Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 752341, 25 pages
doi:10.1155/2011/752341

Research Article

Some Properties of Certain Multivalent Analytic Functions Involving the Cătas Operator

E. A. Elrifai, H. E. Darwish, and A. R. Ahmed

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 35516, Egypt

Correspondence should be addressed to H. E. Darwish, darwish333@yahoo.com

Received 2 March 2011; Accepted 8 May 2011

Academic Editor: Marco Squassina

We introduce a certain subclass of multivalent analytic functions by making use of the principle of subordination between these functions and Cătas operator. Such results as subordination and superordination properties, convolution properties, inclusion relationships, distortion theorems, inequality properties, and sufficient conditions for multivalent starlikeness are provide. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

1. Introduction

Let $A_p(n)$ denote the class of functions of the following form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (p, n \in N := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disk $U := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

For simplicity, we write

$$A_p(1) := A_p, \qquad A_1(1) = A. \tag{1.2}$$

A function $f(z) \in A_p(n)$ is said to be in the class $S^*_{p,n}(\gamma)$ of p-valent starlike functions of order γ in U if it satisfies the following inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (0 \le \gamma < p;\ z \in U). \tag{1.3}$$

Let $H[a,n]$ be the class of analytic functions of the following form:

$$F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in U). \tag{1.4}$$

Let $f, g \in A_p(n)$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}. \tag{1.5}$$

Then the Hadanard product (or convolution) $f * g$ of the functions $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) := z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} =: (g * f)(z). \tag{1.6}$$

We consider the following multiplier transformations.

Definition 1.1 (see [1]). Let $f(z) \in A_p(n)$. For $p, n \in N$, $\delta, \lambda \ge 0, \ell \ge 0$, define the multiplier transformations $I_p(\delta, \lambda, \ell)$ on $A_p(n)$ by the following infinite series:

$$I_p(\delta, \lambda, \ell) f(z) := z^p + \sum_{k=n}^{\infty} \left[\frac{p + \lambda k + \ell}{p + \ell}\right]^{\delta} a_{k+p} z^{k+p}. \tag{1.7}$$

It is easily verified from (1.7), that

$$(p+\ell) I_p(\delta+1, \lambda, \ell) f(z) = [p(1-\lambda) + \ell] I_p(\delta, \lambda, \ell) f(z) + \lambda z \left(I_p(\delta, \lambda, \ell) f(z)\right)'. \tag{1.8}$$

It should be remarked that the class of multiplier transforms $I_p(\delta, \lambda, \ell)$ is a generalization of several other linear operators considered, in earlier investigations (see [2–12]).

If $f(z)$ is given by (1.1), then we have

$$I_p(\delta, \lambda, \ell) f(z) = \left(f * \varphi^{\delta}_{p,\lambda,\ell}\right)(z), \tag{1.9}$$

where

$$\varphi^{\delta}_{p,\lambda,\ell}(z) = z^p + \sum_{k=n}^{\infty} \left[\frac{p + \lambda k + \ell}{p + \ell}\right]^{\delta} z^{p+k}. \tag{1.10}$$

In particular, we set

$$I_1(\delta,\lambda,\ell)f(z) := I(\delta,\lambda,\ell)f(z). \tag{1.11}$$

For two functions $f(z)$ and $g(z)$, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$ $(z \in U)$ if there exists a Schwarz function $w(z)$, which is analytic in U with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in U) \tag{1.12}$$

such that

$$f(z) = g(w(z)) \quad (z \in U). \tag{1.13}$$

Indeed, it is known that

$$f(z) \prec g(z), \quad (z \in U) \Longrightarrow f(0) = g(0), \qquad f(U) \subset g(U). \tag{1.14}$$

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z), \quad (z \in U) \Longleftrightarrow f(0) = g(0), \qquad f(U) \subset g(U). \tag{1.15}$$

By making use of the linear operator $I_p(\delta,\lambda,\ell)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class $A_p(n)$ of p-valent analytic functions.

Definition 1.2. A function $f(z) \in A_p(n)$ is said to be in the class $ß_p^{\alpha,\beta}(\delta,\lambda,\ell,n;A,B)$ if it satisfies the following subordination condition:

$$(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.16}$$

where (and throughout this paper unless otherwise mentioned) the parameters $\alpha,\beta,p,n,\lambda,\delta,$ $\ell, A,$ and B are constrained as follows:

$$\alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0, \quad \lambda,\ell \geq 0, \quad \lambda,\ell \in R, \quad \delta \geq 0, \quad -1 \leq B \leq 1, A \neq B \in R, \quad p,n \in N. \tag{1.17}$$

For simplicity, we write

$$ß_1^{1,\beta}(1,1,1;1;1,-1) = ß(\beta). \tag{1.18}$$

Clearly, the class $ß(\beta)$ is a subclass of the familiar class of Bazilevič functions of type β.

If we set $\delta = 0$, $\lambda = \ell = p = 1$ in the class $ß_p^{\alpha,\beta}(\delta, \lambda, \ell, n; A, B)$, which was studied by Liu [13]. In particular, Zhu [14] determined the sufficient conditions such that $ß_n(\alpha, \beta, A, 0) \subset S^*_{p,n}(\rho)$.

Cătas [1, 5, 15], Cho and Srivastava [6], Cho and Kim [7], and Kumar et al. [10] obtained many interesting results associated with the multiplier operator.

In the present paper, we aim at proving such results as subordination and superordination properties, convolution properties, inclusion relationships, distortion theorems, inequality properties, and sufficient conditions for multivalent starlikeness of the class $ß_p^{\alpha,\beta}(\delta, \lambda, \ell, n; A, B)$. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

2. Preliminary Results

In order to establish our main results, we need the following definition and lemmas.

Definition 2.1 (see [16]). Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty \right\}, \tag{2.1}$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U - E(f)$.

Lemma 2.2 (see [17]). *Let the function h be analytic and univalent (convex) in U with $h(0) = 1$. Suppose also that the function k given by*

$$k(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots \tag{2.2}$$

is analytic in U. If

$$k(z) + \frac{zk'(z)}{\zeta} \prec h(z) \quad (\operatorname{Re}(\zeta) > 0;\ \zeta \neq 0;\ z \in U), \tag{2.3}$$

then

$$k(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\zeta/n} \int_0^z t^{(\zeta/n)-1} h(t) dt \prec h(z) \quad (z \in U), \tag{2.4}$$

and $\chi(z)$ is the best dominant of (2.3).

Lemma 2.3 (see [18]). *Let $q(z)$ be a convex univalent function in U and let $\sigma, \eta \in \mathbb{C}$ with*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{\sigma}{\eta}\right)\right\}. \tag{2.5}$$

If the function p is analytic in U and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z), \tag{2.6}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.4 (see [16]). *Let q be convex univalent in U and $k \in C$. Further assume that* $\operatorname{Re}(\overline{k}) > 0$ *if*

$$p(z) \in H[q(0), 1] \cap Q, \tag{2.7}$$

and $p(z) + kzp'(z)$ is univalent in U, then

$$q(z) + kzq'(z) \prec p(z) + kzp'(z) \tag{2.8}$$

implies that $q(z) \prec p(z)$ and $q(z)$ is the best subdominant.

