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Research Article

Rough Filters in *BL*-Algebras

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We apply the rough set theory to *BL*-algebras. As a generalization of filters (subalgebras) of *BL*-algebras, we introduce the notion of rough filters (subalgebras) of *BL*-algebras and investigate some of their properties.

1. Introduction

The rough sets theory introduced by Pawlak [13] has often proved to be an excellent mathematical tool for the analysis of a vague description of objects (called actions in decision problems). Many different problems can be addressed by rough sets theory. During the last few years this formalism has been approached as a tool used in connection with many different areas of research. There have been investigations of the relations between rough sets theory and the Dempster-Shafer theory and between rough sets and fuzzy sets. Rough sets theory has also provided the necessary formalism and ideas for the development of some propositional machine learning systems. It has also been used for, among many others, knowledge representation; data mining; dealing with imperfect data; reducing knowledge representation and for analyzing attribute dependencies. The notions of rough relations and rough functions are based on rough sets theory and can be applied as a theoretical basis for rough controllers, among others. An algebraic approach to rough sets has been given by Iwinski [1]. Rough set theory is applied to semigroups and groups (see [2, 3]). In 1994, Biswas and Nanda [4] introduced and discussed the concept of rough groups and rough subgroups. Jun [5] applied rough set theory to BCK-algebras. Recently, Rasouli [6] introduced and studied the notion of roughness in MV-algebras.

BL-algebras are the algebraic structures for Hàjek Basic Logic (BL-logic) [7], arising from the continuous triangular norms (t-norms), familiar in the frameworks of fuzzy set theory. The language of propositional Hàjek basic logic [7] contains the binary connectives $\circ$, $\Rightarrow$ and the constant $\overline{0}$.

Axioms of BL are as follows:

$(A1)$ $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$,

$(A2)$ $(\varphi \circ \psi) \Rightarrow \varphi$,

$(A3)$ $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$,

$(A4)$ $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$,

$(A5a)$ $(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$,

$(A5b)$ $((\varphi \circ \psi) \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$,

$(A6)$ $((\varphi \Rightarrow \psi) \Rightarrow \omega) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega)$,

$(A7)$ $\overline{0} \Rightarrow \omega$.

BL-algebras rise as Lindenbaum algebras from the above logical axioms in a similar manner that Boolean algebras or MV-algebras do from Classical logic or Lukasiewicz logic, respectively. MV-algebras are BL-algebras while the converse, in general, is not true. Indeed, BL-algebras with involutory complement are MV-algebras. Moreover, Boolean algebras are MV-algebras and MV-algebras with idempotent product are Boolean algebras. Filters theory plays an important role in studying these logical algebras. From logical point of view, various filters correspond to various sets of provable formula.

In this paper, we apply the rough set theory to BL-algebras, and we introduce the notion of (lower) upper rough subalgebras and (lower) upper rough filters of BL-algebras and obtain some related results.

2. Preliminaries

Definition 2.1. A BL-algebra is an algebra $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

$(BL1)$ $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,

$(BL2)$ $(L, *, 1)$ is a commutative monoid,

$(BL3)$ $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in L$, (i.e., $*$ and $\rightarrow$ form an adjoint pair),

$(BL4)$ $a \wedge b = a * (a \rightarrow b)$,

$(BL5)$ $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

Examples of BL-algebras [7] are t-algebras $([0,1], \wedge, \vee, *_t, \rightarrow_t, 0, 1)$, where $([0,1], \wedge, \vee, 0, 1)$ is the usual lattice on $[0,1]$ and $*_t$ is a continuous t-norm, whereas $\rightarrow_t$ is the corresponding residuum.

If $(L, \wedge, \vee, ^-, 0, 1)$ is a Boolean algebra, then $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra where the operation $*$ coincides with $\wedge$ and $x \rightarrow y = x^- \vee y$, for all $x, y \in L$.

From now $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ or simply L is a BL-algebra.

A BL-algebra is called an *MV-algebra* if $\neg\neg x = x$, for all $x \in L$, where $\neg x = x \rightarrow 0$.

A BL-algebra is nontrivial if $0 \neq 1$. For any BL-algebra L, $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. We denote the set of natural numbers by N and define $a^0 = 1$ and

$a^n = a^{n-1} * a$, for $n \in N \setminus \{0\}$. The *order* of $a \in L$, $a \neq 1$, in symbols $\mathrm{ord}(a)$ is the smallest $n \in N$ such that $a^n = 0$; if no such n exists, then $\mathrm{ord}(a) = \infty$.

Lemma 2.2 (see [7–11]). *In any BL-algebra L, the following properties hold for all $x, y, z \in L$:*

(1) $x \leq y$ *if and only if* $x \to y = 1$,

(2) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$,

(3) *if* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,

(4) $y \to x \leq (z \to y) \to (z \to x)$ *and* $x \to y \leq (y \to z) \to (x \to z)$,

(5) $x \leq y$ *implies* $x * z \leq y * z$,

(6) $1 \to x = x$, $x \to x = 1$, $x \leq y \to x$, $x \to 1 = 1$, $0 \to x = 1$,

(7) $(x \to y) \to (x \to z) = (x \wedge y) \to z$,

(8) $x \to y \leq (x * z) \to (y * z)$,

(9) $x \leq y \to x$ *and* $x * y \leq x, y$,

(10) $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$.

For any BL-algebra L, $B(L)$ denotes the Boolean algebra of all complemented elements in lattice of L.

Proposition 2.3 (see [7, 11]). *For $e \in L$, the following are equivalent:*

(i) $e \in B(L)$,

(ii) $e * e = e$ *and* $e = e^{--}$,

(iii) $e * e = e$ *and* $e^{-} \to e = e$,

(iv) $e \vee e^{-} = 1$,

(v) $(e \to x) \to e = e$, *for every* $x \in L$.

Hàjek [7] defined a *filter* of a BL-algebra L to be a nonempty subset F of L such that (i) if $a, b \in F$ implies $a * b \in F$ and (ii) if $a \in F$, $a \leq b$, then $b \in F$. Turunen [8] defined a *deductive system* of a BL-algebra L to be a nonempty subset D of L such that (i) $1 \in D$ and (ii) $x \in D$ and $x \to y \in D$ imply $y \in D$. Note that a subset F of a BL-algebra L is a deductive system of L if and only if F is a filter of L [8].

