

Research Article

The Semi-Difference Entire Sequence Space $cs \cap d_1$

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Let Γ denote the space of all entire sequences. Let Λ denote the space of all analytic sequences. In this paper, we introduce a new class of sequence space, namely, the semi-difference entire sequence space $cs \cap d_1$. It is shown that the intersection of all semi-difference entire sequence spaces $cs \cap d_1$ is $I \subset cs \cap d_1$ and $\Gamma(\Delta) \subset I$.

1. Introduction

A complex sequence, whose k th term is x_k , is denoted by $\{x_k\}$ or simply x . Let w be the set of all sequences and ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the classes of bounded, convergent, and null sequence, respectively. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ . A sequence x is called entire sequence if $\lim_{k \rightarrow \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ .

Given a sequence $x = \{x_k\}$, its n th section is the sequence $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$. Let $\delta^{(n)} = (0, 0, \dots, 1, 0, 0, \dots)$, 1 in the n th place and zeros elsewhere, $s^{(k)} = (0, 0, \dots, 1, -1, 0, \dots)$, 1 in the n th place, -1 in the $(n + 1)$ th place and zeros elsewhere. An FK-space (Fréchet coordinate space) is a Fréchet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ ($k = 1, 2, 3, \dots$) are continuous.

We recall the following definitions (one may refer to Wilansky [1]).

An FK-space is a locally convex Fréchet space which is made up of sequences and has the property that coordinate projections are continuous. A metric space (X, d) is said to have AK (or sectional convergence) if and only if $d(x^{(n)}, x) \rightarrow 0$ as $n \rightarrow \infty$, (see [1]). The space is said to have AD (or) be an AD-space if ϕ is dense in X , where ϕ denotes the set of all finitely nonzero sequences. We note that AK implies AD (one may refer to Brown [2]).

If X is a sequence space, we define

- (i) X' = the continuous dual of X ;
- (ii) $X^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\}$;
- (iii) $X^\beta = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\}$;
- (iv) $X^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^n a_k x_k| < \infty, \text{ for each } x \in X\}$;
- (v) let X be an FK-space $\supset \phi$. Then, $X^f = \{f(\delta^{(n)}) : f \in X'\}$.

$X^\alpha, X^\beta, X^\gamma$ are called the α (or Köthe-Töeplitz) dual of X , β —(or generalized Köthe-Töeplitz) dual of X , γ dual of X . Note that $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\mu \subset X^\mu$, for $\mu = \alpha, \beta$, or γ .

Let $p = (p_k)$ be a sequence of positive real numbers with $\sup_k p_k = G$ and $D = \max\{1, 2^{G-1}\}$. Then, it is well known that for all $a_k, b_k \in \mathbb{C}$, the field of complex numbers, for all $k \in \mathbb{N}$,

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \quad (1.1)$$

Lemma 1.1 (Wilansky [1, Theorem 7.2.7]). *Let X be an FK-space $\supset \phi$. Then,*

- (i) $X^\gamma \subset X^f$;
- (ii) if X has AK, $X^\beta = X^f$;
- (iii) if X has AD, $X^\beta = X^\gamma$.

2. Definitions and Preliminaries

Let $\Delta : w \rightarrow w$ be the difference operator defined by $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$. Let

$$\begin{aligned} \Gamma &= \left\{ x \in w : \lim_{k \rightarrow \infty} (|x_k|^{1/k}) = 0 \right\}, \\ \Lambda &= \left\{ x \in w : \sup_k (|x_k|^{1/k}) < \infty \right\}. \end{aligned} \quad (2.1)$$

Define the sets $\Gamma(\Delta) = \{x \in w : \Delta x \in \Gamma\}$ and $\Lambda(\Delta) = \{x \in w : \Delta x \in \Lambda\}$.

The spaces $\Gamma(\Delta)$ and $\Lambda(\Delta)$ are the metric spaces with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k (|\Delta x_k - \Delta y_k|^{1/k}) \leq \rho \right\}. \quad (2.2)$$

Because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi-conservative. (See Wilansky [1]).

