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Research Article

Morita Equivalence of Brandt Semigroup Algebras

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We show that Banach semigroup algebras of any two Brandt semigroups over a fixed group are Morita equivalence with respect to the Morita theory of self-induced Banach algebras introduced by Grønbæk. As applications, we show that the bounded Hochschild (co)homology groups of Brandt semigroup algebras over amenable groups are trivial and prove that the notion of approximate amenability is not Morita invariant.

1. Introduction

Morita theory is a very useful tool in the study of rings and algebras. In the area of topological algebras, there are different notions of Morita theory in the literature, but all of these notions are simple variants of the original one, defined by Kiiti Morita, in the pure algebraic case. Niels Grønbæk defined a Morita theory for Banach algebras with bounded approximate identities in [1]. Then he extended his theory to the larger class of self-induced Banach algebras in [2]. In the first theory, the only class of algebras that are Morita equivalent to the algebra of complex numbers $\mathbb{C}$ is the class of finite dimensional matrix algebras, but in the second one we find many infinite dimensional Banach algebras that are Morita equivalent to $\mathbb{C}$. In this paper we construct a new class of infinite dimensional self-induced Banach algebras Morita equivalent to $\mathbb{C}$. Then by this construction, we show that for a discrete group G, and every two nonempty sets I, J, the Banach convolution algebras $\ell^1(B(I,G))$ and $\ell^1(B(J,G))$ are Morita equivalent, where $B(I,G)$ denotes the Brandt semigroup over G with index set I, [3]. Brandt semigroups are one of the most important classes of inverse semigroups paid. Some authors have studied the bounded Hochschild cohomology and amenability properties of inverse semigroup algebras, [4–9]. As a corollary of Morita equivalence of Brandt semigroup algebras and a strong result of [2], we show that if G is an amenable semigroup and I is an arbitrary nonempty set, then the topological Hochschild homology and cohomology groups $\mathscr{H}_n(\ell^1(B(I,G)),E)$ and $\mathscr{H}^n(\ell^1(B(I,G)),E^*)$, for any $n > 0$ and every $\ell^1(B(I,G))$-induced Banach bimodule E, are trivial. Also by a specific example, we show that the notion of approximate amenability of Banach algebras, introduced by Ghahramani and Loy [10], is not Morita invariant. This result is in contrast to Grønbæk's corollary on amenability [1, Corollary 6.5] that says for Banach algebras with bounded approximate identities the notion of amenability is Morita invariant.

2. Preliminaries

Throughout this paper, for an element x of a set X, δ_x is its point mass measure in $\ell^1(X)$. Let E and F be Banach spaces. The Banach space which is the completed projective tensor product of E and F is denoted by $E\widehat{\otimes}F$; for $z \in E\widehat{\otimes}F$, there are sequences $(x_n) \in E$ and $(y_n) \in F$ such that $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty$ and $z = \sum_{n=1}^{\infty} x_n \otimes y_n$. Analogous to the pure algebraic case, the key property of $E\widehat{\otimes}F$ is that, for each continuous bilinear map $B : E \times F \to D$, where D is a Banach space, there is a unique continuous linear map $T : E\widehat{\otimes}F \to D$ with $\|T\| = \|B\|$ and $T(x \otimes y) = B(x,y)$ $(x \in E, y \in F)$, see [11] for more details.

It is well known that for nonempty sets X and Y, the map $\delta_x\otimes\delta_y \mapsto \delta_{(x,y)}$ $(x \in X, y \in Y)$ defines an isometric isomorphism between Banach spaces $\ell^1(X)\widehat{\otimes}\ell^1(Y)$ and $\ell^1(X\times Y)$; we use frequently this identification.

Let A, B, and C be Banach algebras. A left Banach A-module E is an ordinary left A-module which is a Banach space and there is a constant $M > 0$ such that $\|a \cdot x\| \le M\|a\|\|x\|$ $(a \in A, x \in E)$. Similarly, right Banach modules and Banach bimodules are defined. The category of left Banach A-modules and bounded module homomorphisms is denoted by A-mod. Similarly, one can define the category of right Banach A-modules mod-A, and the category of left-A right-B Banach bimodules A-mod-B. The notations ${}_AE$, E_A, and ${}_AE_B$ are shorthand indications that E is in A-mod, mod-A, and A-mod-B, respectively.

