Research Article

Bifurcation of Gradient Mappings Possessing the Palais-Smale Condition

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This paper considers bifurcation at the principal eigenvalue of a class of gradient operators which possess the Palais-Smale condition. The existence of the bifurcation branch and the asymptotic nature of the bifurcation is verified by using the compactness in the Palais Smale condition and the order of the nonlinearity in the operator. The main result is applied to estimate the asyptotic behaviour of solutions to a class of semilinear elliptic equations with a critical Sobolev exponent.

1. Introduction

Let H be a real Hilbert space endowed with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$. Let $\Phi \in C^1(H, \mathbf{R})$ and consider the bifurcation problem

$$\Phi'(w) = \mu w. \tag{1.1}$$

If μ_* is a bifurcation point for (1.1), then for sufficiently small $r > 0$ there exists a solution w_r to (1.1), with $\|w_r\| = r$ and $\mu_r \to \mu_*$ as $r \to 0$.

Krasnosel'skii [1] has shown that if Φ is weakly continuous and Φ' completely continuous with a selfadjoint completely continuous Fréchet derivative T at zero, then the smallest positive characteristic value μ_0 of T forms a bifurcation point for Φ'. A local bifurcation branch exists in a neighbourhood of $(\mu, u) = (\mu_0, 0)$.

The condition that Φ be weakly continuous is relaxed in [2]. For a slightly more specific structure of Φ, authors such as [3, 4] (see [5]) have eliminated the requirement that Φ' be completely continuous.

In this paper, it is shown that the compactness inherent in the Palais-Smale condition is an adequate substitute for the requirement of complete continuity of the perturbation. Our

technique follows the original path laid in [1] and does not implement Lyapunov-Schmidt reduction. This allows us to obtain specific estimates for the size of the solution and extend the the local estimates of Chiappinelli [6]. By adopting an alternative approach to identify the bifurcation branch, we are able to identify the point at which compactness is required and weaken it to a local Palais-Smale condition.

Problems such as (1.1) are related to the variational formulation of elliptic partial differential equations. In the final section, some applications of bifurcation theorems are presented.

1.1. Existing Results

We say that μ is an eigenvalue (and w an eigenfunction) for the operator A if

$$A(w) = \mu w. \tag{1.2}$$

We call $\lambda = 1/\mu$ a characteristic value for A. Stating that $(\lambda, u) \in \mathbf{R} \times H$ is a solution means that $\lambda A(w) = w$.

Recall by spectral theory [7] that if A is a linear completely continuous operator, then the eigenvalues $\{\mu_j\}$ are countable and form a bounded sequence with 0 as the only possible accumulation point.

The following is the basic bifurcation result by Krasnosel'skii for gradient mappings. Later, this theorem is modified and proven.

Theorem 1.1. *Assume that $\Phi : H \mapsto \mathbf{R}$ is weakly continuous and uniformly differentiable in a neighbourhood of 0 and assume that $A = \Phi' : H \mapsto H$ is completely continuous. Then, if A is differentiable at 0, every eigenvalue $\mu_* \neq 0$ of the derivative $A'(0)$ is a bifurcation point for (1.2).*

More precisely, for any sufficiently small $r > 0$ there exists $\mu_r \in \mathbf{R}$, $w_r \in H$ with $\|w_r\| = r$ such that $A(w_r) = \mu_r w_r$ and furthermore $\mu_r \to \mu_*$ as $r \to 0$.

Chiappinelli [6] developed Theorem 1.2 which improved Krasnosel'skii's result for gradient mappings by a quantitative estimate of local bifurcation properties. Suppose that $A(w) = Tw + R(w)$ where $T = A'(0)$ and $R(w) = o(\|w\|)$ as $\|w\| \to 0$.