Snyder and Wilansky [3] introduced the concept of semi-conservative spaces. Snyder [4] studied the properties of semi-conservative spaces. Later on, in the year 1996 the semi-replete spaces were introduced by Rao and Srinivasalu [5].

In a similar way, in this paper, we define semi-difference entire sequence space $cs \cap d_1$, and show that semi-difference entire sequence space $cs \cap d_1$ is $I \subset cs \cap d_1$ and $\Gamma(\Delta) \subset I$.

3. Main Results

Proposition 3.1. $\Gamma \subset \Gamma(\Delta)$ and the inclusion is strict.

Proof. Let $x \in \Gamma$. Then, we have

$$|x_k|^{1/k} \rightarrow 0, \quad \text{as } k \rightarrow \infty, \tag{3.1}$$

$$\begin{aligned} \frac{|\Delta x_k|^{1/k}}{2} &\leq \frac{1}{2}(|x_k|^{1/k}) + \frac{1}{2}(|x_{k+1}|^{1/k}), \quad \text{by (1.1)} \\ &\rightarrow 0, \quad \text{as } k \rightarrow \infty \quad \text{by (3.1)}. \end{aligned} \tag{3.2}$$

Let $x \in \Gamma$. Then, we have

$$\left(|x_k|^{1/k}\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{3.3}$$

Then, $(x_k) \in \Gamma(\Delta)$ follows from the inequality (1.1) and (3.3).

Consider the sequence $e = (1, 1, \dots)$. Then, $e \in \Gamma(\Delta)$ but $e \notin \Gamma$. Hence, the inclusion $\Gamma \subset \Gamma(\Delta)$ is strict. \square

Lemma 3.2. $A \in (\Gamma, c)$ if and only if

$$\lim_{n \rightarrow \infty} a_{nk} \quad \text{exists for each } k \in N, \tag{3.4}$$

$$\sup_{n,k} \left| \sum_{i=0}^k a_{ni} \right| < \infty. \tag{3.5}$$

Proposition 3.3. Define the set $d_1 = \{a = (a_k) \in w : \sup_{n,k \in N} |\sum_{j=0}^k (\sum_{i=j}^n a_i)| < \infty\}$. Then, $[\Gamma(\Delta)]^\beta = cs \cap d_1$.

Proof. Consider the equation

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^n a_k \left(\sum_{j=0}^k y_j \right) = \sum_{k=0}^n \left(\sum_{j=k}^n a_j \right) y_k = (Cy)_n, \tag{3.6}$$

where $C = (C_{nk})$ is defined by

$$C_{nk} = \begin{cases} \sum_{j=k}^n a_j, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n; \quad n, k \in N. \end{cases} \tag{3.7}$$

Thus, we deduce from Lemma 3.2 with (3.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \Gamma(\Delta)$ if and only if $Cy \in c$ whenever $y = (y_k) \in \Gamma$, that is $C \in (\Gamma, c)$. Thus, $(a_k) \in cs$ and $(a_k) \in d_1$ by Lemma 3.2 and (3.5) and (3.6), respectively. This completes the proof. \square

Proposition 3.4. $\Gamma(\Delta)$ has AK.

Proof. Let $x = \{x_k\} \in \Gamma(\Delta)$. Then, $(|\Delta x_k|^{1/k}) \in \Gamma$. Hence,

$$\sup_{k \geq n+1} (|\Delta x_k|^{1/k}) \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad (3.8)$$

$$\begin{aligned} d(x, x^{[n]}) &= \inf \left\{ \rho > 0 : \sup_{k \geq n+1} (|\Delta x_k|^{1/k}) \leq 1 \right\} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \text{ by using (3.8)} \\ &\implies x^{[n]} \rightarrow x \quad \text{as } n \rightarrow \infty, \\ &\implies \Gamma(\Delta) \text{ has AK.} \end{aligned} \quad (3.9)$$

This completes the proof. \square

Proposition 3.5. $\Gamma(\Delta)$ is not solid.