Lemma 2.5 (Jach's Lemma [19]). *Let $w(z)$ be a noncostant analytic function in U with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0 w'(z_0) = k w(z_0), \tag{2.9}$$

where $k \geq 1$ is a real number.

Lemma 2.6 (see [20]). *Let F be analytic and convex in U. If $f(z), g(z) \in A$ and $f(z), g(z) \prec F(z)$; then*

$$\lambda f(z) + (1 - \lambda) g(z) \prec F(z) \quad (0 \leq \lambda \leq 1). \tag{2.10}$$

Lemma 2.7 (see [21, 22]). *Let $k, v \in \mathbb{C}$. Suppose also that m is convex and univalent in U with*

$$m(0) = 1, \qquad \operatorname{Re}(km(z) + v) > 0 \quad (z \in U). \tag{2.11}$$

If u is analytic in U with $u(0) = 1$, then the following subordination:

$$u(z) + \frac{zu'(z)}{ku(z) + v} \prec m(z), \quad (z \in U) \tag{2.12}$$

implies that

$$u(z) \prec m(z) \quad (z \in U). \tag{2.13}$$

Lemma 2.8 (see [23]). *Let $f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$ analytic in U and $g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ be analytic and convex in U. If $f(z) \prec g(z)$, then $|a_k| \leq |b_1|$, $(k \in N)$.*

Lemma 2.9 (see [24]). *Let $\delta \neq 0$, $\delta \in R$, $v/\delta > 0, 0 \leq \rho < 1$, $p \in H[1,n]$, and $p(z) \prec 1 + kz$ $(k := vM/(n\delta + v))$, where*

$$M := M_n(\delta, \nu, \rho) = \frac{(1-\rho)|\delta|(1 + n\delta/v)}{\left|1 - \delta + \rho\delta\right| + \sqrt{1 + (n\delta/v)^2}}. \tag{2.14}$$

If $q(z) \in H[1,n]$ satisfies the following subordination condition:

$$p(z)\left[1 - \delta + \delta\left((1-\rho)q(z) + \rho\right)\right] \prec 1 + Mz, \tag{2.15}$$

then

$$\operatorname{Re}\left(q(z)\right) > 0 \quad (z \in U). \tag{2.16}$$

3. Main Results

We begin by presenting our first subordination property given by Theorem 3.1 below.

Theorem 3.1. *Let $f(z) \in ß_p^{\alpha,\beta}(\delta, \lambda, \ell; n; A, B)$ with $\operatorname{Re}(\alpha) > 0$. Then*

$$\left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p}\right)^{\beta} \prec \frac{(p+\ell)\beta}{\lambda n \alpha} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{(p+\ell)\beta/\lambda n\alpha - 1} du \prec \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{3.1}$$

Proof. Define the function $P(z)$ by

$$P(z) := \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p}\right)^{\beta} \quad (z \in U). \tag{3.2}$$

Then $P(z)$ is analytic in U with $P(0) = 1$. By taking the derivatives in the both sides in equality (3.2) and using (1.8), we get

$$\begin{aligned}(1-\alpha)&\left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1, \lambda, \ell) f(z)}{I_p(\delta, \lambda, \ell) f(z)}\right)\left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p}\right)^{\beta} \\ &= P(z) + \frac{\lambda\alpha z P'(z)}{\beta(p+\ell)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U).\end{aligned} \tag{3.3}$$

An application of Lemma 2.2 to (3.3) yields

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec \frac{(p+\ell)\beta}{\lambda n\alpha} z^{-(p+\ell)\beta/\lambda n\alpha} \int_0^z t^{(p+\ell)\beta/\lambda n\alpha-1} \frac{1+At}{1+Bt} dt$$
$$= \frac{\zeta}{n}\int_0^1 u^{\zeta/n-1}\frac{1+Azu}{1+Bzu}du \prec \frac{1+Az}{1+Bz} \quad (z\in U), \tag{3.4}$$

where

$$\zeta = \frac{(p+\ell)\beta}{\lambda\alpha}. \tag{3.5}$$

The proof of Theorem 3.1 is thus completed. □

Theorem 3.2. *Let $q(z)$ be univalent in U, $0 \neq \alpha \in \mathbb{C}$. Suppose also that $q(z)$ satisfies*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{(p+\ell)\beta}{\lambda\alpha}\right)\right\}. \tag{3.6}$$

If $f(z) \in A_p(n)$ satisfying the following subordination:

$$(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec q(z) + \frac{\alpha\lambda z q'(z)}{(p+\ell)\beta}, \tag{3.7}$$

then

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec q(z), \tag{3.8}$$

and $q(z)$ is the best dominant.

Proof. Let the function $P(z)$ be defined by (3.2). We know that (3.3) holds true. Combining (3.3) and (3.7), we find that

$$P(z) + \frac{\lambda\alpha z P'(z)}{(p+\ell)\beta} \prec q(z) + \frac{\lambda\alpha z q'(z)}{(p+\ell)\beta}. \tag{3.9}$$

□

By Lemma 2.3 and (3.9), we easily get the assertion of Theorem 3.2.

Taking $q(z) = (1 + Az)/(1 + Bz)$ in Theorem 3.2, we get the following result.

Corollary 3.3. *Let $\alpha \in \mathbb{C}$ and $-1 \leq B < A \leq 1$. Suppose also that $(1 + Az)/(1 + Bz)$ satisfies the condition (3.6). If $f(z) \in A_p(n)$ satisfies the following subordination:*

$$(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\lambda\alpha(A-B)z}{(p+\ell)\beta(1+Bz)^2}, \tag{3.10}$$

then

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz}, \tag{3.11}$$

and $(1 + Az)/(1 + Bz)$ is the best dominant.

If $f(z)$ is subordinate to $F(z)$, then $F(z)$ is superordinate to $f(z)$. We now derive the following superordination result for the class $\mathfrak{B}_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$.

Theorem 3.4. *Let $q(z)$ be convex univalent in U, $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$. Also let*

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \in H[q(0),1] \cap Q,$$
$$(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \tag{3.12}$$

be univalent in U. If

$$q(z) + \frac{\lambda\alpha z q'(z)}{(p+\ell)\beta} \prec (1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}, \tag{3.13}$$

then

$$q(z) \prec \left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}, \tag{3.14}$$

and $q(z)$ is the best subdominant.

Proof. Let the function $P(z)$ be defined by (3.2). Then

$$q(z)+\frac{\lambda\alpha zq'(z)}{(p+\ell)\beta}\prec(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}+\alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}$$
$$=P(z)+\frac{\lambda\alpha zP'(z)}{(p+\ell)\beta}. \tag{3.15}$$

An application of Lemma 2.4 yields the assertion of Theorem 3.4. □

Taking $q(z)=(1+Az)/(1+Bz)$ in Theorem 3.4, we get the following corollary.