Theorem 1.2. *Under the same assumptions as Theorem 1.1, suppose that R satisfies $R(w) = O(\|w\|^p)$ for $w \to 0$ with $p > 1$. Then as $r \to 0$ the eigenvalues μ_r satisfy*

$$\mu_r = \mu_* + O\left(r^{p-1}\right). \tag{1.3}$$

Recent innovations have allowed authors to produce still sharper estimates on the asymptotic nature of the bifurcation branches. Chiappinelli [8] enhanced expression (1.3), where μ_* was confined to an isolated eigenvalue of finite multiplicity:

$$kr^{p-1} + o\left(r^{p-1}\right) \leq \mu - \mu_* \leq Kr^{p-1} + o\left(r^{p-1}\right). \tag{1.4}$$

The method of proof dispensed with the requirement that $R(\cdot)$ be a gradient operator. Further refinement was achieved in [9], where the coefficients of the asymptotic bounds

were expressed explicitly in terms of the domain volume and other fundamental constants, including the Sobolev constant. Those publications further extended the results to apply for eigenvalues apart from the principal eigenvalue by careful decomposition of the Hilbert space into subspaces of the eigenfunction and its complement. In [8, 9], applications of the abstract result are made to semilinear elliptic differential equations, and in each case the representative examples are confined to equations exhibiting full compactness, that is, an exponent p in the nonlinearity which is a subcritical Sobolev exponent.

Chabrowski et al. formulated the problem with an (S+) condition to relax the requirement of compactness and extended the semilinear problem to a quasilinear formulation. As stated in [10], although the (S+) and Palais Smale conditions appear similar, there is no direct relationship between the two. Kandilakis et al. [11] have also explored quasilinear operators bifurcating from the principal eigenvalue.

In the current paper, we focus on the semilinear problem with a critical Sobolev exponent. The proof relies upon a positive eigenfunction, and we assume bifurcation only around the principal eigenvalue. Our method has not performed the detailed asymptotic analysis to yield an expression in the vein of (1.4), but we believe by a careful analysis that it would be possible to recover similar bounds.

2. Main Results

In partial differential equations, critical Sobolev exponents sometimes arise which generate functionals without compactness. With consistent notation, $R(u)$ is no longer compact, and weak continuity of the functional is lost. In variational methods, the Palais-Smale condition is often used as a substitute for compactness. Following the arguments of Krasnosel'skii, we follow a similar philosophy in this paper. Reference is made to [12, Theorem 8.9] where progress along a different route has produced broadly similar outcomes.

Definition 2.1. Let $\Phi \in C^1(H)$, and suppose $\{u_n\}$ is a sequence in H satisfying $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in H^*. Then $\{u_n\}$ is termed a *Palais-Smale sequence* at level c. If every Palais-Smale sequence at level c contains a strongly convergent subsequence, then Φ is said to satisfy the Palais-Smale condition at level c, $(\text{PS})_c$.

The following theorem improves upon Theorem 1.1 by removing the requirement of complete continuity of Φ' and weak continuity of Φ.

Theorem 2.2. *Let Φ' be a linear operator which is the gradient of a C^1 functional $\Phi(u)$. Let Φ' have a Fréchet derivative T at the origin in H, where T is a selfadjoint, completely continuous operator. Suppose μ_0 is the largest eigenvalue (i.e., $\lambda_0 = 1/\mu_0$ is the smallest positive characteristic value) of T. Suppose that for some $\xi > 0$ the family of functionals*

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda\Phi(u) \tag{2.1}$$

satisfies the $(\text{PS})_c$-condition for $\lambda_0 - \xi < \lambda < \lambda_0 + \xi$ and for $c \in \mathbf{R}$ in a neighbourhood of 0.
Then λ_0 is a bifurcation point for Φ'.

The following result expands Chiappinelli's result [6].

Theorem 2.3. *Assume* $\Phi \in C^1(H, \mathbf{R})$ *is uniformly differentiable in a neighbourhood of 0. Let* $A \equiv \Phi' : H \mapsto H$ *and suppose* $A(u) = Tu + R(u)$ *where* $T = A'(0)$ *is selfadjoint and completely continuous. Let* $\mu_0 = 1/\lambda_0$ *be the principal eigenvalue of T, (or equivalently,* λ_0 *the smallest characteristic value). Suppose that for some* $\xi > 0$, *the family of functionals* $I_\lambda(u) = (1/2)\|u\|^2 - \lambda\Phi(u)$, $\lambda_0 - \xi < \lambda < \lambda_0 + \xi$ *satisfies the* $(PS)_c$*-condition for* $-\overline{c} < c < \overline{c}$. *Assuming that* $R(u) = O(\|u\|^p)$ *as* $\|u\| \to 0$, *with* $p > 1$, *it follows that*

$$\lambda_r = \lambda_0 + O\left(r^{p-1}\right) \quad \left(\text{equivalently } \mu_r = \mu_0 + O\left(r^{p-1}\right)\right) \quad \text{as } r \to 0. \tag{2.2}$$