Let U denote a nonempty set of objects called the univers, and let $\theta \subseteq U \times U$ be an equivalence relation on U. The pair $(U;\theta)$ is called a Pawlak approximation space. The equivalence relation θ partitions the set U into disjoint subsets. Let U/θ denote the quotient set consisting of all the equivalence classes of θ. The empty set $\emptyset$ and the elements of U/θ are called elementary sets. A finite union of elementary sets, that is, the union of one or more elementary sets, is called a composed set [12]. The family of all composed sets is denoted by Com(Apr). It is a subalgebra of the Boolean algebra 2^U formed by the power set of U. A set which is a union of elementary sets is called a definable set [12]. The family of all definable sets is denoted by Def(Apr). For a finite universe, the family of definable sets is the same as the family of composed sets. A Pawlak approximation space defines uniquely a topological space $(U;\mathrm{Def(Apr)})$, in which Def(Apr) is the family of all open and closed sets [13]. In connection to rough set theory there exist two views. The operator-oriented view interprets rough set theory as an extension of set theory with two additional unary operators.

Under such a view, lower and upper approximations are related to the interior and closure operators in topological spaces, the necessity and possibility operators in modal logic, and lower and upper approximations in interval structures. The set-oriented view focuses on the interpretation and characterization of members of rough sets. Both operator-oriented and set-oriented views are useful in the understanding and application of the theory of rough sets.

Definition 2.4. For an approximation space $(U;\theta)$, by a rough approximation in $(U;\theta)$ we mean a mapping $\mathrm{Apr} : P(U) \to P(U) \times P(U)$ defined for every $X \in P(U)$ by $\mathrm{Apr}(X) = (\underline{\mathrm{Apr}}(X); \overline{\mathrm{Apr}}(X))$, where

$$\underline{\mathrm{Apr}}(X) = \{x \in U : [x]_\theta \cap X \neq \emptyset\}, \qquad \overline{\mathrm{Apr}}(X) = \{x \in U : [x]_\theta \subseteq X\}. \tag{2.1}$$

$\underline{\mathrm{Apr}}(X)$ is called a lower rough approximation of X in $(U;\theta)$, whereas $\overline{\mathrm{Apr}}(X)$ is called an upper rough approximation of X in $(U;\theta)$.

Let F be a filter of a BL-algebra L. Define relation $\equiv_F$ on L as follows:

$$x \equiv_F y \quad \text{iff } x \longrightarrow y \in F,\ y \longrightarrow x \in F. \tag{2.2}$$

Then $\equiv_F$ is a congruence relation on L. L/F denotes the set of all congruence classes of $\equiv_F$, that is, $L/F = \{[x]_F : x \in L\}$, thus L/F is a BL-algebra.

3. Lower and Upper Approximations in BL-Algebras

Definition 3.1. Let L be a BL-algebra and F a filter of L. For any nonempty subset X of L, the sets

$$\underline{\mathrm{Apr}_F}(X) = \{x \in L : [x]_F \subseteq X\}, \qquad \overline{\mathrm{Apr}_F}(X) = \{x \in L : [x]_F \cap X \neq \emptyset\} \tag{3.1}$$

are called, respectively, the *lower and upper approximations* of the set X with respect to the filter F. Therefore, when $U = L$ and θ is the induced congruence relation by filter F, then we use the pair (L, F) instead of the approximation space (U, θ). Also, in this case we use the symbols $\underline{\mathrm{Apr}_F}(X)$ and $\overline{\mathrm{Apr}_F}(X)$ instead of $\underline{\mathrm{Apr}}(X)$ and $\overline{\mathrm{Apr}}(X)$.

Proposition 3.2. *Let (L, F) be an approximation space and $X, Y \subseteq L$. Then the following hold:*

(1) $\underline{\mathrm{Apr}_F}(X) \subseteq X \subseteq \overline{\mathrm{Apr}_F}(X)$,

(2) $\underline{\mathrm{Apr}_F}(X \cap Y) = \underline{\mathrm{Apr}_F}(X) \cap \underline{\mathrm{Apr}_F}(Y)$,

(3) $\underline{\mathrm{Apr}_F}(X) \cup \underline{\mathrm{Apr}_F}(Y) \subseteq \underline{\mathrm{Apr}_F}(X \cup Y)$,

(4) $\overline{\mathrm{Apr}_F}(X \cap Y) \subseteq \overline{\mathrm{Apr}_F}(X) \cap \overline{\mathrm{Apr}_F}(Y)$,

(5) $\overline{\mathrm{Apr}_F}(X) \cup \overline{\mathrm{Apr}_F}(Y) = \overline{\mathrm{Apr}_F}(X \cup Y)$,

(6) $\underline{\mathrm{Apr}_F}(\overline{\mathrm{Apr}_F}(X)) \subseteq \overline{\mathrm{Apr}_F}(\overline{\mathrm{Apr}_F}(X))$,

(7) $\underline{\mathrm{Apr}}_F(\underline{\mathrm{Apr}}_F(X)) \subseteq \overline{\mathrm{Apr}}_F(\underline{\mathrm{Apr}}_F(X))$,

(8) $\underline{\mathrm{Apr}}_F(X^c) = (\overline{\mathrm{Apr}}_F(X))^c$, $\overline{\mathrm{Apr}}_F(X^c) = (\underline{\mathrm{Apr}}_F(X))^c$,

(9) *if* $X \neq \emptyset$, *then* $\overline{\mathrm{Apr}}_L(X) = L$,

(10) *if* $X \neq L$, *then* $\underline{\mathrm{Apr}}_L(X) = \emptyset$.

Proof. The proof is similar to the proof of Theorem 2.1 of [14]. □

Proposition 3.3. *Let* (L, F) *be an approximation space. Then,* $\overline{\mathrm{Apr}}_F$ *is a closure operator and* $\underline{\mathrm{Apr}}_F$ *is an interior operator.*

Proof. Let X be an arbitrary subset of L.

(i) By Proposition 3.2 part (1), we have $X \subseteq \overline{\mathrm{Apr}}_F(X)$.

