2} \right\} E_k(x) \\ &\quad + \binom{n+2}{2} E_n(x). \end{aligned} \quad (2.15)$$

Remark 2.3. By the same method, we obtain the following identities.

(I)

$$\begin{aligned} &\sum_{r+s+t=n} B_r(x) E_s(x) E_t(x) \\ &= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ 2 \sum_{r+s+t=n-k} B_r E_s E_t + \sum_{s+t=n-k-1} E_s E_t - 4 \sum_{r+s=n-k} B_r E_s + 4B_{n-k} - 4E_{n-k-1} \right\} E_k(x) \\ &\quad + \binom{n+2}{2} E_n(x). \end{aligned} \quad (2.16)$$

(II)

$$\begin{aligned} &\sum_{r+s+t=n} E_r(x) E_s(x) E_t(x) \\ &= 3 \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ \sum_{r+s=n-k} E_r E_s - 2E_{n-k} \right\} E_k(x) + \binom{n+2}{2} E_n(x). \end{aligned} \quad (2.17)$$

Let us consider the polynomial $p(x)$ as follows:

$$p(x) = \sum_{r+s+t=n} B_r(x) B_s(x) x^t. \quad (2.18)$$

Thus, by (2.18), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3) \sum_{r+s+t=n-k} B_r(x)B_s(x)x^t. \quad (2.19)$$

From (1.6), (1.7), (2.18), and (2.19), we have

$$\begin{aligned} b_k &= \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} = \frac{\binom{n+2}{k}}{2} \sum_{r+s+t=n-k} \{ B_r(1)B_s(1) + B_rB_s0^t \} \\ &= \frac{\binom{n+2}{k}}{2} \sum_{r+s+t=n-k} \{ (B_r + \delta_{1,r})(B_s + \delta_{1,s}) + B_rB_s0^t \} \\ &= \frac{\binom{n+2}{k}}{2} \left\{ \sum_{r+s+t=n-k} B_rB_s + \sum_{r+s+t=n-k} B_r\delta_{1,s} + \sum_{r+s+t=n-k} \delta_{1,r}B_s + \sum_{r+s+t=n-k} \delta_{1,r}\delta_{1,s} + \sum_{r+s+t=n-k} B_rB_s0^t \right\}. \end{aligned} \quad (2.20)$$

Here we note that

$$\begin{aligned} \sum_{r+s+t=n-k} B_rB_s &= \sum_{t=0}^{n-k} \sum_{r+s=n-k-t} B_rB_s = \sum_{t=0}^{n-k} \sum_{r+s=t} B_rB_s \\ \sum_{r+s+t=n-k} B_r\delta_{1,s} &= \begin{cases} \sum_{r=0}^{n-k-1} B_r, & \text{if } k \leq n-1, \\ 0, & \text{if } k = n, \end{cases} \\ \sum_{r+s+t=n-k} B_s\delta_{1,r} &= \begin{cases} \sum_{r=0}^{n-k-1} B_r, & \text{if } k \leq n-1, \\ 0, & \text{if } k = n, \end{cases} \\ \sum_{r+s+t=n-k} \delta_{1,r}\delta_{1,s} &= \begin{cases} 1, & \text{if } k \leq n-2, \\ 0, & \text{if } k = n-1 \text{ or } n, \end{cases} \\ \sum_{r+s+t=n-k} B_rB_s0^t &= \sum_{r+s=n-k} B_rB_s, \quad \forall k. \end{aligned} \quad (2.21)$$

It is easy to show that

$$\begin{aligned} b_{n-1} &= \frac{1}{2} \binom{n+2}{n-1} \left\{ \sum_{r+s=0} B_rB_s + 2 \sum_{r+s=1} B_rB_s + 2B_0 \right\} \\ &= \frac{1}{2} \binom{n+2}{n-1} \{ 1 + 2(B_1 + B_2) + 2 \} = \frac{1}{2} \binom{n+2}{n-1}. \end{aligned} \quad (2.22)$$

Therefore, by (1.6), (2.18), (2.20), and (2.22), we obtain the following theorem.