3. Proof of the Main Results

We firstly recall a result of Lusternik [13] expressed in modern notation, [14, Theorem 8.2]

Theorem 3.1. *Let* Φ *and* ψ *lie in* $C^1(H, \mathbf{R})$. *Denote* $M_c = \{u \in H : \psi(u) = c\}$ *and suppose that* $\nabla\psi(u) \neq 0$ *for* $u \in M_c$. *Let* $\mathcal{T}M_c(w)$ *be the tangent manifold to* M_c *at* w. *Suppose that* $(\Phi'(u_0), v) = 0$ *for some* $u_0 \in M_c$ *and all* $v \in \mathcal{T}M_c(u_0)$. *Then* $\Phi'(u_0) = k\psi'(u_0)$ *for some* $k \in \mathbf{R}$.

The notion of functionals approximating a quadratic is related to the linearisation of an operator.

Definition 3.2. The functional $\Phi(u)$ defined in some neighbourhood of the origin in H is said to approximate the quadratic $(1/2)(Tu, u)$ if, for any $\epsilon > 0$ there is a $\delta > 0$ such that, for all $\|u\| < \delta$, the following inequality holds:

$$\left|\Phi(u) - \frac{1}{2}(Tu, u)\right| \leq \epsilon\|u\|^2. \tag{3.1}$$

The following lemma is from Krasnosel'skii [1].

Lemma 3.3. *Let* Φ' *be a linear operator which is the gradient of the functional* $\Phi(u)$ *defined in some neighbourhood of the origin in* H. *Let* Φ' *have a Fréchet derivate* T *at zero. Then the quadratic* $(1/2)(Tu, u)$ *approximates the functional* $\Phi(u)$.

An important preliminary result is derived from the Ekeland variational principle (Lemma 3.4).

Lemma 3.4. *Let* M *be a complete metric space with metric* d *and let* $E : M \mapsto \mathbf{R}$ *be lower semicontinuous and bounded below. Then for any* ϵ *and* $\delta > 0$ *and any* $u \in M$ *with* $E(u) \leq \inf_M E + \epsilon$ *there is* $v \in M$ *strictly minimising*

$$E_v(w) \equiv E(w) + \frac{\epsilon}{\delta}d(v, w). \tag{3.2}$$

Moreover, $E(v) \leq E(u)$, $d(u, v) \leq \delta$.

Corollary 3.5. *Let V be a Banach space and suppose $S_\rho \subset V$ is the sphere of radius ρ. Suppose $E \in C^1(V)$ is bounded from below. Then there exists a minimising sequence $\{v_m\}$ for E in S_ρ such that*

$$E(v_m) \longrightarrow \inf_{S_\rho} E, \quad DE|'_{S_\rho}(v_m) \longrightarrow 0 \quad in\ V^* \tag{3.3}$$

as $m \to \infty$, where $E|_{S_\rho}$ is the restriction of E to S_ρ. Note that for $v \in S_\rho$,

$$\left\| DE|'_{S_\rho}(v) \right\| = \sup_{w \in \mathcal{T} S_\rho(v)} \frac{|(E'(v), w)|}{\|w\|}. \tag{3.4}$$

Proof. A standard technique in the variational method applies the Ekeland Variational Principle to a minimising sequence on a lower semicontinuous functional coercive in a region where the functional is bounded below. The Ekeland Variational Principle then guarantees a Palais-Smale sequence, which yields an almost critical point. See for example [15].

Here we apply the same approach, adapted for a restriction to a manifold, rather than on an open set containing a local minimum. We confine the analysis to directional derivatives associated with the tangent manifold. The manifold here is the sphere, so the tangency condition is simple to verify.