(ii) We will show that $\overline{\mathrm{Apr}}_F(\overline{\mathrm{Apr}}_F(X)) = \overline{\mathrm{Apr}}_F(X)$. Suppose that $x \in \overline{\mathrm{Apr}}_F(\overline{\mathrm{Apr}}_F(X))$. By Definition 3.1, we have $[x]_F \cap \overline{\mathrm{Apr}}_F(X) \neq \emptyset$. Hence there exists $y \in \overline{\mathrm{Apr}}_F(X)$ such that $[x]_F = [y]_F$. By Definition 3.1, $[y]_F \cap X \neq \emptyset$, and so we get that $[x]_F \cap X \neq \emptyset$. Hence $x \in \overline{\mathrm{Apr}}_F(X)$ and then $\overline{\mathrm{Apr}}_F(\overline{\mathrm{Apr}}_F(X)) \subseteq \overline{\mathrm{Apr}}_F(X)$. By part (1) of Proposition 3.2, we have $\overline{\mathrm{Apr}}_F(X) \subseteq \overline{\mathrm{Apr}}_F(\overline{\mathrm{Apr}}_F(X))$.

(iii) Suppose that $X \subseteq Y$. We will show that $\overline{\mathrm{Apr}}_F(X) \subseteq \overline{\mathrm{Apr}}_F(Y)$. Let $x \in \overline{\mathrm{Apr}}_F(X)$. Then $[x]_F \cap X \neq \emptyset$. Since $X \subseteq Y$, we get that $[x]_F \cap Y \neq \emptyset$, that is, $x \in \overline{\mathrm{Apr}}_F(Y)$. Analogously, we can prove that $\underline{\mathrm{Apr}}_F$ is an interior operator. □

Definition 3.4. (L, F) is an approximation space and $X \subseteq L$. X is called *definable with respect to* F, if $\underline{\mathrm{Apr}}_F(X) = \overline{\mathrm{Apr}}_F(X)$.

Proposition 3.5. *Let* (L, F) *be an approximation space. Then* $\emptyset$, L, *and* $[x]_F$ *are definable respect to* F.

Proof. The proof is straightforward. □

Proposition 3.6. *Let* (L, F) *be an approximation space. If* $F = \{1\}$, *then every subset of* L *is definable.*

Proof. Let X be an arbitrary subset of L. We have

$$[x]_F = \{y \in L : x \longrightarrow y = 1, y \longrightarrow x = 1\} = \{y \in L : x = y\} = \{x\}, \tag{3.2}$$

for all $x \in L$. We get that

$$\begin{aligned}\underline{\mathrm{Apr}}_F(X) &= \{x \in L : [x]_F \subseteq X\} = \{x \in L : \{x\} \subseteq X\} = X,\\ \overline{\mathrm{Apr}}_F(X) &= \{x \in L : [x]_F \cap X \neq \emptyset\} = \{x \in L : \{x\} \cap X \neq \emptyset\} = X.\end{aligned} \tag{3.3}$$

Hence X is definable. □

Let X and Y be nonempty subsets of L. Then we define two sets

$$X * Y = \{x * y : x \in X, y \in Y\},$$
$$X \longrightarrow Y = \{x \longrightarrow y : x \in X, y \in Y\}. \tag{3.4}$$

If either X or Y is empty, then we define $X * Y = \emptyset$ and $X \to Y = \emptyset$. Clearly, $X * Y = Y * X$, for every $X, Y \in L$. $X * Y$ and $X \to Y$ are called, respectively, Minkowski product and Minkowski arrow. However, Minkowski arrow is not the residuum of Minkowski product.

Lemma 3.7. *Let (L, F) be an approximation space and $X, Y \subseteq L$. Then the following hold:*

(1) $\overline{\text{Apr}_F}(X) * \overline{\text{Apr}_F}(Y) \subseteq \overline{\text{Apr}_F}(X * Y)$,

(2) $\overline{\text{Apr}_F}(X) \to \overline{\text{Apr}_F}(Y) \subseteq \overline{\text{Apr}_F}(X \to Y)$.

Proof. (1) Let $z \in \overline{\text{Apr}_F}(X) * \overline{\text{Apr}_F}(Y)$, and so there exist $x \in \overline{\text{Apr}_F}(X)$ and $y \in \overline{\text{Apr}_F}(Y)$ such that $z = x * y$. We have $[x]_F \cap X \neq \emptyset$ and $[y]_F \cap Y \neq \emptyset$. There exist $a \in X$ and $b \in Y$ such that $[x]_F = [a]_F$ and $[y]_F = [b]_F$. We get that $a * b \in X * Y$ such that $[a * b]_F = [x * y]_F$. Hence $[x * y]_F \cap (X * Y) \neq \emptyset$, that is, $\overline{\text{Apr}_F}(X) * \overline{\text{Apr}_F}(Y) \subseteq \overline{\text{Apr}_F}(X * Y)$.

Similarly, we can prove (2). □

Definition 3.8. Let (L, F) be an approximation space. A nonempty subset S of L is called an *upper* (resp., a lower) *rough subalgebra* (or filter) of L, if the upper (resp., the lower) approximation of S is a subalgebra (or filter) of L. If S is both an upper and a lower rough subalgebra (or filter) of L, we say S is a rough subalgebra (or filter) of L.

Proposition 3.9. *Let (L, F) be an approximation space. If S is a subalgebra of L, then S is an upper rough subalgebra of L.*

Proof. We will show that $\overline{\text{Apr}_F}(S)$ is a subalgebra of L. Since $0, 1 \in S$, $0 \in [0]_F$ and $1 \in [1]_F$, then $[0]_F \cap S \neq \emptyset$ and $[1]_F \cap S \neq \emptyset$. Hence $0, 1 \in \overline{\text{Apr}_F}(S)$. Taking $X = Y = S$ in Lemma 3.7, by S is a subalgebra of L, we obtain

$$\overline{\text{Apr}_F}(S) * \overline{\text{Apr}_F}(S) \subseteq \overline{\text{Apr}_F}(S * S) \subseteq \overline{\text{Apr}_F}(S),$$
$$\overline{\text{Apr}_F}(S) \longrightarrow \overline{\text{Apr}_F}(S) \subseteq \overline{\text{Apr}_F}(S \longrightarrow S) \subseteq \overline{\text{Apr}_F}(S \longrightarrow S). \tag{3.5}$$

Hence S is an upper rough subalgebra. □

Let (L, F) be an approximation space and S a subalgebra of L. The following example shows that S may not be a lower rough subalgebra of L in general.