Theorem 2.4. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\begin{aligned} & \sum_{r+s+t=n} B_r(x)B_s(x)x^t \\ &= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ \sum_{t=0}^{n-k-1} \sum_{r+s=t} B_r B_s + 2 \sum_{r+s=n-k} B_r B_s + 2 \sum_{r=0}^{n-k-1} B_r + 1 \right\} E_k(x) \\ &+ \frac{1}{2} \binom{n+2}{n-1} E_{n-1}(x) + \binom{n+2}{n} E_n(x). \end{aligned} \quad (2.23)$$

Remark 2.5. By the same method, we can derive the following identities.

(I)

$$\begin{aligned} & \sum_{r+s+t=n} B_r(x)E_s(x)x^t \\ &= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ - \sum_{t=0}^{n-k-1} \sum_{r+s=t} B_r E_s - \sum_{s=0}^{n-k-1} E_s + 2 \sum_{r=0}^{n-k} B_r + 2 \right\} E_k(x) \\ &+ \frac{1}{2} \binom{n+2}{n-1} E_{n-1}(x) + \binom{n+2}{n} E_n(x). \end{aligned} \quad (2.24)$$

(II)

$$\begin{aligned} & \sum_{r+s+t=n} E_r(x)E_s(x)x^t \\ &= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+2}{k} \left\{ \sum_{t=0}^{n-k-1} \sum_{r+s=t} E_r E_s + 2 \sum_{r+s=n-k} E_r E_s - 4 \sum_{r=0}^{n-k} E_r + 4 \right\} E_k(x) \\ &+ \frac{1}{2} \binom{n+2}{n-1} E_{n-1}(x) + \binom{n+2}{2} E_n(x). \end{aligned} \quad (2.25)$$

Now we generalize the above consideration to the completely arbitrary case. Let

$$p(x) = \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x), \quad (2.26)$$

where the sum runs over all nonnegative integers $i_1, i_2, \dots, i_r, j_1, \dots, j_s$ satisfying $i_1 + i_2 + \dots + i_r + j_1 + \dots + j_s = n$. From (2.26), we note that

$$p^{(k)}(x) = (n+r+s-1) \cdots (n+r+s-k) \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} B_{i_1}(x) \cdots B_{i_r}(x) \times E_{j_1}(x) \cdots E_{j_s}(x). \quad (2.27)$$

By (1.6), (1.7), (2.18), and (2.27), we get

$$\begin{aligned}
b_k &= \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \\
&= \frac{1}{2} \binom{n+r+s-1}{k} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \{ B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) + B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \} \\
&= \frac{1}{2} \binom{n+r+s-1}{k} \\
&\quad \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \{ (B_{i_1} + \delta_{1,i_1}) \cdots (B_{i_r} + \delta_{1,i_r}) \\
&\quad \times (-E_{j_1} + 2\delta_{0,j_1}) \cdots (-E_{j_s} + 2\delta_{0,j_s}) + B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \} \\
&= \frac{1}{2} \binom{n+r+s-1}{k} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a-k-r} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}. \tag{2.28}
\end{aligned}$$

Note that

$$\begin{aligned}
b_n &= \frac{1}{2} \binom{n+r+s-1}{n} \left\{ \sum_{0 \leq c \leq s} \binom{s}{c} (-1)^c 2^{s-c} \times \sum_{i_1+\dots+i_r+j_1+\dots+j_c=0} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_c} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=0} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} \\
&= \frac{1}{2} \binom{n+r+s-1}{n} ((2-1)^s + 1) = \binom{n+r+s-1}{n}, \\
b_{n-1} &= \frac{1}{2} \binom{n+r+s-1}{n-1} \left\{ \sum_{\substack{r-1 \leq a \leq r \\ 0 \leq c \leq s}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\
&\quad \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=1+a-r} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=1} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \binom{n+r+s-1}{n-1} \left\{ r(2-1)^s + \sum_{0 \leq c \leq s} \binom{s}{c} (-1)^c 2^{s-c} \left[-\frac{1}{2}(r+c) \right] - \frac{1}{2}(r+s) \right\} \\
&= \frac{1}{2} \binom{n+r+s-1}{n-1} \left\{ r - \frac{1}{2}r + \frac{1}{2}s - \frac{1}{2}r - \frac{1}{2}s \right\} = 0.
\end{aligned} \tag{2.29}$$