Choose an arbitrary sequence $\{\epsilon_m\}$, $\epsilon_m > 0$, $\epsilon_m \to 0$. Define the metric space S_ρ with metric $d(u,v) = \|u - v\|$. For $m \in \mathbf{N}$, choose $u_m \in S_\rho$ such that

$$E(u_m) \le \inf_{S_\rho} E + \epsilon_m^2. \tag{3.5}$$

Let $\epsilon = \epsilon_m^2$, $\delta = \epsilon_m$ and determine $v_m = v$ according to the Ekeland Variational Principle, satisfying $E(v_m) \le E(v_m + w) + \epsilon_m \|w\|_V$ for all $v_m + w \in S_\rho$. Hence

$$\sup_{\substack{\|w\|_V \le \delta, \\ v_m + w \in S_\rho,\, w \ne 0}} \frac{E(v_m) - E(v_m + w)}{\|w\|_V} \le \epsilon_m. \tag{3.6}$$

We deal only with the restriction of E onto the manifold which confines its domain of definition to S_ρ. Expressing the derivative in a Fréchet sense we have

$$\lim_{\substack{h \to 0, \\ v_m + h \in S_\rho}} \frac{\left(E|_{S_\rho}(v_m + h) - E|_{S_\rho}(v_m) - DE|'_{S_\rho}(v_m), h \right)}{\|h\|} = 0 \tag{3.7}$$

which rearranged and using (3.6) gives

$$\lim_{\substack{h \to 0, \\ v_m + h \in S_\rho}} \left(DE|'_{S_\rho}(v_m), \frac{h}{\|h\|} \right) = \lim_{\substack{h \to 0, \\ v_m + h \in S_\rho}} \frac{\left(E|_{S_\rho}(v_m + h) - E|_{S_\rho}(v_m) \right)}{\|h\|} \ge -\epsilon_m. \tag{3.8}$$

Since the limit holds for any path to zero, we can replace h with $-h$ to yield

$$\lim_{\substack{h \to 0, \\ v_m + h \in S_\rho}} \left| \left(DE|'_{S_\rho}(v_m), \frac{h}{\|h\|} \right) \right| \leq \epsilon_m. \tag{3.9}$$

Now, for $h \in S_\rho(v_m)$, let $h = T(h) + N(h)$ where $T(h) \in \mathcal{T}S_\rho(v_m)$ and $N(h) \perp \mathcal{T}S_\rho(v_m)$. Owing to the tangency condition, we have

$$\lim_{h \to 0} \frac{N(h)}{\|h\|} = 0. \tag{3.10}$$

It follows that

$$\sup_{\substack{\phi \in \mathcal{T}S_\rho(v_m), \\ \|\phi\|=1}} \left(DE|'_{S_\rho}(v_m), \phi \right) < \epsilon'_m \longrightarrow 0 \tag{3.11}$$

as $m \to \infty$, yielding

$$\left\| DE|'_{S_\rho}(v_m) \right\| \longrightarrow 0 \tag{3.12}$$

proving the result.   □

Proof of Theorem 2.2. Let H_0 be the eigenspace corresponding to μ_0 and define H_1 as the orthogonal complement to H_0 in H. Let P_0 be the projector of H onto H_0, and P_1 project H onto H_1. Let ν be the largest positive eigenvalue of T different to μ_0, letting $\nu = 0$ if this is nonexistent.

Since T is completely continuous, we have the standard decomposition for any $u \in H$:

$$Tu = \sum_{i=0}^{\infty} \mu_i(u, e_i)e_i. \tag{3.13}$$

In particular, this means that $(TP_1u, u) \leq \nu\|P_1u\|^2$ for all $u \in H$.

From Lemma 3.3, we know that $\Phi(u)$ is approximated to $(1/2)(Tu, u)$. For some small $\epsilon > 0$, let δ be a suitably small number from the definition of quadratic approximation.

Denote by $\rho \in (0, \delta)$ a number such that for all $\|u\| \leq \rho$,

$$\left| \Phi(u) - \frac{1}{2}(Tu, u) \right| \leq \epsilon_1(u, u), \tag{3.14}$$

$$\|\Phi'(u) - Tu\| \leq \epsilon_2\|u\|, \tag{3.15}$$

where ϵ_1 and ϵ_2 are chosen sufficiently small that

$$\epsilon_1 < \frac{\mu_0 - \nu}{6}, \qquad \epsilon_2 < \mu_0 \sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}}.$$

$$\lambda_0 - \xi < \left(\frac{1}{\lambda_0} + \epsilon_2\right)^{-1}, \qquad \lambda_0 + \xi > \left(\frac{1}{\lambda_0}\sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}} - \epsilon_2\right)^{-1}. \tag{3.16}$$