Example 3.10. Let $L = \{0, a, b, c, d, 1\}$, where $0 < d < c < a, b < 1$. Define $*$ and $\to$ as follow:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

(3.6)

Then $(L, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. It is easy to check that $F = \{1, b\}$ is a filter of L and $S = \{0, a, 1\}$ is a subalgebra of L. Since $[1]_F = F \nsubseteq S$, then S is not be a lower rough subalgebra of L.

Theorem 3.11. *Let (L, F) be an approximation space and X a nonempty subset of L. Then the following hold:*

(i) $X \subseteq F$ *if and only if* $\overline{\mathrm{Apr}_F}(X) = F$,

(ii) $F \subseteq X$ *if and only if* $F \subseteq \underline{\mathrm{Apr}_F}(X)$.

Proof. (i) Let $X \subseteq F$ and $z \in \overline{\mathrm{Apr}_F}(X)$. Then $[z]_F \cap X \neq \emptyset$, and so there exists $x \in X$ such that $[x]_F = [z]_F$. Since $X \subseteq F$, then $[x]_F = F$. So $z \in F$, that is, $\overline{\mathrm{Apr}_F}(X) \subseteq F$. Now let $z \in F$. Then $[z]_F = F$ and so $[z]_F \cap X = F \cap X = X \neq \emptyset$. Thus $z \in \overline{\mathrm{Apr}_F}(X)$, and hence $\overline{\mathrm{Apr}_F}(X) = F$.

The converse follows from Proposition 3.2 part (1).

(ii) Let $F \subseteq X$ and $x \in F$. Then $[x]_F = F \subseteq X$, and so $x \in \underline{\mathrm{Apr}_F}(X)$. Therefore $F \subseteq \underline{\mathrm{Apr}_F}(X)$. □

Theorem 3.12. *Let (L, F) be an approximation space and J a filter of L. Then the following hold:*

(1) $F \subseteq J$ *if and only if* $\overline{\mathrm{Apr}_F}(J) = J = \underline{\mathrm{Apr}_F}(J)$,

(2) $\underline{\mathrm{Apr}_F}(F) = F = \underline{\mathrm{Apr}_F}(F)$,

(3) $F \subseteq \overline{\mathrm{Apr}_F}(J)$.

Proof. (1) Suppose that $F \subseteq J$. By Proposition 3.2 part (1), $J \subseteq \overline{\mathrm{Apr}_F}(J)$ and $\underline{\mathrm{Apr}_F}(J) \subseteq J$. Now let $x \in \overline{\mathrm{Apr}_F}(J)$. Then $[x]_F \cap J \neq \emptyset$ and so there exists $y \in J$ such that $[x]_F = [y]_F$. We get that $x \to y, y \to x \in F$. Since $F \subseteq J$, J is a filter and $y \in J$, then $x \in J$, that is, $\overline{\mathrm{Apr}_F}(J) \subseteq J$. Hence $\overline{\mathrm{Apr}_F}(J) = J$. Similarly, we can obtain $J \subseteq \underline{\mathrm{Apr}_F}(J)$. Conversely, suppose that $\overline{\mathrm{Apr}_F}(J) = J = \underline{\mathrm{Apr}_F}(J)$ and $x \in F$. We have $1 \in [x]_F \cap J$. Hence $x \in \overline{\mathrm{Apr}_F}(J)$. Thus $F \subseteq \overline{\mathrm{Apr}_F}(J)$. We get that $F \subseteq J$.

(2) The result follows from (1).

(3) Let $x \in F$. Then $1 \in F \cap J = [x]_F \cap J$ and so $x \in \overline{\mathrm{Apr}_F}(J)$. Therefore $F \subseteq \overline{\mathrm{Apr}_F}(J)$. □

Lemma 3.13. *Let L be linearly ordered and F a filter of L. If $x \leq y$ and $[x]_F \neq [y]_F$, then for each $t \in [x]_F$ and $s \in [y]_F$, $t \leq s$.*

Proof. Let there exist $t \in [x]_F$ and $s \in [y]_F$ such that $s < t$. Then $t \to x \le s \to x$, and also $t \to x \in F$. So $s \to x \in F$. By $x \le y$ we get that $y \to s \le x \to s$, also we have $y \to s \in F$, and so $x \to s \in F$. Thus $s \in [x]_F \cap [y]_F$, that is, $[x]_F = [y]_F$, it is a contradiction. Thus for each $t \in [x]_F$ and $s \in [y]_F$, $t \le s$. □

Theorem 3.14. *Let (L, F) be an approximation space and J a filter of L. Then the following hold.*

(1) *If $F \subseteq J$, then J is a rough filter of L.*

(2) *If $J \subseteq F$, then J is an upper rough filter of L.*

(3) *If L is linearly ordered, then J is an upper rough filter of L.*

Proof. (1) The proof follows from Theorem 3.12 part (1).

(2) The proof is easy by Theorem 3.11 part (i).

(3) Let $x, y \in \overline{\mathrm{Apr}}_F(J)$. Then it is easy to see that $x * y \in \overline{\mathrm{Apr}}_F(J)$. If $x \le y$ and $x \in \overline{\mathrm{Apr}}_F(J)$, then $[x]_F \cap J \neq \emptyset$ and so there is $t \in J$ such that $[x]_F = [t]_F$. If $[x]_F = [y]_F$, we get that $y \in \overline{\mathrm{Apr}}_F(J)$. If $[x]_F \neq [y]_F$, then by Lemma 3.13 we obtain $t \le y$. So by $t \in J$ we get that $y \in J$, that is, $y \in \overline{\mathrm{Apr}}_F(J)$. □

If X is a nonempty subset of a BL-algebra L, we let $\neg X = \{\neg x \mid x \in X\}$. It is easy to see that for every nonempty subset X, Y of L, $X \subseteq Y$ implies that $\neg X \subseteq \neg Y$.