Corollary 3.5. *Let $q(z)$ be convex univalent in U and $-1\le B<A\le 1,\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$. Also let*

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}\in H[q(0),1]\cap Q,$$
$$(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}+\alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \tag{3.16}$$

be univalent in U. If

$$\frac{1+Az}{1+Bz}+\frac{\lambda\alpha(A-B)z}{(p+\ell)\beta(1+Bz)^2}$$
$$\prec(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}+\alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}, \tag{3.17}$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}, \tag{3.18}$$

and $(1+Az)/(1+Bz)$ is best subdominant.

Combining the above results of subordination and superordination. We easily get the following "Sandwich-type result".

Corollary 3.6. *Let $q_1(z)$ be convex univalent and let $q_2(z)$ be univalent in U, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$. Let $q_2(z)$ satisfies (3.6). If*

$$
\begin{gathered}
0 \neq \left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \in H[q_1(0),1] \cap Q, \\
(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}
\end{gathered}
\tag{3.19}
$$

is univalent in U, also

$$
\begin{aligned}
q_1(z) + \frac{\lambda\alpha z q_1'(z)}{(p+\ell)\beta} &\prec (1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \\
&\prec q_2(z) + \frac{\lambda\alpha z q_2'(z)}{(p+\ell)\beta},
\end{aligned}
\tag{3.20}
$$

then

$$
q_1(z) \prec \left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec q_2(z), \tag{3.21}
$$

and $q_1(z)$, $q_2(z)$ are, respectively, the best subordinate, and dominant.

Theorem 3.7. *Let $f(z) \in A_p(n)$, $\xi \in \mathbb{C} \setminus \{0\}$, and $0 \le \gamma < 1$. Also let the function φ be defined by*

$$
\varphi(z) = \frac{z\Big(\big(I_p(\delta+1,\lambda,\ell)f(z)/I_p(\delta,\lambda,\ell)f(z)\big)\big(I_p(\delta,\lambda,\ell)f(z)/z^p\big)^{\beta} - 1\Big)'}{\big(I_p(\delta+1,\lambda,\ell)f(z)/I_p(\delta,\lambda,\ell)f(z)\big)\big(I_p(\delta,\lambda,\ell)f(z)/z^p\big)^{\beta} - 1} \quad (z \in U). \tag{3.22}
$$

If φ satisfies one of the following conditions:

$$
\mathrm{Re}(\varphi(z)) \begin{cases} < \dfrac{1}{|\xi|^2}\mathrm{Re}(\xi) & (\mathrm{Re}(\xi) > 0), \\ \neq 0 & (\mathrm{Re}(\xi) = 0), \\ > \dfrac{1}{|\xi|^2}\mathrm{Re}(\xi) & (\mathrm{Re}(\xi) < 0), \end{cases} \tag{3.23}
$$

or

$$I(\varphi(z))\begin{cases} > -\dfrac{1}{|\xi|^2}I(\xi) & (I(\xi) > 0),\\ \neq 0 & (I(\xi) = 0),\\ < -\dfrac{1}{|\xi|^2}I(\xi) & (I(\xi) < 0),\end{cases} \tag{3.24}$$

then

$$\left|\left(\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}-1\right)^{\xi}\right| < 1-\gamma. \tag{3.25}$$

Proof. We define the function $\phi(z)$ by

$$\begin{aligned}&\left(\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}-1\right)^{\xi}\\ &\quad= (1-\gamma)\phi(z) \quad (0 \leq \gamma < 1; \xi \in \mathbb{C}\setminus\{0\}; z \in U).\end{aligned} \tag{3.26}$$

It is easy to see that the function ϕ is analytic in U with $\phi(0) = 0$.

Differentiating both sides of (3.26) with respect to z logarithmically, we get

$$\begin{aligned}&\frac{z\phi'(z)}{\phi(z)}\\ &= \xi\frac{z\left(\left(I_p(\delta+1,\lambda,\ell)f(z)/I_p(\delta,\lambda,\ell)f(z)\right)\left(I_p(\delta,\lambda,\ell)f(z)/z^p\right)^{\beta}-1\right)'}{\left(I_p(\delta+1,\lambda,\ell)f(z)/I_p(\delta,\lambda,\ell)f(z)\right)\left(I_p(\delta,\lambda,\ell)f(z)/z^p\right)^{\beta}-1} \quad (z \in U;\ \xi \in \mathbb{C}\setminus\{0\}).\end{aligned} \tag{3.27}$$

We now consider the function φ defined by

$$\varphi := \frac{\overline{\xi}}{|\xi|^2}\frac{z\phi'(z)}{\phi(z)} \quad (z \in U,\ \xi \in \mathbb{C}\setminus\{0\}). \tag{3.28}$$

Assume that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|\phi(z)| = |\phi(z_0)| = 1, \tag{3.29}$$

by Lemma 2.5, we know that

$$z_0\phi'(z) = k\phi(z_0) \quad (k \geq 1). \tag{3.30}$$

If follows from (3.28) and (3.30) that

$$\operatorname{Re}(\varphi(z_0)) = \operatorname{Re}\left(\frac{\bar{\xi}}{|\xi|^2}\frac{z\phi'(z_0)}{\phi(z_0)}\right) = \frac{k}{|\xi|^2}\operatorname{Re}\left(\bar{\xi}\right)$$
$$= \frac{k}{|\xi|^2}\operatorname{Re}(\xi)\begin{cases} \geq \dfrac{1}{|\xi|^2}\operatorname{Re}(\xi) & (\operatorname{Re}(\xi) > 0), \\ = 0 & (\operatorname{Re}(\xi) = 0) \\ \leq \dfrac{1}{|\xi|^2}\operatorname{Re}(\xi) & (\operatorname{Re}(\xi) < 0), \end{cases} \tag{3.31}$$

$$I(\varphi(z_0)) = I\left(\frac{\bar{\xi}}{|\xi|^2}\frac{z\phi'(z_0)}{\phi(z_0)}\right) = \frac{k}{|\xi|^2}I\left(\bar{\xi}\right)$$
$$= \frac{-k}{|\xi|^2}I(\xi)\begin{cases} \leq -\dfrac{1}{|\xi|^2}I(\xi) & (I(\xi) > 0), \\ = 0 & (I(\xi) = 0), \\ \geq -\dfrac{1}{|\xi|^2}I(\xi) & (I(\xi) < 0). \end{cases} \tag{3.32}$$

But the inequalities in (3.31) and (3.32) contradict, respectively, the inequalities in (3.23) and (3.24). Therefore, we can conclude that

$$\left|\phi(z)\right| < 1 \quad (z \in U), \tag{3.33}$$

which implies that

$$\left|\left(\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} - 1\right)^{\xi}\right| = (1-\gamma)\left|\phi(z)\right| < 1-\gamma. \tag{3.34}$$

We thus complete the proof of Theorem 3.7. □

From Theorem 3.7, we easily get the following result for the class ß(β) of Bazilevič functions of type β.