Consequently, for any $u \in H_0$, $\|u\| \le \rho$, we have that

$$\Phi(u) \ge \frac{1}{2}(Tu, u) - \left|\Phi(u) - \frac{1}{2}(Tu, u)\right|$$

$$\ge \left(\frac{\mu_0}{2} - \epsilon_1\right)(u, u). \tag{3.17}$$

For a normalised eigenfunction of T, $\varphi_0 \in H_0$, define

$$c_\rho \equiv \sup_{u \in S_\rho} \Phi(u) \ge \Phi(\rho\varphi_0) \ge \left(\frac{\mu_0}{2} - \epsilon_1\right)\rho^2. \tag{3.18}$$

Since Φ is not weakly continuous, we cannot immediately guarantee that the supremum is achieved. Instead of relying upon complete continuity, we will invoke the (PS) condition.

Applying Corollary 3.5 to $-\Phi$, there exists a sequence $\{u_n\} \subset S_\rho$ such that $\Phi(u_n) \to c_\rho$ and $\|\Phi'|_{S_\rho}(u_n)\| \to 0$ as $n \to \infty$. By definition, for any sequence $\{v_n\}$ satisfying $\lim\|v_n\| < C < \infty$ and $v_n \in \mathcal{T}S_\rho(u_n)$, it follows that $(\Phi'(u_n), v_n) \to 0$.

Taking a subsequence if necessary, we have that for all n sufficiently large,

$$\Phi(u_n) \ge \left(\frac{\mu_0}{2} - 2\epsilon_1\right)\rho^2. \tag{3.19}$$

Letting

$$\alpha_n = \frac{(u_n, u_n)}{(\Phi'(u_n), u_n)} = \frac{\rho^2}{(\Phi'(u_n), u_n)}, \tag{3.20}$$

it follows that

$$\left(I'_{\alpha_n}(u_n), u_n\right) = 0. \tag{3.21}$$

In the next part of the argument, bounds are placed on $\lim_{n \to \infty} \alpha_n$. For all n sufficiently large (3.19) holds. However by (3.14),

$$
\begin{aligned}
\Phi(u_n) &\leq \frac{1}{2}(Tu_n, u_n) + \left| \Phi(u_n) - \frac{1}{2}(Tu_n, u_n) \right| \\
&\leq \frac{\mu_0}{2}\|P_0 u_n\|^2 + \frac{\nu}{2}\|P_1 u_n\|^2 + \epsilon_1 \rho^2.
\end{aligned}
\tag{3.22}
$$

Hence,

$$
\left(\frac{\mu_0}{2} - 2\epsilon_1 \right)\rho^2 \leq \epsilon_1 \rho^2 + \frac{\mu_0}{2}\|P_0 u_n\|^2 + \frac{\nu}{2}\|P_1 u_n\|^2.
\tag{3.23}
$$

But $\|u_n\|^2 = \|P_0 u_n\|^2 + \|P_1 u_n\|^2$ so

$$
\|P_0 u_n\|^2 \geq \rho^2 - \frac{6\epsilon_1 \rho^2}{\mu_0 - \nu},
\tag{3.24}
$$

$$
\|P_1 u_n\|^2 \leq \frac{6\epsilon_1 \rho^2}{\mu_0 - \nu}.
\tag{3.25}
$$

From (3.24)

$$
\|Tu_n\|^2 \geq \|TP_0 u_n\|^2 = \mu_0^2\|P_0 u_n\|^2 \geq \mu_0^2 \rho^2 - \frac{6\epsilon_1 \mu_0^2 \rho^2}{\mu_0 - \nu}
\tag{3.26}
$$

and by (3.15),

$$
\begin{aligned}
\left\| \Phi'(u_n) \right\| &\geq \|Tu_n\| - \left\| \Phi'(u_n) - Tu_n \right\| \\
&\geq \mu_0 \rho \sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}} - \epsilon_2 \|u_n\| \\
&= \rho \left\{ \mu_0 \sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}} - \epsilon_2 \right\}.
\end{aligned}
\tag{3.27}
$$