Proposition 3.15. *Let F be a filter of L and X a nonempty subset of L. Then $\neg\overline{\mathrm{Apr}}_F(X) \subseteq \overline{\mathrm{Apr}}_F(\neg X)$.*

Proof. Let $z \in \neg\overline{\mathrm{Apr}}_F(X)$. Then there is $t \in \overline{\mathrm{Apr}}_F(X)$ such that $z = \neg t$ and so $[t]_F \cap X \neq \emptyset$. Thus there exists $h \in X$ such that $[t]_F = [h]_F$, hence $[z]_F = [\neg t]_F = [\neg h]_F$. Also $h \in X$ implies that $\neg h \in \neg X$ and so $[z]_F \cap \neg X = [\neg h]_F \cap \neg X \neq \emptyset$. Therefore $z \in \overline{\mathrm{Apr}}_F(\neg X)$ and hence $\neg\overline{\mathrm{Apr}}_F(X) \subseteq \overline{\mathrm{Apr}}_F(\neg X)$. □

Remark 3.16. (1) We cannot replace the inclusion symbol $\subseteq$ by an equal sign in Proposition 3.15. Consider filter $F = \{1, b\}$ in Example 3.10. Let subset $X = \{0, c\}$ of L; we have $\neg X = \{1, 0\}$, $\neg\overline{\mathrm{Apr}}_F(X) = \neg\{0, a, c\} = \{0, 1\}$ and $\overline{\mathrm{Apr}}_F(\neg X) = \{0, b, 1\}$. Therefore, $\overline{\mathrm{Apr}}_F(\neg X) \nsubseteq \neg\overline{\mathrm{Apr}}_F(X)$.

(2) We can show that Proposition 3.15 may not be true for $\underline{\mathrm{Apr}}_F$. Consider Example 3.10, filter $F = \{1, b\}$ and subset $X = \{0, a, c\}$ of L. We can get that $\neg X = \{0, 1\}$, $\underline{\mathrm{Apr}}_F(\neg X) = \{0\}$ and $\neg\underline{\mathrm{Apr}}_F(X) = \neg\{0, a, c\} = \{0, 1\}$. Therefore, $\neg\underline{\mathrm{Apr}}_F(X) \nsubseteq \underline{\mathrm{Apr}}_F(\neg X)$. Also by considering $X = \{1\}$ we can check that $\underline{\mathrm{Apr}}_F(\neg X) = \{0\}$ and $\neg\underline{\mathrm{Apr}}_F(X) = \emptyset$. Thus $\underline{\mathrm{Apr}}_F(\neg X) \nsubseteq \neg\underline{\mathrm{Apr}}_F(X)$.

Let L be a BL-algebra. An element a of L is said to be regular if and only if $\neg\neg a = a$. The set of all regular elements of L is denoted by $\mathrm{Reg}(L)$. The set of regular elements is also denoted by $MV(L)$ in [10] where it is proved that it is the largest sub MV-algebra of L.

Lemma 3.17. *Let F be a filter of L and $\emptyset \neq X \subseteq L$. Then,*

(i) $\mathrm{Reg}(L) \cap \overline{\mathrm{Apr}}_F(\neg X) \subseteq \neg\overline{\mathrm{Apr}}_F(\neg\neg X)$,

(ii) $\mathrm{Reg}(L) \cap \overline{\mathrm{Apr}}_F \neg(X \cap \mathrm{Reg}(L)) \subseteq \neg\overline{Apr}_F(X)$.

Proof. (i) Let $z \in \operatorname{Reg}(L) \cap \overline{\operatorname{Apr}}_F(\neg X)$. Then $[z]_F \cap \neg X \neq \emptyset$ and $\neg\neg z = z$ and so there exists $x \in X$ such that $[\neg x]_F = [z]_F$. Thus we have $z = \neg\neg z = \neg(\neg z)$ and $[\neg z]_F \cap \neg\neg X = [\neg\neg x]_F \cap \neg\neg X \neq \emptyset$ and hence $\neg z \in \overline{\operatorname{Apr}}_F(\neg\neg X)$. Therefore, $z \in \neg\overline{\operatorname{Apr}}_F(\neg\neg X)$, and this implies that $\operatorname{Reg}(L) \cap \overline{\operatorname{Apr}}_F(\neg X) \subseteq \neg\overline{\operatorname{Apr}}_F(\neg\neg X)$.

(ii) Let $z \in \operatorname{Reg}(L) \cap \overline{\operatorname{Apr}}_F \neg(X \cap \operatorname{Reg}(L))$. Then $\neg\neg z = z$ and $[z]_F \cap \neg(X \cap \operatorname{Reg}(L)) \neq \emptyset$. So there exists $x \in X \cap \operatorname{Reg}(L)$ such that $[\neg x]_F = [z]_F$ and $\neg\neg x = x$. We get that $z = \neg(\neg z)$ and $[\neg z]_F \cap X = [x]_F \cap X \neq \emptyset$ and so $z \in \neg\overline{\operatorname{Apr}}_F(X)$. Therefore, $\operatorname{Reg}(L) \cap \overline{\operatorname{Apr}}_F \neg(X \cap \operatorname{Reg}(L)) \subseteq \neg\overline{\operatorname{Apr}}_F(X)$. □

An element a of L is said to be dense if and only if $\neg a = 0$. We denote by $Ds(L)$ the set of the dense elements of L. $Ds(L)$ is a filter of L [15].

Lemma 3.18. *Let F be a filter of L. Then $F \subseteq \overline{\operatorname{Apr}}_F(Ds(L)) \subseteq \{z \in L \mid \neg\neg z \in F\}$.*

Proof. Let $z \in \overline{\operatorname{Apr}}_F(Ds(L))$. Then $[z]_F \cap Ds(L) \neq \emptyset$ and so there is $t \in [z]_F$ such that $\neg t = 0$. Thus $[0]_F = [\neg t]_F = [\neg z]_F$ and hence $\neg\neg z \in F$. Also since $Ds(L)$ is a filter of L, then by Theorem 3.12 part (3), $F \subseteq \overline{\operatorname{Apr}}_F(Ds(L))$. □

Lemma 3.19. *Let F be a filter of L and X a nonempty set of L. Then X is definable respect to F if and only if $\overline{\operatorname{Apr}}_F(X) = X$ or $\underline{\operatorname{Apr}}_F(X) = X$.*