Corollary 3.8. *Let $f(z) \in A$, $\delta = 0$, $p = \lambda = \ell = \xi = 1$, and $\gamma = 0$. Also let the function φ be defined by* (3.22). *If φ satisfies one of the following conditions:*

$$\operatorname{Re}(\varphi(z)) < 1 \quad \text{or} \quad I(\varphi(z)) \neq 0, \tag{3.35}$$

then $f(z) \in$ ß(β).

Theorem 3.9. *Let* $\operatorname{Re}(\alpha) > 0$, $\beta > 0$, *and* $f(z) \in \text{ß}_p^{0,\beta}(\delta,\lambda,\ell;n;1-2\rho,-1)$ $(0 \leq \rho < 1)$. *Then* $f(z) \in \text{ß}_p^{\alpha,\beta}(\delta,\lambda,\ell;n,1-2\rho,-1)$ *for* $|z| < R(\alpha,\beta,\ell,\lambda,p)$, *where*

$$R(\alpha,\beta,\ell,\lambda,p) = \frac{-\lambda\alpha + \sqrt{\lambda^2\alpha^2 + (p+\ell)^2\beta^2}}{(\ell+\rho)\beta}. \tag{3.36}$$

The bound $R(\alpha,\beta,\ell,\lambda,p)$ *is the best possible.*

Proof. Suppose that

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} = \rho + (1-\rho)h(z) \quad (z \in U;\ 0 \leq \rho < 1), \tag{3.37}$$

where h is analytic and has a positive real part in U. By taking the derivatives in the both sides in equality (3.37) and using (1.9), we get

$$\begin{aligned}
&\operatorname{Re}\left((1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}\right) - \rho \\
&\quad = (1-\rho)\operatorname{Re}\left(h(z) + \frac{\lambda\alpha z h'(z)}{(p+\ell)\beta}\right) \\
&\quad \geq (1-\rho)\operatorname{Re}\left(h(z) - \frac{\lambda\alpha |z h'(z)|}{(p+\ell)\beta}\right).
\end{aligned} \tag{3.38}$$

By making use of the following well-known estimate (see [25]):

$$\left|z h'(z)\right| \leq \frac{2r}{1-r^2}\operatorname{Re}(h(z)) \quad (|z| = r < 1) \tag{3.39}$$

in (3.38), we obtain that

$$\begin{aligned}
&\operatorname{Re}\left[(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} - \rho\right] \\
&\quad = (1-\rho)\left(1 - \frac{2\lambda\alpha r}{(p+\ell)\beta(1-r^2)}\right)\operatorname{Re}(h(z)) > 0
\end{aligned} \tag{3.40}$$

for $r < R(\alpha,\beta,\ell,\lambda,p)$, where $R(\alpha,\beta,\ell,\lambda,p)$ is given by (3.36).

To show that the bound $R(\alpha,\beta,\ell,\lambda,p)$ is the best possible, we consider the function $f(z) \in A_p(n)$ defined by

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} = \rho + (1-\rho)\frac{1+z}{1-z} \quad (z \in U). \tag{3.41}$$

By noting that

$$\begin{aligned} &\operatorname{Re}\left((1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}\right) - \rho \\ &\quad = (1-\rho)\operatorname{Re}\left(\frac{1+z}{1-z} + \frac{2\lambda\alpha}{(p+\ell)\beta}\frac{z}{(1-z)^2}\right) = 0 \end{aligned} \tag{3.42}$$

for $z = R(\alpha,\beta,\ell,\lambda,p)$, we conclude that the bound is the best possible. Theorem 3.9 is thus proved. □

Theorem 3.10. *Let $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ with $\operatorname{Re}(\alpha) > 0$. Then*

$$f(z) = \left(z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{1/\beta}\right) * \left(z^p + \sum_{k=n}^{\infty}\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{-\delta} z^{k+p}\right), \tag{3.43}$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.

Proof. Suppose that $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$. It follows from (3.1) that

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} = \frac{1+Aw(z)}{1+Bw(z)}, \tag{3.44}$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.

By virtue of (3.44), we easily find that

$$I_p(\delta,\lambda,\ell)f(z) = z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{1/\beta}. \tag{3.45}$$

Combining (1.10), (1.16), and (3.45), we have

$$\left(z^p + \sum_{k=n}^{\infty}\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{\delta} z^{k+p}\right) * f(z) = z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{1/\beta}. \tag{3.46}$$

The assertion (3.43) of Theorem 3.10 can now easily be derived from (3.46). □

Theorem 3.11. *Let* $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\mathrm{Re}(\alpha) > 0$. *Then*

$$\frac{1}{z}\left[\left(1+B^{i\theta}\right)^{1/\beta}\left(z^p+\sum_{k=n}^{\infty}\left(\frac{p+\lambda k+\ell}{p+\ell}\right)^{-\delta}z^{k+p}\right)*f(z)-z^p\left(1+Ae^{i\theta}\right)^{1/\beta}\right] \neq 0 \quad (z\in U;\ 0<\theta<2\pi). \tag{3.47}$$

Proof. Suppose that $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ with $\mathrm{Re}(\alpha) > 0$. We know that (3.1) holds true, which implies that

$$\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}} \quad (z\in U;\ 0<\theta<2\pi). \tag{3.48}$$

It is easy to see that the condition (3.48) can be written as follows:

$$\frac{1}{z}\left[I_p(\delta,\lambda,\ell)f(z)\left(1+Be^{i\theta}\right)^{1/\beta}-z^p\left(1+A^{i\theta}\right)^{1/\beta}\right]\neq 0 \quad (z\in U; 0<\theta<2\pi). \tag{3.49}$$

Combining (1.9), (1.10), and (3.49), we easily get the convolution property (3.47) asserted by Theorem 3.11. □

Theorem 3.12. *Let* $\alpha_2 \geq \alpha_1 \geq 0$ *and* $-1 \leq B_1 \leq B_2 < A_2 < A_1 \leq 1$. *Then*

$$ß_p^{\alpha_2,\beta}(\delta,\lambda,\ell;n;A_2,B_2) \subset ß_p^{\alpha_1,\beta}(\delta,\lambda,\ell;n;A_1,B_1). \tag{3.50}$$

Proof. Suppose that $f(z) \in \beta_p^{\alpha_2,\beta}(\delta,\lambda,\ell;n;A_2,B_2)$. We know that

$$(1-\alpha_2)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}+\alpha_2\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec \frac{1+A_2z}{1+B_2z}. \tag{3.51}$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$(1-\alpha_2)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}+\alpha_2\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z} \tag{3.52}$$

that is, $f(z) \in ß_p^{\alpha_2,\beta}(\delta,\lambda,\ell;n;A_1,B_1)$. Thus the assertion of Theorem 3.12 holds for $\alpha_2 = \alpha_1 \geq 0$.