Now, for each n, any $w \in H$ may be expressed as $w = t_n u_n + v_n$ where $t_n \in \mathbf{R}$ and $v_n \in TS_\rho(u_n)$. Using the information that $(\Phi'(u_n), v_n) = o(1)$ as $n \to \infty$,

$$
\begin{aligned}
\left\| \Phi'(u_n) \right\| &= \sup_{w \in S_1} \left| (\Phi'(u_n), w) \right| \\
&= \sup \left\{ \left| (\Phi'(u_n), t_n u_n + v_n) \right| : t_n \in \mathbf{R}, v_n \in TS_\rho(u_n), \|t_n u_n + v_n\| = 1 \right\} \\
&= \sup \left\{ \left| (\Phi'(u_n), t_n u_n) \right| : \|t_n u_n\| \leq 1 \right\} + o(1) \\
&= \frac{1}{\rho}(\Phi'(u_n), u_n) + o(1).
\end{aligned}
\tag{3.28}
$$

Consequently,

$$\lim_{n \to \infty} (\Phi'(u_n), u_n) = \lim_{n \to \infty} \rho \|\Phi'(u_n)\| \geq \rho^2 \left\{ \mu_0 \sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}} - \epsilon_2 \right\} > 0. \tag{3.29}$$

Thus,

$$\lim_{n \to \infty} \alpha_n \leq \left(\mu_0 \sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}} - \epsilon_2 \right)^{-1}. \tag{3.30}$$

For the other bound,

$$
\begin{aligned}
\|\Phi'(u_n)\| &\leq \|\Phi'(u_n) - Tu_n\| + \|Tu_n\| \\
&\leq \epsilon_2 \|u_n\| + \left\| \sum_{i=0}^{\infty} \mu_i (u_n, e_i) e_i \right\| \\
&\leq (\epsilon_2 + \mu_0)\rho.
\end{aligned}
\tag{3.31}
$$

Thus,

$$\lim_{n \to \infty} (\Phi'(u_n), u_n) = \lim_{n \to \infty} \rho \|\Phi'(u_n)\| \leq \rho^2 (\epsilon_2 + \mu_0). \tag{3.32}$$

In combination with (3.30),

$$[\mu_0 + \epsilon_2]^{-1} \leq \lim_{n \to \infty} \alpha_n \equiv \alpha_0 \leq \left[\mu_0 \sqrt{1 - \frac{6\epsilon_1}{\mu_0 - \nu}} - \epsilon_2 \right]^{-1}. \tag{3.33}$$

As $\epsilon \to 0$, ρ can be chosen small so that ϵ_1 and $\epsilon_2 \to 0$.

We now consider the sequence $\{u_n\}$ acting on the functional $I_{\alpha_0}(u)$. Again decomposing any $w \in H$ as $w = t_n u_n + v_n$, we have that

$$
\begin{aligned}
\|I'_{\alpha_0}(u_n)\| &= \sup_{\|w\|=1} \left| (I'_{\alpha_0}(u_n), w) \right| \\
&= \sup \left\{ \left| (u_n, t_n u_n + v_n) - \alpha_0 (\Phi'(u_n), t_n u_n + v_n) \right| : t_n u_n + v_n \in S_1 \right\}.
\end{aligned}
\tag{3.34}
$$

Now, $u_n \perp v_n$, so $(u_n, v_n) = 0$. Also, u_n is a maximising sequence for Φ on S_ρ, so $(\Phi'(u_n), v_n) = o(1)$ as $n \to \infty$, leaving us with

$$
\begin{aligned}
\left\| I'_{\alpha_0}(u_n) \right\| &= \sup \left\{ \left| (u_n, t_n u_n) - \alpha_0 (\Phi'(u_n), t_n u_n) \right| : \|t_n u_n\| \leq 1 \right\} + o(1) \\
&= \sup \left\{ t_n \left| (u_n, u_n) - \alpha_0 (\Phi'(u_n), u_n) \right| : \|t_n u_n\| \leq 1 \right\} + o(1) \\
&= \sup \left\{ t_n \left| (u_n, u_n) - \alpha_0 (\Phi'(u_n), u_n) \right| : t_n \leq \frac{1}{\rho} \right\} + o(1) \\
&= \frac{1}{\rho} \left| (u_n, u_n) - \alpha_0 (\Phi'(u_n), u_n) \right| + o(1).
\end{aligned}
\tag{3.35}
$$

But $\|u_n\|^2 - \alpha_n(\Phi'(u_n), u_n) = 0$ so

$$
\begin{aligned}
\left\| I'_{\alpha_0}(u_n) \right\| &= \frac{1}{\rho} \left| \|u_n\|^2 - \alpha_0 (\Phi'(u_n), u_n) \right| + o(1) \\
&= \frac{1}{\rho} \left| \|u_n\|^2 - \alpha_n (\Phi'(u_n), u_n) \right| + o(1) \longrightarrow 0.
\end{aligned}
\tag{3.36}
$$