Therefore, by (1.6), (2.28), and (2.29), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
&= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+r+s-1}{k} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\
&\quad \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a-k-r} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} E_k(x) \\
&+ \binom{n+r+s-1}{n} E_n(x).
\end{aligned} \tag{2.30}$$

Let us consider the polynomial $p(x)$ of degree n as

$$p(x) = \sum_{t+i_1+\dots+i_r+j_1+\dots+j_s=n} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t. \tag{2.31}$$

Then, from (2.31), we have

$$\begin{aligned}
p^{(k)}(x) &= (n+r+s)(n+r+s-1) \cdots (n+r+s-k+1) \\
&\quad \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t.
\end{aligned} \tag{2.32}$$

By (1.7) and (2.32), we get

$$\begin{aligned}
b_k &= \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \\
&= \frac{1}{2} \binom{n+r+s}{k} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k} \left\{ B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) + B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} 0^t \right\} \\
&= \frac{1}{2} \binom{n+r+s}{k} \\
&\times \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k} \left\{ (B_{i_1} + \delta_{1,i_1}) \cdots (B_{i_r} + \delta_{1,i_r}) \right. \\
&\quad \left. \times (-E_{j_0} + 2\delta_{0,j_1}) \cdots (-E_{j_s} + 2\delta_{1,j_s}) + B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} 0^t \right\} \tag{2.33}
\end{aligned}$$

From (2.33), we can derive the following equation:

$$\begin{aligned}
b_k &= \frac{1}{2} \binom{n+r+s}{k} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \times \sum_{t=0}^{n+a-k-r} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}. \tag{2.34}
\end{aligned}$$

Observe now that

$$\begin{aligned}
b_n &= \frac{1}{2} \binom{n+r+s}{n} \left\{ \sum_{c=0}^s \binom{s}{c} (-1)^c 2^{s-c} \times \sum_{i_1+\dots+i_r+j_1+\dots+j_c=0} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_c} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=0} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} \tag{2.35} \\
&= \frac{1}{2} \binom{n+r+s}{n} [(2-1)^s + 1] = \binom{n+r+s}{n},
\end{aligned}$$

$$\begin{aligned}
b_{n-1} &= \frac{1}{2} \binom{n+r+s}{n-1} \left\{ \sum_{\substack{0 \leq c \leq s \\ 0 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \times \sum_{t=0}^{1+a-r} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=1} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} \\
&= \frac{1}{2} \binom{n+r+s}{n-1} \left\{ r+1 - \frac{1}{2}r + \frac{1}{2}s - \frac{1}{2}r - \frac{1}{2}s \right\} = \frac{1}{2} \binom{n+r+s}{n-1}. \tag{2.36}
\end{aligned}$$

Therefore, by (1.6), (2.31), (2.34), (2.35), and (2.36), we obtain the following theorem.

Theorem 2.7. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t \\
&= \frac{1}{2} \sum_{k=0}^{n-2} \binom{n+r+s}{k} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\
&\quad \times \sum_{t=0}^{n+a-k-r} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} E_k(x) \tag{2.37} \\
&+ \frac{1}{2} \binom{n+r+s}{n-1} E_{n-1}(x) + \binom{n+r+s}{n} E_n(x).
\end{aligned}$$

Let us consider the following polynomial of degree n .

$$p(x) = \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} \frac{1}{i_1! \cdots i_r! j_1! \cdots j_s!} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x). \tag{2.38}$$

Thus, by (2.38), we get

$$p^{(k)}(x) = (r+s)^k \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{1}{i_1! \cdots i_r! j_1! \cdots j_s!} \times B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x). \tag{2.39}$$