If $\alpha_2 > \alpha_1 \geq 0$, by Theorem 3.1 and (3.52), we know that $f(z) \in \mathfrak{B}_p^{0,\beta}(\delta, \lambda, \ell; n; A_1, B_1)$, that is,

$$\left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} \prec \frac{1 + A_1 z}{1 + B_1 z}. \tag{3.53}$$

At the same time, we have

$$\begin{aligned}
&(1 - \alpha_1) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} + \alpha_1 \left(\frac{I_p(\delta + 1, \lambda, \ell) f(z)}{I_p(\delta, \lambda, \ell) f(z)} \right) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} \\
&\quad = \left(1 - \frac{\alpha_1}{\alpha_2} \right) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} \\
&\qquad + \frac{\alpha_1}{\alpha_2} \left[(1 - \alpha_2) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} + \alpha_2 \left(\frac{I_p(\delta + 1, \lambda, \ell) f(z)}{I_p(\delta, \lambda, \ell) f(z)} \right) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} \right].
\end{aligned} \tag{3.54}$$

Moreover, since $0 \leq (\alpha_1/\alpha_2) < 1$ and $h_1(z) := (1 + A_1 z)/(1 + B_1 z)$, is analytic and convex in U. Combining (3.52)–(3.54) and Lemma 2.6, we find that

$$(1 - \alpha_1) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} + \alpha_1 \left(\frac{I_p(\delta + 1, \lambda, \ell) f(z)}{I_p(\delta, \lambda, \ell) f(z)} \right) \left(\frac{I_p(\delta, \lambda, \ell) f(z)}{z^p} \right)^{\beta} \prec \frac{1 + A_1 z}{1 + B_1 z}, \tag{3.55}$$

that is, $f(z) \in \mathfrak{B}_p^{\alpha_1,\beta}(\delta, \lambda, \ell; n; A_1, B_1)$, which implies that the assertion (3.50) of Theorem 3.12 holds.

Let ρ denote the class of functions of the following form:

$$t(z) = 1 + \sum_{k=n}^{\infty} t_k z^k \quad (n \in N), \tag{3.56}$$

which are analytic and convex in U and satisfy the following condition:

$$\operatorname{Re}(t(z)) > 0 \quad (z \in U). \tag{3.57}$$

By making use of the principle of subordination between analytic functions, we introduce the subclasses $S_{p,n}^{*}(\mu, \phi)$ and $C_{p,n}(\mu, \phi)$ of the class $A_p(n)$:

$$S^*_{p,n}(\mu,\phi) := \left\{ f \in A_p(n) : \frac{1}{p-\mu}\left(\frac{zf'(z)}{f(z)} - \mu\right) \prec \phi(z) \ (\phi \in \rho;\ 0 \le \mu < p; z \in U)\right\},$$
$$C_{p,n}(\mu;\phi) = \left\{ f(z) \in A_p(n) : \frac{1}{p-\mu}\left(1 + \frac{zf''(z)}{f'(z)} - \mu\right) \prec \phi(z) \ (\phi \in \rho;\ 0 \le \mu < p;\ z \in U)\right\}. \tag{3.58}$$

Next, by using the operator defined by (1.7), we define the following two subclasses $S_{p,n}(\delta,\lambda,\ell;\mu;\phi)$ and $C_{p,n}(\delta,\lambda,\ell;\mu;\phi)$ of the class $A_p(n)$:

$$\begin{gathered} S_{p,n}(\delta,\lambda,\ell;\mu;\phi) := \left\{ f(z) \in A_p(n) : I_p(\delta,\lambda,\ell)f(z) \in S^*_{p,n}(\mu;\phi)\right\}, \\ C_{p,n}(\delta,\lambda,\ell;\mu;\phi) := \left\{ f(z) \in A_p(n) : I_p(\delta,\lambda,\ell)f(z) \in C_{p,n}(\mu;\phi)\right\}. \end{gathered} \tag{3.59}$$

Clearly, we know that

$$f \in C_{p,n}(\delta,\lambda,\ell;\mu;\phi) \Longleftrightarrow \frac{zf'(z)}{p} \in S_{p,n}(\delta,\lambda,\ell;\mu;\phi). \tag{3.60}$$

We now derive some inclusion relationships for the classes $S_{p,n}(\delta,\lambda,\ell;\mu;\phi)$ and $C_{p,n}(\delta,\lambda,\ell;\mu;\phi)$, by similarly applying the method of proof of Proposition 1 obtained by Cho et al. [26] and Wang et al. [27]. □

Theorem 3.13. *Let $0 \le \mu < p, \lambda > -p$, and $\phi \in \rho$ with*

$$\operatorname{Re}(\phi(z)) > \max\left\{0, -\frac{\delta+\mu}{p-\mu}, -\frac{\lambda+\mu-p}{p-\mu}\right\} \quad (z \in U). \tag{3.61}$$

Then

$$S_{p,n}(\delta+1,\lambda,\ell;\mu;\phi) \subset S_{p,n}(\delta,\lambda,\ell;\mu;\phi) \subset S_{p,n}(\delta,\lambda+1,\ell;\mu;\phi). \tag{3.62}$$

Theorem 3.14. *Let $0 \le \mu < p$, $\lambda > -p$ and $\phi \in \rho$ with* (3.61) *holds. Then*

$$C_{p,n}(\delta+1,\lambda,\ell;n;\mu;\phi) \subset C_{p,n}(\delta,\lambda,\ell;\mu;\phi) \subset C_{p,n}(\delta,\lambda+1,\ell;\mu;\phi). \tag{3.63}$$

Proof. By virtue of (3.60) and Theorem 3.13, we observe that

$$\begin{aligned} f(z) \in C_{p,n}(\delta+1,\lambda,\ell;\mu;\phi) &\Longleftrightarrow I_p(\delta+1,\lambda,\ell)f(z) \in C_{p,n}(\mu;\phi) \\ &\Longleftrightarrow \frac{z\left(I_p(\delta+1,\lambda,\ell)f(z)\right)'}{p} \in S^*_{p,n}(\mu;\phi) \\ &\Longleftrightarrow I_p(\delta+1,\lambda,\ell)\left(\frac{zf'(z)}{p}\right) \in S^*_{p,n}(\mu;\phi) \\ &\Longrightarrow \frac{zf'(z)}{p} \in S_{p,n}(\delta+1,\lambda,\ell;\mu;\phi) \end{aligned}$$