For E in mod-A and F in A-mod, let $E\widehat{\otimes}_AF$ be the universal object for A-balanced bounded bilinear maps from $E \times F$. This can be realized as the Banach space $E\widehat{\otimes}F/N$ where N is the closed linear span of $\{x \cdot a \otimes y - x \otimes a \cdot y : x \in E, y \in F, a \in A\}$ [11, 12]. For ${}_BE_A$ and ${}_AF_C$ the tensor product $E\widehat{\otimes}_AF$ is in B-mod-C. For ${}_AE$, define a left module homomorphism $\mu_E : A\widehat{\otimes}_AE \to E$ by $\mu_E(a \otimes x) = a \cdot x$ $(a \in A, x \in E)$.

The Banach algebra A is called self-induced if the multiplication map $a \otimes b \to ab$, from $A\widehat{\otimes}_AA$ to A, is an isomorphism (between A bimodules). More generally, a left Banach A-module E, is A induced if μ_E is an isomorphism [13]. Similarly, right-and two-sided-induced modules are defined. The category of left Banach A-induced modules is denoted by ind-A-mod.

Let A and B be two self-induced Banach algebras. Then A and B are Morita equivalent if ind-A-mod and ind-B-mod are equivalent, i.e., there are covariant functors

$$\Phi : \text{ind-}A\text{-mod} \longrightarrow \text{ind-}B\text{-mod} \qquad \Psi : \text{ind-}B\text{-mod} \longrightarrow \text{ind-}A\text{-mod} \tag{2.1}$$

such that $\Psi\Phi$ and $\Phi\Psi$ are natural isomorphic to the identity functors on ind-A-mod and ind-B-mod, respectively. For complete definitions see the original paper [2]. We only need the following characterization of Morita equivalence [2].

Theorem 2.1. *Let A and B be self-induced Banach algebras. Then A and B are Morita equivalent if and only if there are two-sided-induced modules $P \in B$-mod-A and $Q \in A$-mod-B such that $P\widehat{\otimes}_A Q \cong B$ and $Q\widehat{\otimes}_B P \cong A$, where $\cong$ denotes topological isomorphism of bimodules.*

3. A Banach Algebra Morita Equivalent to $\mathbb{C}$

In this section, for any set I, we define a *matrix-like* Banach algebra $\mathcal{M}_I$ and prove that $\mathcal{M}_I$ is Morita equivalent to the algebra of complex numbers $\mathbb{C}$.

Let I be a nonempty set. Let the underlying Banach space of $\mathcal{M}_I$ be $\ell^1(I \times I)$ and let its multiplication be the convolution product

$$(ab)(i,j) = \sum_{k\in I} a(i,k)b(k,j) \quad (a,b \in \mathcal{M}_I,\ i,j \in I). \tag{3.1}$$

Note that if I is a finite set, then $\mathcal{M}_I$ is isomorphic to an ordinary matrix algebra. Also, for any $i,p,q,j \in I$, we have the following identity in $\mathcal{M}_I$:

$$\delta_{(i,p)}\delta_{(q,j)} = \begin{cases} \delta_{(i,j)} & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases} \tag{3.2}$$

Define a two-sided Banach module action of $\mathcal{M}_I$ on $\ell^1(I)$ by

$$(a\cdot b)(i) = \sum_{k\in I} a(i,k)b(k), \qquad (b\cdot a)(i) = \sum_{k\in I} b(k)a(k,i), \tag{3.3}$$

for $a \in \mathcal{M}_I,\ b \in \ell^1(I),\ i \in I$.

Lemma 3.1. *The map $\nu : \ell^1(I)\widehat{\otimes}_{\mathcal{M}_I}\ell^1(I) \to \mathbb{C}$, defined by $\nu(a\otimes b) = \sum_{i\in I} a(i)b(i)$, is an isomorphism of Banach spaces.*

Proof. By definition, it is enough to prove that the map

$$\overline{\nu} : \ell^1(I)\widehat{\otimes}\ell^1(I) \longrightarrow \mathbb{C}, \tag{3.4}$$

defined by $\overline{\nu}(a\otimes b) = \sum_{i\in I} a(i)b(i)$, is nonzero and N, the closed linear span of $\{\delta_i\cdot\delta_{(j,k)}\otimes\delta_{i'} - \delta_i\otimes\delta_{(j,k)}\cdot\delta_{i'} : i,i',j,k \in I\}$, is equal to $\ker\overline{\nu}$.