From (1.7), we have

$$\begin{aligned}
b_k &= \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \\
&= \frac{(r+s)^k}{2k!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{1}{i_1! \cdots i_r! j_1! \cdots j_s!} \\
&\quad \times \{ B_{i_1}(1) \cdots B_{i_r}(1) \times E_{j_1}(1) \cdots E_{j_s}(1) + B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \} \\
&= \frac{(r+s)^k}{2k!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{1}{i_1! \cdots i_r! j_1! \cdots j_s!} \\
&\quad \times \{ (B_{i_1} + \delta_{1,i_1}) \cdots (B_{i_r} + \delta_{1,i_r}) \times (-E_{j_1} + 2\delta_{0,j_1}) \cdots (-E_{j_s} + 2\delta_{0,j_s}) + B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \}. \tag{2.40}
\end{aligned}$$

Thus, by (2.40), we get

$$\begin{aligned}
b_k &= \frac{(r+s)^k}{2k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a-k-r} \frac{B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c}}{i_1! \cdots i_a! j_1! \cdots j_c!} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1! \cdots i_r! j_1! \cdots j_s!} \right\}. \tag{2.41}
\end{aligned}$$

Now, we note that

$$\begin{aligned}
b_n &= \frac{(r+s)^n}{2n!} \left\{ \sum_{c=0}^s \binom{s}{c} (-1)^c 2^{s-c} \right. \\
&\quad \times \sum_{i_1+\dots+i_r+j_1+\dots+j_c=0} \frac{B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_c}}{i_1! \cdots i_r! j_1! \cdots j_c!} + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=0} \frac{B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1! \cdots i_r! j_1! \cdots j_s!} \left. \right\} \\
&= \frac{(r+s)^n}{2n!} [(2-1)^s + 1] = \frac{(r+s)^n}{n!},
\end{aligned}$$

$$\begin{aligned}
b_{n-1} &= \frac{(r+s)^{n-1}}{2(n-1)!} \left\{ \sum_{\substack{r-1 \leq a \leq r \\ 0 \leq c \leq s}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\
&\quad \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=1+a-r} \frac{B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c}}{i_1! \cdots i_a! j_1! \cdots j_c!} + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=1} \frac{B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1! \cdots i_r! j_1! \cdots j_s!} \Bigg\} \\
&= \frac{(r+s)^{n-1}}{2(n-1)!} \left\{ r(2-1)^s + \sum_{c=0}^s \binom{s}{c} (-1)^c 2^{s-c} \left[-\frac{1}{2}(r+c) \right] - \frac{1}{2}(r+s) \right\} = 0.
\end{aligned} \tag{2.42}$$

Therefore, by (1.6), (2.38), (2.41), and (2.42), we obtain the following theorem.

Theorem 2.8. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} \frac{B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x)}{i_1! \cdots i_r! j_1! \cdots j_s!} \\
&= \frac{1}{2} \sum_{k=0}^{n-2} \frac{(r+s)^k}{k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \times \right. \\
&\quad \left. \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a-k-r} \frac{B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c}}{i_1! \cdots i_a! j_1! \cdots j_c!} \right. \\
&\quad \left. + \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1! \cdots i_r! j_1! \cdots j_s!} \right\} E_k(x) \\
&+ \frac{(r+s)^n}{n!} E_n(x).
\end{aligned} \tag{2.43}$$

By the same method, we can obtain the following identity:

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \frac{B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t}{i_1! \cdots i_r! j_1! \cdots j_s! t!} \\
&= \frac{1}{2} \sum_{k=0}^{n-2} \frac{(r+s+1)^k}{k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ a \geq k+r-n}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\
&\quad \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{t=0}^{n+a-k-r} \frac{1}{(n+a-k-r-t)!} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} \frac{B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c}}{i_1! \cdots i_a! j_1! \cdots j_c!} \\
& + \left. \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1! \cdots i_r! j_1! \cdots j_s!} \right\} E_k(x) \\
& + \frac{(r+s+1)^{n-1}}{2(n-1)!} E_{n-1}(x) + \frac{(r+s+1)^n}{n!} E_n(x).
\end{aligned} \tag{2.44}$$