To prove Proposition 3.5, consider $(x_k) = (1) \in \Gamma(\Delta)$ and $\alpha_k = \{(-1)^k\}$. Then $(\alpha_k x_k) \notin \Gamma(\Delta)$. Hence, $\Gamma(\Delta)$ is not solid.

Proposition 3.6. $(\Gamma(\Delta))^\mu = cs \cap d_1$ for $\mu = \alpha, \beta, \gamma, f$.

Proof.

Step 1. $\Gamma(\Delta)$ has AK by Proposition 3.4. Hence, by Lemma 1.1(ii), we get $(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^f$. But $(\Gamma(\Delta))^\beta = cs \cap d_1$. Hence,

$$(\Gamma(\Delta))^f = cs \cap d_1. \quad (3.10)$$

Step 2. Since $AK \Rightarrow AD$. Hence, by Lemma 1.1(iii), we get $(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^\gamma$. Therefore,

$$(\Gamma(\Delta))^\gamma = cs \cap d_1. \quad (3.11)$$

Step 3. $\Gamma(\Delta)$ is not normal by Proposition 3.5. Hence by Proposition 2.7 of Kamthan and Gupta [6], we get

$$(\Gamma(\Delta))^\alpha \neq (\Gamma(\Delta))^\gamma \neq cs \cap d_1. \quad (3.12)$$

From (3.10) and (3.11), we have

$$(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^\gamma = (\Gamma(\Delta))^f = cs \cap d_1. \quad (3.13)$$

\square

Lemma 3.7 (Wilansky [1, Theorem 8.6.1]). $Y \supset X \Leftrightarrow Y^f \subset X^f$ where X is an AD-space and Y an FK-space.

Proposition 3.8. Let Y be any FK-space $\supset \phi$. Then, $Y \supset \Gamma(\Delta)$ if and only if the sequence $\delta^{(k)}$ is weakly converges in $cs \cap d_1$.

Proof. The following implications establish the result.

$$Y \supset \Gamma(\Delta) \Leftrightarrow Y^f \subset (\Gamma(\Delta))^f, \text{ since } \Gamma(\Delta) \text{ has AD by Lemma 3.7.}$$

$$\Leftrightarrow Y^f \subset cs \cap d_1, \text{ since } (\Gamma(\Delta))^f = cs \cap d_1.$$

$$\Leftrightarrow \text{for each } f \in Y', \text{ the topological dual of } Y.$$

$$\Leftrightarrow f(\delta^{(k)}) \in cs \cap d_1.$$

$$\Leftrightarrow \delta^{(k)} \text{ is weakly converges in } cs \cap d_1.$$

This completes the proof. \square

4. Properties of Semi-Difference Entire Sequence Space $cs \cap d_1$

Definition 4.1. An FK-space ΔX is called “semi-difference entire sequence space $cs \cap d_1$ ” if its dual $(\Delta X)^f \subset cs \cap d_1$.

In other words ΔX is semi-difference entire sequence space $cs \cap d_1$ if $f(\delta^{(k)}) \in cs \cap d_1$ for all $f \in (\Delta X)'$ for each fixed k .

Example 4.2. $\Gamma(\Delta)$ is semi-difference entire sequence space $cs \cap d_1$. Indeed, if $\Gamma(\Delta)$ is the space of all difference of entire sequences, then by Lemma 4.3, $(\Gamma(\Delta))^f = cs \cap d_1$.

Lemma 4.3 (Wilansky [1, Theorem 4.3.7]). Let z be a sequence. Then (z^β, P) is an AK space with $P = (P_k : k = 0, 1, 2, \dots)$, where $P_0(x) = \sup_m |\sum_{k=1}^m z_k x_k|$, and $P_n(x) = |x_n|$. For any k such that $z_k \neq 0$, P_k may be omitted. If $z \in \phi$, P_0 may be omitted.