$$
\begin{aligned}
&\Longleftrightarrow \frac{zf'(z)}{p} \in S_{p,n}(\delta,\lambda,\ell;\mu;\phi)\\
&\Longleftrightarrow I_p(\delta,\lambda,\ell)\left(\frac{zf'(z)}{p}\right) \in S^*_{p,n}(\mu,\phi)\\
&\Longleftrightarrow \frac{z\left(I_p(\delta,\lambda,\ell)f(z)\right)'}{p} \in S^*_{p,n}(\mu;\phi)\\
&\Longleftrightarrow I_p(\delta,\lambda,\ell)f(z) \in C_{p,n}(\mu;\phi)\\
&\Longleftrightarrow f(z) \in C_{p,n}(\delta,\lambda,\ell;\mu;\phi),\\
f(z) \in C_{p,n}(\delta,\lambda,\ell;\mu;\phi) &\Longleftrightarrow \frac{zf'(z)}{p} \in S_{p,n}(\delta,\lambda,\ell;\mu;\phi)\\
&\Longrightarrow \frac{zf'(z)}{p} \in S_{p,n}(\delta,\lambda+1,\ell;\mu;\phi)\\
&\Longleftrightarrow \frac{z\left(I_p(\delta,\lambda+1,\ell)f(z)\right)'}{p} \in S^*_{p,n}(\mu;\phi)\\
&\Longleftrightarrow I_p(\delta,\lambda+1,\ell)f(z) \in C_{p,n}(\mu;\phi)\\
&\Longleftrightarrow f(z) \in C_{p,n}(\delta,\lambda+1,\ell;\mu;\phi).
\end{aligned}
\tag{3.64}
$$

From (3.64), we conclude that the assertion of Theorem 3.14 holds true. □

Taking $\phi(z) = (1+Az)/(1+Bz)$ in Theorems 3.13 and 3.14, we get the following results.

Corollary 3.15. *Let* $0 \le \mu < p$, $\lambda > -p$, *and* $-1 \le B < A \le 1$. *Then*

$$
\begin{aligned}
&S_{p,n}\left(\delta+1,\lambda,\ell;\mu;\frac{1+Az}{1+Bz}\right) \subset S_{p,n}\left(\delta,\lambda,\ell;\mu,\frac{1+Az}{1+Bz}\right) \subset S_{p,n}\left(\delta,\lambda+1,\ell;\mu;\frac{1+Az}{1+Bz}\right),\\
&C_{p,n}\left(\delta+1,\lambda,\ell;\mu;\frac{1+Az}{1+Bz}\right) \subset C_{p,n}\left(\delta,\lambda,\ell;\mu,\frac{1+Az}{1+Bz}\right) \subset C_{p,n}\left(\delta,\lambda+1,\ell;\mu;\frac{1+Az}{1+Bz}\right).
\end{aligned}
\tag{3.65}
$$

Theorem 3.16. *Let* $f(z) \in \text{ß}_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\mathrm{Re}(\alpha) > 0$ *and* $-1 \le B < A \le 1$. *Then*

$$
\begin{aligned}
&\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du < Re\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta}\\
&\quad < \frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du.
\end{aligned}
\tag{3.66}
$$

The extremal function of (3.66) *is defined by*

$$I_p(\delta,\lambda,\ell)F_{\alpha,\beta,A,B}(z) = z^p\left(\frac{(p+\ell)\beta}{\lambda\alpha n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/\beta}. \tag{3.67}$$

Proof. Let $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ with $\mathrm{Re}(\alpha) > 0$. From Theorem 3.1, we know that (3.1) holds, which implies that

$$\begin{aligned}
\mathrm{Re}\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} &< \sup_{z\in U}\mathrm{Re}\left(\frac{(p+\ell)\beta}{\lambda\alpha n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)\\
&\le \frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \sup_{z\in U}\mathrm{Re}\left(\frac{1+Azu}{1+Bzu}\right)u^{(p+\ell)\beta/\lambda n\alpha-1}du\\
&< \frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du,\\
\mathrm{Re}\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} &> \inf_{z\in U}\mathrm{Re}\left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)\\
&\ge \frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \inf_{z\in U}\mathrm{Re}\,\frac{1+Azu}{1+Bzu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\\
&> \frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du.
\end{aligned} \tag{3.68}$$

Combining both equations of (3.68), we get (3.66). By noting that the function $I_p(\delta,\lambda,\ell)F_{\alpha,\beta,A,B}(z)$ defined by (3.67) belongs to the class $ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$, we obtain that the equality (3.66) is sharp. The proof of Theorem 3.16 is evidently completed. □

By similarly applying the method of proof of Theorem 3.16, we easily get the following result.

Corollary 3.17. *Let* $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\mathrm{Re}(\alpha) > 0$ *and* $-1 \le A < B \le 1$. *Then*

$$\begin{aligned}
&\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\\
&\quad< \mathrm{Re}\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} < \frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du.
\end{aligned} \tag{3.69}$$

The extremal function of (3.69) *is defined by* (3.67).

In view of Theorem 3.16 and Corollary 3.17, we easily derive the following distortion theorems for the class $ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$.

Corollary 3.18. *Let* $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\operatorname{Re}(\alpha) > 0$ *and* $-1 \le B < A \le 1$*. Then for* $|z| = r < 1$*, we have*

$$\begin{aligned} & r^p\left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Aur}{1-Bur}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/\beta} \\ & \quad < \left|I_p(\delta,\lambda,\ell)f(z)\right| < r^p\left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Aur}{1+Bur}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/\beta}. \end{aligned} \tag{3.70}$$

The extremal function of (3.70) *is defined by* (3.67).

Corollary 3.19. *Let* $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\operatorname{Re}(\alpha) > 0$ *and* $-1 \le A < B \le 1$*. Then for* $|z| = r < 1$*, we have*

$$\begin{aligned} & r^p\left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Aur}{1+Bur}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/\beta} \\ & \quad < \left|I_p(\delta,\lambda,\ell)f(z)\right| < r^p\left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Aur}{1-Bur}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/\beta}. \end{aligned} \tag{3.71}$$

The extremal function of (3.71) *is defined by* (3.67).

By noting that

$$(\operatorname{Re}(\nu))^{1/2} \le \operatorname{Re}\left(\nu^{1/2}\right) \le |\nu|^{1/2} \quad (\nu \in \mathbb{C};\ \operatorname{Re}(\nu) \ge 0). \tag{3.72}$$

From Theorem 3.16 and Corollary 3.17, we easily get the following results.

Corollary 3.20. *Let* $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\operatorname{Re}(\alpha) > 0$ *and* $-1 \le B < A \le 1$*. Then*

$$\begin{aligned} & \left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/2} \\ & \quad < \operatorname{Re}\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta/2} < \left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/2}. \end{aligned} \tag{3.73}$$

Corollary 3.21. *Let* $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,B)$ *with* $\operatorname{Re}(\alpha) > 0$ *and* $-1 \le A < B \le 1$*. Then*

$$\begin{aligned} & \left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1+Au}{1+Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/2} \\ & \quad < \operatorname{Re}\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta/2} < \left(\frac{(p+\ell)\beta}{\lambda n\alpha}\int_0^1 \frac{1-Au}{1-Bu}u^{(p+\ell)\beta/\lambda n\alpha-1}du\right)^{1/2}. \end{aligned} \tag{3.74}$$

Theorem 3.22. *Let*

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \in \mathfrak{B}_p^{\alpha,\beta}(\delta, \lambda, \ell; n; A, B). \tag{3.75}$$

Then

$$\left|a_{p+n}\right| \leq \frac{(\ell + p)^{\delta+1}}{(p + \lambda n + \ell)^{\delta}} \left| \frac{A - B}{(\ell + p)\beta + \lambda n \alpha} \right|. \tag{3.76}$$

The inequality (3.76) *is sharp, with extremal function defined by* (3.67).