If $k_0 \in I$ is arbitrary, then $\overline{\nu}(\delta_{k_0}\otimes\delta_{k_0}) = 1$. This shows that $\overline{\nu}$ is not zero.

A simple computation shows that for every $i,i',j,k \in I$, $\overline{\nu}(\delta_i\cdot\delta_{(j,k)}\otimes\delta_{i'}) = \overline{\nu}(\delta_i\otimes\delta_{(j,k)}\cdot\delta_{i'})$. This implies that $N \subseteq \ker\overline{\nu}$.

For the converse, we have

$$\delta_{(i,j)} \in N, \quad \text{if } i \neq j, \tag{3.5}$$

since $\delta_i\cdot\delta_{(j,k)}\otimes\delta_k - \delta_i\otimes\delta_{(j,k)}\cdot\delta_k = -\delta_i\otimes\delta_j$. Also for every $i,j \in I$, we have

$$\delta_{(j,j)} - \delta_{(i,i)} \in N, \tag{3.6}$$

since $\delta_i \cdot \delta_{(i,j)} \otimes \delta_j - \delta_i \otimes \delta_{(i,j)} \cdot \delta_j = \delta_j \otimes \delta_j - \delta_i \otimes \delta_i$. Now suppose that $c = \sum_{i,j\in I} c(i,j)\delta_{(i,j)}$ is in $\ker \overline{\nu}$. Thus we have

$$\overline{\nu}(c) = \sum_{i\in I} c(i,i) = 0. \tag{3.7}$$

Consider the following decomposition of c:

$$c = \sum_{i,j\in I, i\neq j} c(i,j)\delta_{(i,j)} + \sum_{i\in I} c(i,i)\delta_{(i,i)} = a + b. \tag{3.8}$$

Then by (3.5), a is in N. Let $k_0 \in I$ be arbitrary and fixed, then by (3.7), we have $b = \sum_{i\in I} c(i,i)\delta_{(i,i)} - \sum_{i\in I} c(i,i)\delta_{(k_0,k_0)}$. Thus by (3.6), b is also in N. Therefore c is in N and $\ker \overline{\nu} \subseteq N$. □

Proposition 3.2. *(i) $\ell^1(I)$ is a two-sided $\mathcal{M}_I$-induced module.*
(ii) $\mathcal{M}_I$ is a self-induced Banach algebra.

Proof. The canonical map $\delta_i \otimes \delta_j \mapsto \delta_{(i,j)}$ $(i,j \in I)$, from $\ell^1(I)\widehat{\otimes}\ell^1(I)$ to $\mathcal{M}_I$, is an isomorphism of Banach $\mathcal{M}_I$ bimodules. Thus we have,

$$\begin{aligned}
\mathcal{M}_I \widehat{\otimes}_{\mathcal{M}_I} \ell^1(I) &\cong \left(\ell^1(I)\widehat{\otimes}\ell^1(I)\right)\widehat{\otimes}_{\mathcal{M}_I} \ell^1(I) \\
&\cong \ell^1(I)\widehat{\otimes}\left(\ell^1(I)\widehat{\otimes}_{\mathcal{M}_I} \ell^1(I)\right) \\
&\cong \ell^1(I)\widehat{\otimes}\mathbb{C} \quad (\text{by Lemma 3.1}) \\
&\cong \ell^1(I).
\end{aligned} \tag{3.9}$$

This proves $\ell^1(I)$ is left $\mathcal{M}_I$ induced. Similarly, it is proved that $\ell^1(I)$ is right $\mathcal{M}_I$ induced. For (ii), we have

$$\begin{aligned}
\mathcal{M}_I \widehat{\otimes}_{\mathcal{M}_I} \mathcal{M}_I &\cong \left(\ell^1(I)\widehat{\otimes}\ell^1(I)\right)\widehat{\otimes}_{\mathcal{M}_I} \mathcal{M}_I \\
&\cong \ell^1(I)\widehat{\otimes}\left(\ell^1(I)\widehat{\otimes}_{\mathcal{M}_I} \mathcal{M}_I\right) \\
&\cong \ell^1(I)\widehat{\otimes}\ell^1(I) \quad (\text{by (i)}) \\
&\cong \mathcal{M}_I.
\end{aligned} \tag{3.10}$$