Proof. Suppose that X is definable. Then $X \subseteq \overline{\operatorname{Apr}}_F(X) = \underline{\operatorname{Apr}}_F(X) \subseteq X$ and so $\overline{\operatorname{Apr}}_F(X) = X = \underline{\operatorname{Apr}}_F(X)$. Conversely, let $\overline{\operatorname{Apr}}_F(X) = X$. We show that $\underline{\operatorname{Apr}}_F(X) = X$. We have $\underline{\operatorname{Apr}}_F(X) \subseteq X$. Now let $x \in X$. and $z \in [x]_F$. Then $[z]_F \cap X = [x]_F \cap X \neq \emptyset$ and so $z \in \overline{\operatorname{Apr}}_F(X) = X$. Therefore $\underline{\operatorname{Apr}}_F(X) = X$. If $\underline{\operatorname{Apr}}_F(X) = X$, then we show that $\overline{\operatorname{Apr}}_F(X) = X$. Let $x \in \overline{\operatorname{Apr}}_F(X)$. Then $[x]_F \cap X \neq \emptyset$ and so there is $z \in X$ such that $[x]_F = [z]_F$. By hypothesis we get that $[z]_F \subseteq X$, hence $x \in X$. Therefore $\overline{\operatorname{Apr}}_F(X) \subseteq X$. Since $X \subseteq \overline{\operatorname{Apr}}_F(X)$, then $\overline{\operatorname{Apr}}_F(X) = X$. □

By Theorem 3.12 part (1) and Lemma 3.19 we have the following.

Corollary 3.20. *Let F and J be two filters of L. Then J is definable respect to F if and only if $F \subseteq J$.*

Let X and Y be two nonempty subsets of L. Then we define

$$X \odot Y = \{t \in L \mid x * y \leq t, \text{ for some } x \in X, y \in Y\}. \tag{3.7}$$

If either X or Y are empty, then we define $X \odot Y = \emptyset$. If A and B are two filters of L, then it is clear that $A \odot B$ is the smallest filter containing of A and B. For any subsets X, Y, Z of L we have $X \odot Y = Y \odot X$ and $(X \odot Y) \odot Z = X \odot (Y \odot Z)$. It is easy to see that $\overline{\operatorname{Apr}}_F(X) \cup \overline{\operatorname{Apr}}_E(X) \subseteq \overline{\operatorname{Apr}}_{F \odot E}(X)$, for any filter E and F of L and nonempty subset X of L. Since $1 \in X \odot Y \subseteq \overline{\operatorname{Apr}}_F(X \odot Y)$, for nonempty subsets X, Y of L and filter F of L, then we can conclude that $F \subseteq \overline{\operatorname{Apr}}_F(X \odot Y)$.

Proposition 3.21. *Let F be a filter of L and X, Y be nonempty subsets of L. Then*

(i) $\overline{\operatorname{Apr}}_F(X \odot Y) \subseteq \overline{\operatorname{Apr}}_F(X) \odot \overline{\operatorname{Apr}}_F(Y)$,

(ii) *If $X, Y \subseteq F$, then* $\overline{\operatorname{Apr}}_F(X \odot Y) = \overline{\operatorname{Apr}}_F(X) \odot \overline{\operatorname{Apr}}_F(Y)$,

(iii) *If L is linearly ordered, then* $\overline{\operatorname{Apr}}_F(X \odot Y) = \overline{\operatorname{Apr}}_F(X) \odot \overline{\operatorname{Apr}}_F(Y)$.

Proof. (i) Let $z \in \overline{\mathrm{Apr}}_F(X \odot Y)$. Then $[z]_F \cap (X \odot Y) \neq \emptyset$ and so there is $t \in X \odot Y$ such that $[z]_F = [t]_F$. Hence there are $x \in X$ and $y \in Y$ such that $x * y \leq t$. On the other hand, $[z]_F = [t]_F$ implies that $t \to z \in F$, then there is $f \in F$ such that $t \leq f \to z$. By hypothesis we can conclude that $x * y \leq t \leq f \to z$, and hence $x * (y * f) \leq z$. Also by Lemma 2.2, we have $f \leq y * f \to y$ and $f \leq y \to y * f$, thus $f \in F$ implies that $(y * f \to y) \in F$ and $(y \to y * f) \in F$. Therefore $[y * f]_F = [y]_F$, hence $y * f \in \overline{\mathrm{Apr}}_F(Y)$, and also we have $x \in \overline{\mathrm{Apr}}_F(X)$. So $z \in \overline{\mathrm{Apr}}_F(X) \odot \overline{\mathrm{Apr}}_F(Y)$.

(ii) If $X, Y \subseteq F$, then $X \odot Y \subseteq F$. So by Theorem 3.11$\overline{\mathrm{Apr}}_F(X) = \overline{\mathrm{Apr}}_F(Y) = \overline{\mathrm{Apr}}_F(X \odot Y) = F$. Therefore $\overline{\mathrm{Apr}}_F(X \odot Y) = F = F \odot F = \overline{\mathrm{Apr}}_F(X) \odot \overline{\mathrm{Apr}}_F(Y)$.

(iii) Let L be linearly ordered and $z \in \overline{\mathrm{Apr}}_F(X) \odot \overline{\mathrm{Apr}}_F(Y)$. Then there are $h \in \overline{\mathrm{Apr}}_F(X)$ and $k \in \overline{\mathrm{Apr}}_F(Y)$ such that $h * k \leq z$. So $[h]_F \cap X \neq \emptyset$ and $[k]_F \cap Y \neq \emptyset$ imply that there are $x \in X$ and $y \in Y$ such that $[x]_F = [h]_F$ and $[y]_F = [k]_F$. Hence $[x * y]_F = [h * k]_F$ and we have $x * y \in X \odot Y$. If $[z]_F = [h * k]_F$, then by hypothesis we get that $[z]_F \cap X \odot Y \neq \emptyset$ and hence $z \in \overline{\mathrm{Apr}}_F(X \odot Y)$. If $[z]_F \neq [h * k]_F$, then by Lemma 3.13 $x * y \leq z$ and so $z \in X \odot Y$. Therefore $[z]_F \cap (X \odot Y) \neq \emptyset$, that is, $z \in \overline{\mathrm{Apr}}_F(X \odot Y)$. □

Proposition 3.22. *Let F be a filter of L and X, Y be nonempty subsets of L. Then*

(i) $\underline{\mathrm{Apr}}_F(X) \odot \underline{\mathrm{Apr}}_F(Y) \subseteq \underline{\mathrm{Apr}}_F(X \odot Y)$.