We also have that

$$
I_{\alpha_0}(u_n) = \frac{1}{2}\rho^2 - \alpha_0 \Phi(u_n).
\tag{3.37}
$$

For sufficiently small $\rho > 0$ we can ensure that $\lim_{n \to \infty} I_{\alpha_0}(u_n)$ is arbitrarily small. Hence (3.36) and (3.37) imply that u_n is a $(PS)_c$-sequence for I_{α_0}, where c vanishes as ρ tends to 0. Since I_{α_0} satisfies the $(PS)_c$-condition for c in some neighbourhood of zero, we have that u_n is strongly convergent to $u_0 \in S_\rho$, where u_0 must be a maximiser for (3.18).

The Lusternik Theorem 3.1 completes the proof by showing that u_0 must be an eigenfunction of Φ' and α_0 a characteristic value for Φ':

$$
\Phi'(u_0) = \mu u_0,
\tag{3.38}
$$

where $1/\alpha_0 \equiv \mu \in \mathbf{R}$.   □

Proof of Theorem 2.3. Let $k > 0$, $r_0 > 0$, $p > 1$ be such that

$$
\|R(u)\| \leq k\|u\|^p \quad \text{for } \|u\| < r_0.
\tag{3.39}
$$

Now

$$
\Phi(u) = \frac{1}{2}(Tu, u) + \int_0^1 (R(tu), u)\, dt.
\tag{3.40}
$$

From (3.39), and since $p > 1$,

$$\frac{1}{2}(Tu, u) - \frac{k}{2}r^{p+1} \leq \Phi(u) \leq \frac{1}{2}(Tu, u) + \frac{k}{2}r^{p+1} \qquad (3.41)$$

for any $u : \|u\| \leq r < r_0$. We claim that c_r as defined in (3.18) satisfies the following estimate:

$$\left| c_r - \frac{1}{2}\mu_0 r^2 \right| \leq \frac{k}{2}r^{p-1} \quad \text{for } r < r_0. \qquad (3.42)$$

For one half of the estimate, use the normalised eigenfunction φ_1 and note that

$$c_r = \sup_{S_r} \Phi(u) > \Phi(r\varphi_1) \geq \frac{1}{2}\mu_0 r^2 - \frac{k}{2}r^{p+1}. \qquad (3.43)$$

For the other half, by the Rayleigh characterisation of the principal eigenvalue,

$$\mu_0 = \sup_{0 \neq u \in H} \frac{(Tu, u)}{(u, u)} \qquad (3.44)$$

giving that $(Tu, u) \leq \mu_0(u, u)$. Inserting this into (3.41) yields

$$\Phi(u) \leq \frac{1}{2}\mu_0 r^2 + \frac{k}{2}r^{p+1} \qquad (3.45)$$

and the claim follows.

We now use the expression $\Phi(u_r) = c_r$ to estimate μ_r, the eigenvalue corresponding with $u_r \in S_r$. We have $\mu_r r^2 = (Au_r, u_r)$, so

$$\begin{aligned}
\mu_r - \mu_0 &= \frac{1}{r^2}\left[(A(u_r), u_r) - \mu_0(u_r, u_r)\right] \\
&= \frac{1}{r^2}(A(u_r) - Tu_r + Tu_r - \mu_0 u_r, u_r)
\end{aligned} \qquad (3.46)$$

and using (3.39),

$$\begin{aligned}
|\mu_r - \mu_0| &\leq \frac{1}{r^2}\|A(u_r) - Tu_r\|\|u_r\| + \frac{2}{r^2}\left|\frac{(Tu_r, u_r)}{2} - \frac{\mu_0}{2}r^2\right| \\
&\leq \frac{1}{r^2}kr^{p+1} + \frac{2}{r^2}\left|\frac{(Tu_r, u_r)}{2} - \Phi(u_r) + c_r - \frac{\mu_0}{2}r^2\right|.
\end{aligned} \qquad (3.47)$$

Combining this with (3.41) and (3.42) provides the inequality:

$$|\mu_r - \mu_0| \leq kr^{p-1} + 2kr^{p-1} = 3kr^{p-1} \quad \text{for } r < r_0 \qquad (3.48)$$

giving the conclusion. $\qquad \square$

4. Applications

Chiappinelli [6] was able to show that each eigenvalue of $-\Delta$ on a bounded domain $\Omega \subset \mathbf{R}^N$ forms a bifurcation point for the problem

$$-\Delta u = \lambda(u + f(x, u)) \quad \text{for } x \in \Omega; \quad u(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{4.1}$$

where $f : \Omega \times \mathbf{R} \mapsto \mathbf{R}$ is a subcritical Carathéodory function. We extend this result to a family of problems with critical nonlinearities.