Thus $\mathcal{M}_I$ is self-induced. □

Theorem 3.3. *$\mathcal{M}_I$ is Morita equivalent to $\mathbb{C}$.*

Proof. By Lemma 3.1 and Proposition 3.2, the Banach algebras $A = \mathcal{M}_I$, $B = \mathbb{C}$ and Banach bimodules $P = {}_{\mathbb{C}}\ell^1(I)_{\mathcal{M}_I}$, $Q = {}_{\mathcal{M}_I}\ell^1(I)_{\mathbb{C}}$ satisfy conditions of Theorem 2.1. Thus $\mathcal{M}_I$ is Morita equivalent to $\mathbb{C}$. □

Remark 3.4. (I) It is proved in [2] that for any Banach space E, the tensor algebra $E\widehat{\otimes}E^*$ is Morita equivalent to $\mathbb{C}$. Also, it is well known that if E has *bounded approximate property*, then the algebra $\mathcal{N}(E)$ of nuclear operators on E and $E\widehat{\otimes}E^*$ is isomorphic. Thus by Theorem 3.3, $\mathcal{N}(\ell^1(I)) \cong \ell^1(I)\widehat{\otimes}\ell^\infty(I)$ and $\mathcal{M}_I$ are Morita equivalent, but clearly these are not isomorphic if I is an infinite set.

(II) For Morita theory of some other Matrix-like algebras, see [1, 2, 14].

4. The Main Result

Let I be a nonempty set and let G be a discrete group. Consider the set $T = I \times G \times I$, add an extra element $\o$ to T and define a semigroup multiplication on $S = T \cup \{\o\}$, as follows. For $i, i', j, j' \in I$ and $g, g' \in G$, let

$$(i,g,j)(i',g',j') = \begin{cases} (i,gg',j') & \text{if } j = i' \\ \o & \text{if } j \neq i', \end{cases} \tag{4.1}$$

also let $\o(i,g,j) = (i,g,j)\o = \o$ and $\o\o = \o$. Then S becomes a semigroup that is called the Brandt semigroup over G with index set I and usually denoted by $B(I,G)$. For more details see [3].

The Banach space $\ell^1(T)$, with the convolution product

$$(ab)(i,g,j) = \sum_{k\in I, h\in G} a\Big(i, gh^{-1}, k\Big) b(k,h,j), \tag{4.2}$$

for $a, b \in \ell^1(T), i, j \in I, g \in G$ becomes a Banach algebra.

Lemma 4.1. *The Banach algebras $\ell^1(S)$ and $\ell^1(T) \oplus \mathbb{C}$ are homeomorphically isomorphic, where the multiplication of $\ell^1(T) \oplus \mathbb{C}$ is coordinatewise.*

Proof. Consider the following short exact sequence of Banach algebras and continuous algebra homomorphisms.

$$0 \longrightarrow \ell^1(T) \longrightarrow \ell^1(S) \longrightarrow \mathbb{C} \longrightarrow 0, \tag{4.3}$$

where the second arrow $u : \ell^1(T) \to \ell^1(S)$ is defined by $u(b)(t) = b(t)$ and $u(b)(\o) = -\sum_{s\in T} b(s)$, for $b \in \ell^1(T)$ and $t \in T \subset S$, and the third arrow $v : \ell^1(S) \to \mathbb{C}$ is the integral functional, $v(a) = \sum_{s\in S} a(s)$ $(a \in \ell^1(S))$. Now, let $w : \ell^1(S) \to \ell^1(T)$ be the restriction map, $w(a) := a|_T$. Then w is a continuous algebra homomorphism and $wu = Id_{\ell^1(T)}$. Thus the exact sequence splits and we have $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$. □

The following Theorem is our main result.

Theorem 4.2. *Let I and J be nonempty sets and let G be a discrete group. Then $\ell^1(B(I,G))$ and $\ell^1(B(J,G))$ are Morita equivalent self-induced Banach algebras.*