Proof. Combining (1.16) and (3.75), we obtain

$$\begin{aligned} &(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \\ &\quad = 1 + \left(1 + \frac{\lambda n \alpha}{(p+\ell)\beta}\right)\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\delta} \beta a_{p+n} z^n \cdots \prec \frac{1+Az}{1+Bz}. \end{aligned} \tag{3.77}$$

An application of Lemma 2.8 to (3.77) yields

$$\left|\left(1 + \frac{\lambda n \alpha}{(p+\ell)\beta}\right)\left(\frac{p+\lambda n+\ell}{p+\ell}\right)^{\delta} \beta a_{p+n}\right| \leq |A - B|. \tag{3.78}$$

Thus, from (3.78), we easily arrive at (3.76) asserted by Theorem 3.22. □

Theorem 3.23. *Let* $0 \neq \alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\beta/\alpha > 0$, $\lambda > -p$ *and* $0 \leq \rho < 1$. *If* $f(z) \in \mathfrak{B}_p^{\alpha,\beta}(\delta, \lambda, \ell; n; A, 0)$ *with*

$$A = A_n(\alpha, \beta, \lambda, \rho, p) = \frac{(1-\rho)|\alpha|(1 + \lambda n \alpha/(p+\ell)\beta)}{\left|1 - \alpha + \rho\alpha\right| + \sqrt{1 + (1 + \lambda n \alpha/(p+\ell)\beta)^2}}, \tag{3.79}$$

then

$$I_p(\delta, \lambda, \ell) f(z) \in S_{p,n}^{*}(\rho p - (1-\rho)\lambda). \tag{3.80}$$

Proof. Suppose that $f(z) \in \mathfrak{B}_p^{\alpha,\beta}(\delta, \lambda, \ell; n; A, 0)$. By definition, we have

$$\begin{aligned} &(1-\alpha)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} + \alpha\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right)\left(\frac{I_p(\delta,\lambda,\ell)f(z)}{z^p}\right)^{\beta} \\ &\quad \prec 1 + Az \quad (z \in U). \end{aligned} \tag{3.81}$$

Let the function $P(z)$ be defined by (3.2). We then find from (3.1) and (3.81) that

$$P(z) \prec \frac{(p+\ell)\beta}{\lambda n\alpha} z^{-(p+\ell)\beta/\lambda n\alpha} \int_0^z (1+At)t^{(p+\ell)\beta/\lambda n\alpha - 1} dt = 1 + \frac{(\ell+p)\beta A}{\lambda n\alpha + (p+\ell)\beta} z. \tag{3.82}$$

We now suppose that

$$\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)} = (1-\rho)q(z) + \rho \quad (0 \le \rho < 1;\ z \in U). \tag{3.83}$$

Then $q(z) \in H[1,n]$. It follows from (3.81) and (3.83) that

$$P(z)\{(1-\alpha) + \alpha[(1-\rho)q(z) + \rho]\} \prec 1 + Az \quad (z \in U). \tag{3.84}$$

An application of Lemma 2.9 to (3.84) yields

$$\operatorname{Re}(q(z)) > 0 \quad (z \in U). \tag{3.85}$$

Combining (3.83) and (3.85), we find that

$$\operatorname{Re}\left(\frac{I_p(\delta+1,\lambda,\ell)f(z)}{I_p(\delta,\lambda,\ell)f(z)}\right) = (1-\rho)\operatorname{Re}(q(z)) + \rho > \rho \quad (z \in U). \tag{3.86}$$

The assertion of Theorem 3.23 can now easily be derived from (1.8) and (3.86). □

Theorem 3.24. *Let $f(z) \in ß_p^{\alpha,\beta}(\delta,\lambda,\ell;n;A,0)$ with $\beta > 0, A > 0, \operatorname{Re}(\alpha) > 0$, and $|\alpha|(n + \operatorname{Re}(p+\ell)\beta/\lambda\alpha) > A((p+\ell)\beta/\lambda)$. Then*

$$\left|\frac{z(I_p(\delta,\lambda,\ell)f(z))'}{I_p(\delta,\lambda,\ell)f(z)} - p\right| < \frac{A(p+\ell)\beta[\lambda|\alpha|(n+\operatorname{Re}((p+\ell)\beta/\lambda\alpha)) + (p+\ell)\beta]}{\lambda|\alpha|[\lambda|\alpha|(n+\operatorname{Re}((p+\ell)\beta/\lambda\alpha)) - A(p+\ell)\beta]}. \tag{3.87}$$

Proof. Let the function $P(z)$ be defined by (3.2). It follows from (3.3) that

$$p(z) + \frac{\lambda\alpha z p'(z)}{(p+\ell)\beta} = 1 + Aw(z), \tag{3.88}$$

where

$$w(z) = \sum_{k=n}^{\infty} w_k z^k \quad (n \in N) \tag{3.89}$$

is analytic in U with $|w(z)| < 1$ $(z \in U)$. From (3.88), we easily get

$$p(z) = 1 + A\frac{(p+\ell)\beta}{\lambda\alpha}\int_0^1 t^{(p+\ell)\beta/\lambda\alpha - 1} w(tz)\,dt$$
$$= 1 + A\frac{(p+\ell)\beta}{\lambda\alpha}\sum_{k=n}^{\infty}\frac{1}{k + (p+\ell)\beta/\lambda\alpha} w_k z^k. \tag{3.90}$$

It follows from (3.90) that

$$(zp(z))' = 1 + A\frac{(p+\ell)\beta}{\lambda\alpha}\sum_{k=n}^{\infty}\frac{k+1}{k + (p+\ell)\beta/\lambda\alpha} w_k z^k$$
$$= 1 + A\frac{(p+\ell)\beta}{\lambda\alpha}\sum_{k=n}^{\infty}\frac{1}{k + (p+\ell)\beta/\lambda\alpha} w_k z^k \tag{3.91}$$
$$+ A\frac{(\ell+p)\beta}{\lambda\alpha}\left(w(z) - \frac{(p+\ell)\beta}{\lambda\alpha}\int_0^1 t^{(\ell+p)/\lambda\alpha - 1} w(tz)\,dt\right).$$