(ii) *If X and Y are definable respect to F, then* $\underline{\mathrm{Apr}}_F(X) \odot \underline{\mathrm{Apr}}_F(Y) = \underline{\mathrm{Apr}}_F(X \odot Y)$.

Proof. (i) Let $z \in \underline{\mathrm{Apr}}_F(X) \odot \underline{\mathrm{Apr}}_F(Y)$. Then $x * y \leq z$, for some $x \in \underline{\mathrm{Apr}}_F(X)$ and $y \in \underline{\mathrm{Apr}}_F(Y)$. Consider $b \in [z]_F$, so there is $f \in F$ such that $z \leq f \to b$. Hence $x * y \leq f \to b$, we can obtain that $x * (y * f) \leq b$. Thus by hypothesis we have $[y * f]_F = [y]_F \subseteq Y$, $[x]_F \subseteq X$ and $x * (y * f) \leq b$, hence $b \in X \odot Y$. Therefore $z \in \underline{\mathrm{Apr}}_F(X \odot Y)$, it implies that $\underline{\mathrm{Apr}}_F(X) \odot \underline{\mathrm{Apr}}_F(Y) \subseteq \underline{\mathrm{Apr}}_F(X \odot Y)$.

(ii) Since X and Y are definable respect to F, then $\underline{\mathrm{Apr}}_F(X) = X$ and $\underline{\mathrm{Apr}}_F(Y) = Y$. By part (i), we get that $X \odot Y \subseteq \underline{\mathrm{Apr}}_F(X \odot Y) \subseteq X \odot Y$. Therefore $\underline{\mathrm{Apr}}_F(X) \odot \underline{\mathrm{Apr}}_F(Y) = X \odot Y = \underline{\mathrm{Apr}}_F(X \odot Y)$. □

By the following example, we show that we cannot replace the inclusion symbol $\subseteq$ by an equal sign in general in the above proposition part (i).

Example 3.23. Let $L = \{0, a, b, c, d, 1\}$, where $0 < d, b < a < 1$, $0 < d < c < 1$. Define $*$ and $\to$ as follow:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

(3.8)

Then $(L, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. It is easy to check that $F = \{1, c\}$ is a filter of L. By considering subsets $X = \{a\}$ and $Y = \{c\}$ of L, we have $\underline{\mathrm{Apr}}_F(X \odot Y) = \{1, a, d, c\}$ and $\underline{\mathrm{Apr}}_F(X) = \underline{\mathrm{Apr}}_F(Y) = \emptyset$. Therefore $\underline{\mathrm{Apr}}_F(X) \odot \underline{\mathrm{Apr}}_F(Y) \neq \underline{\mathrm{Apr}}_F(X \odot Y)$.

Proposition 3.24. *Let F and J be two filters of L and X be a nonempty subset of L. Then*

(i) *If $X \subseteq B(L)$, then* $\overline{\mathrm{Apr}}_{F\odot J}(X) \subseteq \overline{\mathrm{Apr}}_{F}(X) \odot \overline{\mathrm{Apr}}_{J}(X)$,

(ii) $\overline{\mathrm{Apr}}_{F}(X) \odot \overline{\mathrm{Apr}}_{J}(X) \subseteq \overline{\mathrm{Apr}}_{F\odot J}(X) \odot \overline{\mathrm{Apr}}_{F\odot J}(X)$,

(iii) $\underline{\mathrm{Apr}}_{F\odot J}(X) \subseteq \underline{\mathrm{Apr}}_{F}(X) \odot \underline{\mathrm{Apr}}_{J}(X)$.

Proof. (i) Let $z \in \overline{\mathrm{Apr}}_{F\odot J}(X)$. Then $[z]_{F\odot J} \cap X \neq \emptyset$ and so there is $x \in X$ such that $x \to z \in F \odot J$. Thus there are $f \in F$ and $e \in J$ such that $f * e * x \leq z$, by $X \subseteq B(A)$, we get that $(f*x)*(e*x) \leq z$. Since $[f * x]_F = [x]_F$ and $[e * x]_J = [x]_J$, then we have $z \in \overline{\mathrm{Apr}}_{F}(X) \odot \overline{\mathrm{Apr}}_{J}(X)$.

(ii) Let $h \in \overline{\mathrm{Apr}}_{F}(X) \odot \overline{\mathrm{Apr}}_{J}(X)$. Then $t * s \leq h$, for some $t, s \in L$ such that $[t]_F \cap X \neq \emptyset$ and $[s]_J \cap X \neq \emptyset$. So $[t]_{F\odot J} \cap X \neq \emptyset$ and $[s]_{F\odot J} \cap X \neq \emptyset$ and hence $t, s \in \overline{\mathrm{Apr}}_{F\odot J}(X)$. Thus $h \in \overline{\mathrm{Apr}}_{F\odot J}(X) \odot \overline{\mathrm{Apr}}_{F\odot J}(X)$.

(iii) Let $z \in \underline{\mathrm{Apr}}_{F\odot J}(X)$. Then $[z]_{F\odot J} \subseteq X$. Since $z * z \leq z$ and $[z]_F, [z]_J \subseteq [z]_{F\odot E} \subseteq X$, hence $z \in \underline{\mathrm{Apr}}_{F}(X) \odot \underline{\mathrm{Apr}}_{J}(X)$. □

By Proposition 3.21 part (i) and Proposition 3.24 part (i) we can obtain the following corollary.

Corollary 3.25. *Let F and J be two filters of L and X, Y be nonempty subsets of $B(L)$. Then* $\overline{\mathrm{Apr}}_{F\odot J}(X \odot Y) \subseteq (\overline{\mathrm{Apr}}_{F}(X) \odot \overline{\mathrm{Apr}}_{F}(Y)) \odot (\overline{\mathrm{Apr}}_{J}(X) \odot \overline{\mathrm{Apr}}_{J}(Y))$.