Proposition 4.4. Let z be a sequence. z^β is a semi-difference entire sequence space $cs \cap d_1$ if and only if z is in $cs \cap d_1$.

Proof. Suppose that z^β is a semi-difference entire sequence space $cs \cap d_1$. z^β has AK by Lemma 4.3. Therefore $z^{\beta\beta} = (z^\beta)^f$ by Lemma 1 [1]. So z^β is semi-difference entire sequence space $cs \cap d_1$ if and only if $z^{\beta\beta} \subset cs \cap d_1$. But then $z \in z^{\beta\beta} \subset cs \cap d_1$. Hence, z is in $cs \cap d_1$.

Conversely, suppose that z is in $cs \cap d_1$. Then $z^\beta \supset \{cs \cap d_1\}^\beta$ and $z^{\beta\beta} \subset \{cs \cap d_1\}^{\beta\beta} = cs \cap d_1$. But $(z^\beta)^f = z^{\beta\beta}$. Hence, $(z^\beta)^f \subset cs \cap d_1$. Therefore z^β is semi-difference entire sequence space $cs \cap d_1$. This completes the proof. \square

Proposition 4.5. Every semi-difference entire sequence space $cs \cap d_1$ contains Γ .

Proof. Let ΔX be any semi-difference entire sequence space $cs \cap d_1$. Hence, $(\Delta X)^f \subset cs \cap d_1$. Therefore $f(\delta^{(k)}) \in cs \cap d_1$ for all $f \in (\Delta X)'$. So, $\{\delta^{(k)}\}$ is weakly converges in $cs \cap d_1$ with respect to ΔX . Hence, $\Delta X \supset \Gamma(\Delta)$ by Proposition 3.8. But $\Gamma(\Delta) \supset \Gamma$. Hence, $\Delta X \supset \Gamma$. This completes the proof. \square

Proposition 4.6. ΔX is semi-difference entire sequence space $cs \cap d_1$.

Proof. Let $\Delta X = \bigcap_{n=1}^{\infty} \Delta X_n$. Then ΔX is an FK-space which contains ϕ . Also every $f \in (\Delta X)'$ can be written as $f = g_1 + g_2 + \dots + g_m$, where $g_k \in (\Delta X_n)'$ for some n and for $1 \leq k \leq m$. But then $f(\delta^k) = g_1(\delta^k) + g_2(\delta^k) + \dots + g_m(\delta^k)$. Since ΔX_n ($n = 1, 2, \dots$) are semi-difference entire sequence space $cs \cap d_1$, it follows that $g_i(\delta^k) \in cs \cap d_1$ for all $i = 1, 2, \dots, m$. Therefore $f(\delta^k) \in cs \cap d_1$ for all k and for all f . Hence, ΔX is semi-difference entire sequence space $cs \cap d_1$. This completes the proof. \square

Proposition 4.7. *The intersection of all semi-difference entire sequence space $cs \cap d_1$ is $I \subset (cs \cap d_1)^\beta$ and $\Gamma(\Delta) \subset I$.*

Proof. Let I be the intersection of all semi-difference entire sequence space $cs \cap d_1$. By Proposition 4.4, we see that the intersection

$$I \subset \bigcap \{z^\beta : z \in cs \cap d_1\} = \{cs \cap d_1\}^\beta. \quad (4.1)$$

By Proposition 4.6 it follows that I is semi-difference entire sequence space $cs \cap d_1$. By Proposition 4.5, consequently

$$\Gamma_M = \Gamma(\Delta) \subset I. \quad (4.2)$$

From (4.1) and (4.2), we get $I \subset \{cs \cap d_1\}^\beta$ and $\Gamma(\Delta) \subset I$. This completes the proof. \square

Corollary 4.8. *The smallest semi-difference entire sequence space $cs \cap d_1$ is $I \subset (cs \cap d_1)^\beta$ and $\Gamma(\Delta) \subset I$.*

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