Proof. Let T be as above. It is easily checked that the map $\delta_{(i,j)} \otimes \delta_g \mapsto \delta_{(i,g,j)}$ $(i, j \in I, g \in G)$ is an isometric isomorphism from the Banach algebra $\mathcal{M}_I \widehat{\otimes} \ell^1(G)$ onto $\ell^1(T)$. Thus $\ell^1(T)$ is self-induced, since $\mathcal{M}_I$ and $\ell^1(G)$ are self-induced. Also, since $\mathcal{M}_I \approx \mathbb{C}$, we have $\ell^1(T) \approx \mathbb{C}\widehat{\otimes}\ell^1(G) \approx \ell^1(G)$. By Lemma 4.1, we have $\ell^1(B(I,G)) \cong \ell^1(T) \oplus \mathbb{C}$ thus $\ell^1(B(I,G))$ is self-induced and Morita equivalent to $\ell^1(G) \oplus \mathbb{C}$. Similarly $\ell^1(B(J,G)) \approx \ell^1(G) \oplus \mathbb{C}$, therefore we have $\ell^1(B(I,G)) \approx \ell^1(B(J,G))$. □

5. Some Applications

For the topological Hochschild (co)homology of Banach algebras, we refer the reader to [11]. Recall that a Banach algebra A is amenable if for every Banach A-bimodule E, the first-order bounded Hochschild cohomology group of A with coefficients in the dual Banach bimodule E^* vanishes, $\mathcal{H}^1(A, E^*) = 0$, or equivalently any bounded derivation $D : A \to E^*$ is inner. A famous Theorem proved by Johnson [15] says that for any locally compact group G, amenability of G is equivalent to the amenability of the convolution group algebra $L^1(G)$. For a modern account on amenability see [16].

Proposition 5.1. *Let A and B be Morita equivalent self-induced Banach algebras. Suppose that A is amenable. Then for every two-sided B-induced module $E \in B$-mod-B, and $n \geq 1$, $\mathcal{H}_n(B, E) = 0$, and the complete quotient seminorm of $\mathcal{H}_0(B, E)$ is a norm.*

Proof. Corollary IV.10 of [2]. □

Theorem 5.2. *Let G be an amenable discrete group, I be a nonempty set and $A = \ell^1(B(I,G))$. Then for any two-sided A induced Banach A bimodule E and every $n > 0$, the topological Hochschild homology groups $\mathcal{H}_n(A, E)$ are trivial and $\mathcal{H}_0(A, E)$ is a Banach space.*

Proof. It was proved in the preceding section that $\ell^1(B(I,G)) \approx \ell^1(G) \oplus \mathbb{C}$. By Johnson's Theorem, $\ell^1(G)$ is an amenable Banach algebra and thus so is $\ell^1(G) \oplus \mathbb{C}$. Now, apply Proposition 5.1. □

Note that for any self-induced Banach algebra A, the class of two sided A induced modules is very wide, since for any Banach A bimodule F, the module $A\widehat{\otimes}_A F \widehat{\otimes}_A A$ is two-sided A induced. In fact, any A-induced bimodule is of this form.

The following Theorem directly follows from duality between definitions of Hochschild homology and cohomology, Theorem 5.2, and general results of homology theory in the category of Banach spaces and continuous linear maps, see for instance [11] or Theorem 4.8 of [17].

Theorem 5.3. *Let G be an amenable discrete group, I be a nonempty set and $A = \ell^1(B(I,G))$. Then for any two-sided A induced Banach A bimodule E and every $n > 0$, the bounded Hochschild cohomology groups $\mathcal{H}^n(A, E^*)$ are trivial.*

A Banach algebra A is called *approximately amenable* [8, 10], if for any Banach A bimodule E, every bounded derivation $D : A \to E^*$ is *approximately inner*, that is, for a net $(f_\lambda) \in E^*$ and every $a \in A$, $D(a) = \text{norm-lim}_\lambda a \cdot f_\lambda - f_\lambda \cdot a$. The following Theorem (that corrects! some preceding results on amenability properties of Brandt semigroup algebras) is proved in [9].

Theorem 5.4. *Let $S = B(I, G)$ be a Brandt semigroup. Then the following are equivalent.*

(1) $\ell^1(S)$ *is amenable.*

(2) $\ell^1(S)$ *is approximately amenable.*

(3) I *is finite and* G *is amenable.*

Theorem 5.5. *The notion of approximate amenability of self-induced Banach algebras is not a Morita invariant.*

Proof. Let G be an amenable group, I be a finite nonempty set, and J be an infinite set. Then by Theorem 5.4, $\ell^1(B(I, G))$ is approximately amenable and $\ell^1(B(J, G))$ is not approximately amenable. But by Theorem 4.2, we have $\ell^1(B(I, G)) \approx \ell^1(B(J, G))$. □

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