Let J and F be two filters of L such that $J \subseteq F$ and let X be a nonempty subset of L. Then it is easy to see that $\overline{\mathrm{Apr}}_{J}(X) \subseteq \overline{\mathrm{Apr}}_{F}(X)$ and $\underline{\mathrm{Apr}}_{F}(X) \subseteq \underline{\mathrm{Apr}}_{J}(X)$. So we have $\overline{\mathrm{Apr}}_{F\cap J}(X) \subseteq \overline{\mathrm{Apr}}_{F}(X) \cap \overline{\mathrm{Apr}}_{J}(X)$ and $\underline{\mathrm{Apr}}_{F}(X) \cap \underline{\mathrm{Apr}}_{J}(X) \subseteq \underline{\mathrm{Apr}}_{F\cap J}(X)$.

Consider BL-algebra L in Example 3.10. We can see that $J = \{1, a\}$ and $F = \{1, b\}$ are two filters of L. Put $X = \{a, b\}$. We have $\overline{\mathrm{Apr}}_{F\cap J}(X) = \{a, b\}$ and $\overline{\mathrm{Apr}}_{F}(X) = \overline{\mathrm{Apr}}_{J}(X) = \{1, b, c, a\}$, so $\overline{\mathrm{Apr}}_{F\cap J}(X) \neq \overline{\mathrm{Apr}}_{F}(X) \cap \overline{\mathrm{Apr}}_{J}(X)$. Also $\underline{\mathrm{Apr}}_{F\cap J}(X) = X$ and $\underline{\mathrm{Apr}}_{F}(X) = \underline{\mathrm{Apr}}_{J}(X) = \emptyset$, thus $\underline{\mathrm{Apr}}_{F}(X) \cap \underline{\mathrm{Apr}}_{J}(X) \neq \underline{\mathrm{Apr}}_{F\cap J}(X)$.

By the following proposition we can obtain some conditions that $\overline{\mathrm{Apr}}_{F\cap J}(X) = \overline{\mathrm{Apr}}_{F}(X) \cap \overline{\mathrm{Apr}}_{J}(X)$ or $\underline{\mathrm{Apr}}_{F}(X) \cap \underline{\mathrm{Apr}}_{J}(X) = \underline{\mathrm{Apr}}_{F\cap J}(X)$.

Proposition 3.26. *Let F and J be two filters of L and X be a nonempty subset of L. Then*

(i) *If $X \subseteq F \cap J$ or X is definable respect to J or F, then* $\overline{\mathrm{Apr}}_{F\cap J}(X) = \overline{\mathrm{Apr}}_{F}(X) \cap \overline{\mathrm{Apr}}_{J}(X)$,

(ii) *If X is a filter of L containing J and F, then* $\underline{\mathrm{Apr}}_{F}(X) \cap \underline{\mathrm{Apr}}_{J}(X) = \underline{\mathrm{Apr}}_{F\cap J}(X)$.

Proof. (i) Assume that $X \subseteq F \cap J$. Then by Theorem 3.11 part (i), $\overline{\mathrm{Apr}}_{F\cap J}(X) = F \cap J = \overline{\mathrm{Apr}}_{F}(X) \cap \overline{\mathrm{Apr}}_{J}(X)$. If X is definable respect to J, then $\overline{\mathrm{Apr}}_{F}(X) \cap \overline{\mathrm{Apr}}_{J}(X) = X \subseteq \overline{\mathrm{Apr}}_{F\cap J}(X)$, and it proves theorem.

(ii) If X is a filter of L containing J and F, then by Theorem 3.12 part (1) $\underline{\mathrm{Apr}}_{F\cap J}(X) = X = X \cap X = \underline{\mathrm{Apr}}_{F}(X) \cap \underline{\mathrm{Apr}}_{J}(X)$. □

Proposition 3.27. *Let L and L' be two BL-algebras and $f : L \to L'$ be a homomorphism. Then*

(i) *If X be a nonempty subset of L and F be a filter of L', then* $f^{-1}(\overline{\mathrm{Apr}_F}(f(X)) = \overline{\mathrm{Apr}_{f^{-1}(F)}}(X)$,

(ii) $f(\overline{\mathrm{Apr}_{\ker f}}(X)) = f(X)$, *for any nonempty subset X of L.*

(iii) *Let f be onto, Y be a nonempty subset of L and F a filter of L. If* $\ker f \subseteq F$, *then* $f(\overline{\mathrm{Apr}_F}(Y)) = \overline{\mathrm{Apr}_{f(F)}}(f(Y))$.

Proof. (i) By hypothesis we have

$$z \in f^{-1}(\overline{\mathrm{Apr}_F}(f(X)) \Leftrightarrow f(z) \in (\overline{\mathrm{Apr}_F}(f(X)) \Leftrightarrow [f(z)]_F \cap f(X) \neq \emptyset \Leftrightarrow [f(z)]_F = [f(x)]_F, \text{ for some } x \in X \Leftrightarrow [z]_{f^{-1}(F)} = [x]_{f^{-1}(F)}, \text{ for some } x \in X \Leftrightarrow [z]_{f^{-1}(F)} \cap X \neq \emptyset \Leftrightarrow z \in \overline{\mathrm{Apr}_{f^{-1}(F)}}(X).$$

(ii) The proof is easy by part (i).

(iii) Let $z \in f(\overline{\mathrm{Apr}_F}(Y))$. Then there is $t \in \overline{\mathrm{Apr}_F}(Y)$ such that $z = f(t)$, and so there exists $y \in Y$ such that $[t]_F = [y]_F$. Thus by $[f(t)]_{f(F)} = [f(y)]_{f(F)}$ we get that $[f(t)]_{f(F)} \cap f(Y) \neq \emptyset$. Therefore $z \in \overline{\mathrm{Apr}_{f(F)}}(f(Y))$. Now let $z \in \overline{\mathrm{Apr}_{f(F)}}(f(Y))$. Then $[z]_{f(F)} \cap f(Y) \neq \emptyset$. By hypothesis we have $h \in L$ such that $z = f(h)$, and so there is $y \in Y$ such that $[f(h)]_{f(F)} = [f(y)]_{f(F)}$. Since F is a filter of L and $\ker f \subseteq F$, then we obtain that $[y]_F = [h]_F$. So $h \in \overline{\mathrm{Apr}_F}(Y)$, hence $z = f(h) \in f(\overline{\mathrm{Apr}_F}(Y))$. □

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