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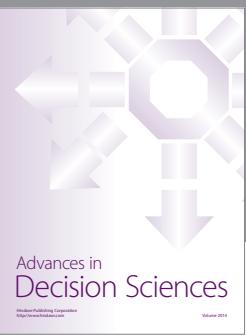
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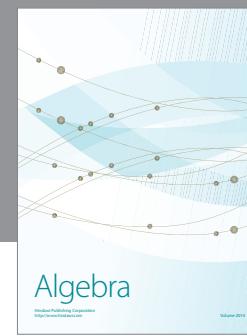
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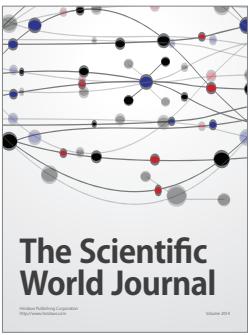
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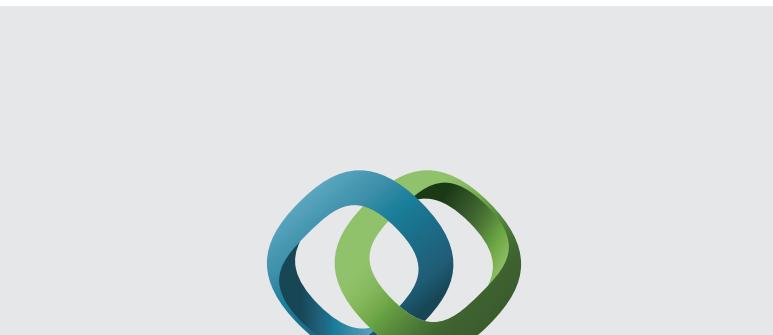
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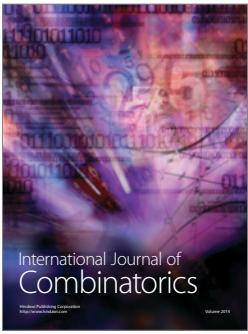


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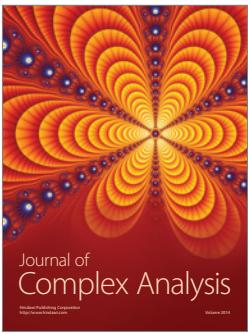
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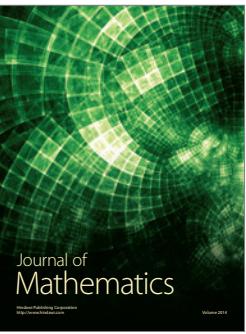
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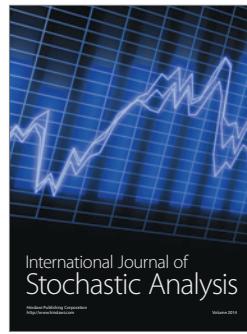
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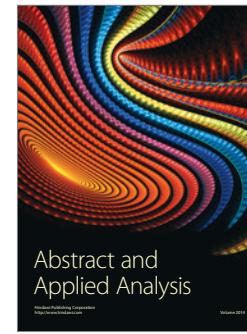
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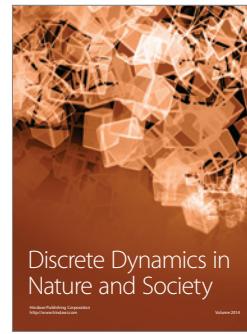
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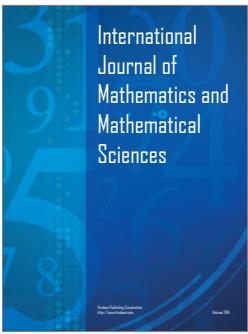
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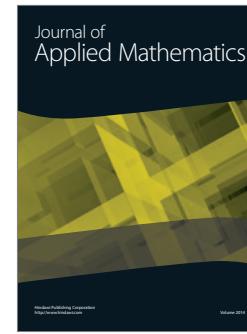
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