Brézis and Nirenberg [16] tackled the problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u \quad \text{on } \Omega; \quad u(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{4.2}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain and solutions are sought in the Sobolev space $W_0^{1,2}(\Omega)$, endowed with norm $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$. Another problem which may be considered is

$$-\Delta u = \lambda\left(u + |u|^{2^*-2}u\right) \quad \text{on } \Omega; \quad u(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{4.3}$$

Solutions to (4.2) and (4.3) correspond to critical points of the functional

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int u^2 - \frac{1}{2^*}\int |u|^{2^*}, \\
\tilde{I}_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda\left(\frac{1}{2}\int u^2 - \frac{1}{2^*}\int |u|^{2^*}\right),
\end{aligned} \tag{4.4}$$

respectively.

The following lemma has been derived in [16].

Lemma 4.1. *For any $\lambda \in \mathbf{R}$, I_λ satisfies the $(PS)_c$-condition for $c < c^* = (1/N)S^{N/2}$.*

Following the same proof, one may derive the following.

Lemma 4.2. *For $\lambda > 0$, $\tilde{I}_\lambda$ satisfies the $(PS)_c$ condition for $c < c_\lambda^* = (1/N)S^{N/2}/\lambda^{(N-2)/2}$.*

Let λ_0 be the principal characteristic value of the linear problem $-\Delta u = \lambda u$, $u \in W_0^{1,2}(\Omega)$. Define the operator T by $(Tu, v) = \int_\Omega uv\, dx$. By the Sobolev embedding theorem, T is a completely continuous operator. Define the operator R by $(R(u), v) = \int |u|^{2^*-2}uv$. The problem (4.3) now becomes $u = \lambda(Tu + R(u))$.

Theorem 4.3. *For sufficiently small $r > 0$, there exists a solution (λ_r, u_r) to (4.3) with $\|u_r\| = r$. One has $\lambda_r \to \lambda_0$ as $r \to 0$ and*

$$\lambda_r = \lambda_0 + O\left(r^{2^*-2}\right) \quad \text{as } r \longrightarrow 0. \tag{4.5}$$

An identical result holds for problem (4.2).

Proof. Clearly u is a weak solution to (4.3) if

$$A(u) \equiv Tu + R(u) = \mu u, \tag{4.6}$$

where $\mu = 1/\lambda$. If $\Phi(u) = (1/2)\int u^2 + (1/2^*)\int |u|^{2^*}$, then $A(u) = \Phi'(u)$. Lemma 4.2 verifies that $\tilde{I}_\lambda$ satisfies the $(PS)_c$ condition in the neighbourhood specified in Theorem 2.2. Theorem 2.3 then gives the first result.

Select any sufficiently small $R > 0$. Owing to the asymptotic nature of λ_r, it is possible to solve $\lambda_r^{1/(2^*-2)} r = R$ for r. With such a solution $r > 0$, there exists u_r with $\|\nabla u_r\| = r$ and $\lambda_r \in \mathbf{R}$ such that (λ_r, u_r) solves (4.3). Now letting $v^R = \lambda_r^{1/(2^*-2)} u_r$, and $\lambda^R = \lambda_r$ it follows that (λ^R, v^R) forms a solution to (4.2). Analysing the asymptotic properties of the bifurcation,

$$\lim_{R\to 0} \frac{\lambda^R - \lambda_0}{R^{2^*-2}} = \lim_{r\to 0} \frac{\lambda_r - \lambda_0}{\left(r\lambda_r^{1/(2^*-2)}\right)^{2^*-2}}$$
$$= \frac{1}{\lambda_0} \lim_{r\to 0} \frac{\lambda_r - \lambda_0}{r^{2^*-2}} = C \tag{4.7}$$

for some constant C providing the second result. $\qquad\square$

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