We next find from (3.90) and (3.91) that

$$zp'(z) = A\frac{(p+\ell)\beta}{\lambda\alpha}\left(w(z) - \frac{(p+\ell)\beta}{\lambda\alpha}\int_0^1 t^{(\ell+p)\beta/\lambda\alpha - 1} w(tz)\,dt\right)$$
$$= A\frac{(p+\ell)\beta}{\lambda\alpha}\sum_{k=n}^{\infty}\frac{k}{k + (p+\ell)\beta/\lambda\alpha} w_k z^k. \tag{3.92}$$

Now from (3.88) we get

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{\left[(n + \operatorname{Re}((p+\ell)\beta/\lambda\alpha)) + (p+\ell)\beta/\lambda|\alpha|\ \right]}{|p(z)|}, \tag{3.93}$$

and from (3.90) we get

$$|p(z)| = \left|1 + A\frac{(p+\ell)\beta}{\lambda\alpha}\sum_{k=n}^{\infty}\frac{1}{k + (p+\ell)\beta/\lambda\alpha} w_k z^k\right|$$
$$> \frac{\left[\lambda|\alpha|(n + \operatorname{Re}((p+\ell)\beta/\lambda\alpha)) - A(p+\ell)\beta\right]}{-\lambda|\alpha|(n + \operatorname{Re}((p+\ell)\beta/\lambda\alpha))}, \tag{3.94}$$

from (3.93) and (3.94), we get

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{A(p+\ell)\beta\left[\lambda|\alpha|(n + \operatorname{Re}((p+\ell)\beta/\lambda\alpha)) + (p+\ell)\beta\right]}{\lambda|\alpha|\left[\lambda|\alpha|(n + \operatorname{Re}((p+\ell)\beta/\lambda\alpha)) - A(p+\ell)\beta\right]}. \tag{3.95}$$

Thus, from (3.2) and (3.95), we easily arrive at the assertion of Theorem 3.24. □

References

[1] A. Cătas, "On certain classes of p-valent functions defined by multiplier transfor presented paper," in *Proceedings of the International Symposium on Geometry Function Theory and Applications*, Istanbul, Turkey, 2007.

[2] M. Acu, Y. Polatoglu, and E. Yavuz, "The radius of starlikeness of the certain classes of p-valent functions defined by multiplier transformations," *Journal of Inequalities and Applications*, vol. 2008, Article ID 280183, 5 pages, 2008.

[3] M. Acu and S. Owa, "Note on a class of starlike functions," in *Proceedings of the Interational Short Joint Work on Study on Calculus Operators in Univalent Functions Theory*, pp. 1–10, RIMS, Kyoto, Japan, August 2006.

[4] F. M. AlObudi, "On univalent functions defined by a generalized Sãlãgean opeator," *International Journal of Mathematics*, vol. 2004, no. 27, pp. 1429–1436, 2004.

[5] A. Cătaş, "Neighborhoods of a certain class of analytic functions with negative coefficients," *Banach Journal of Mathematical Analysis*, vol. 3, no. 1, pp. 111–121, 2009.

[6] N. E. Cho and H. M. Srivastava, "Argument estimates of certain analytic functions defined by a class of multiplier transformations," *Mathematical and Computer Modelling*, vol. 37, no. 1-2, pp. 39–49, 2003.

[7] N. E. Cho and T. H. Kim, "Multiplier transformations and strongly close-to-convex functions," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 3, pp. 399–410, 2003.

[8] J. Patel, "Inclusion relations and convolution properties of certain subclasses of analytic functions defined by generalized Salagean operator," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 15, no. 1, pp. 33–47, 2008.

[9] G. S. Sãlãgean, "Subclasses of univalent functions," in *Proceedings of the 5th Complex Analysis Romanin-Finnish Seminar*, vol. 1013 of *Lecture Notes in Math.*, pp. 362–372, Springer, Berlin, Germany, 1983.

[10] S. S. Kumar, H. C. Taneja, and V. Ravichandran, "Classes of multivalent functions defined by Dziok-Srivastava linear operator and multiplier transformation," *Kyungpook Mathematical Journal*, vol. 46, no. 1, pp. 97–109, 2006.

[11] B. A. Uralegaddi and C. Somanatha, "Certain classes of univalent functions," in *Current Topics in Analytic Function Theory*, pp. 371–374, World Scientific, River Edge, NJ, USA, 1992.

[12] Z. -G. Wang, N. Xu, and M. Acu, "Certain subclasses of multivalent analytic functions defined by multiplier transforms," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 192–204, 2010.

[13] M. -S. Liu, "On certain class of analytic functions defined by differential subordination," *Abstract and Applied Analysis*, vol. 22, pp. 388–392, 2002.

[14] Y. Zhu, "Some starlikeness criterions for analytic functions," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1452–1459, 2007.

[15] A. Cătas, G. I. Oros, and G. Oros, "Differential suborinations associated with multiplier transformations," *Abstract and Applied Analysis*, vol. 2008, Article ID 845724, 11 pages, 2008.

[16] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables*, vol. 48, no. 10, pp. 815–826, 2003.

[17] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, vol. 225 of *Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[18] T. N. Shanmugam, V. Ravichandran, and S. Sivasubramanian, "Differential sandwich theorems for some subclasses of analytic functions," *The Australian Journal of Mathematical Analysis and Applications*, vol. 3, pp. 1–11, 2006.

[19] I. S. Jack, "Functions starlike and convex of order α," *Journal of the London Mathematical Society*, vol. 3, pp. 469–474, 1971.

[20] M. -S. Liu, "A certain subclass of analytic functions," *Journal of South China Normal University*, no. 4, pp. 15–20, 2002.

[21] P. Eenigenburg, M. M. Miller, P. T. Mocanu, and O. Reade, "On a Briot-Bouquet differential suborolination," in *General Mathematics 3*, vol. 64 of *International Series of Numerical Mathematics*, pp. 339–348, Birkhãuser, Basel, Switzerland, 1983.

[22] P. Eenigenburg, S. S. Miller, P. T. Mocanu, and M. O. Reade, "On a Briot- Bouquet differential subordination," *Revue Roumaine de Mathématique Pures et Appliquées* , vol. 29, no. 7, pp. 567–573, 1984.

[23] W. Rogosinski, "On the coefficients of subordinate functions," *Proceedings of the London Mathematical Society (Series 2)*, vol. 48, pp. 48–82, 1943.

[24] M. -S. Liu, “On the starlikeness for certain subclass of analytic functions and a problem of Ruscheweyh,” *Advances in Mathematics*, vol. 34, no. 4, pp. 416–424, 2005.

[25] T. H. MacGregor, “The radius of univalence of certain analytic functions,” *Proceedings of the American Mathematical Society*, vol. 14, pp. 514–520, 1963.

[26] N. E. Cho, O. S. Kwon, and H. M. Srivastava, “Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators,” *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 470–483, 2004.

[27] Z. -G. Wang, R. Aghalary, M. Darus, and R. W. Ibrahim, “Some properties of certain multivalent analytic functions involving the Cho-Kwon-Srivastava operator,” *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1969–1984, 2009.