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Research Article

A Volume Comparison Estimate with Radially Symmetric Ricci Curvature Lower Bound and Its Applications

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We extend the classical Bishop-Gromov volume comparison from constant Ricci curvature lower bound to radially symmetric Ricci curvature lower bound, and apply it to investigate the volume growth, total Betti number, and finite topological type of manifolds with nonasymptotically almost nonnegative Ricci curvature.

1. Introduction

In comparison geometry of Ricci curvature, the classical Bishop-Gromov volume comparison has many applications, such as at least the linear volume growth of complete noncompact Riemannian manifolds with nonnegative Ricci curvature (see [1]), the upper bound of total Betti number (growth) of Riemannian manifolds (see [2–4]), and the finite topological type of complete noncompact Riemannian manifolds with nonnegative Ricci curvature or quadratic Ricci curvature decay (see [3, 5, 6]).

In [7], Lott and Shen establish a volume comparison estimate with quadratic Ricci curvature decay, and apply it to investigate the finite topological type of complete noncompact Riemannian manifolds with quadratic Ricci curvature decay, which generalizes a related result by Sha and Shen in [6].

In [8], we apply the volume comparison with asymptotically nonnegative Ricci curvature to investigate the corresponding topological results for manifolds with asymptotically nonnegative Ricci curvature.

In this paper, we will extend the classical Bishop-Gromov volume comparison from constant Ricci curvature lower bound to general radially symmetric Ricci curvature lower bound, and apply it to investigate the volume growth, total Betti number and finite topological type of manifolds with non-asymptotically almost nonnegative Ricci curvature. (See Definitions 1.1 and 1.2 below for the notions of radially symmetric Ricci curvature lower bound, asymptotically almost nonnegative Ricci curvature, and non-asymptotically almost nonnegative Ricci curvature, resp.)

Note that quadratic Ricci curvature decay is non-asymptotically almost nonnegative Ricci curvature, so our result is a generalization of the corresponding result of Lott and Shen in [7] mentioned above. (See Theorem 1.7.)

Definition 1.1. Let M be a complete n-Riemannian manifold ($n \geq 2$), $p \in M$, and $l := \sup\{d(p,x) \mid x \in M\}$. M has a radially symmetric Ricci curvature lower bound,k at the point p if there exists a continuous function $k : [0,l) \to R$ such that, for any tangent vector $v \in T_xM$ radial from the point p,

$$\mathrm{Ric}(v) \geq (n-1)k\big(d(p,x)\big). \tag{1.1}$$

One can refer to [9] for generalized space forms with radially symmetric curvature and the notion of tangent vector radial from a point.

Definition 1.2. Let M a complete noncompact n-Riemannian manifold ($n \geq 2$), $p \in M$, $k : [0,\infty) \to R$ be a continuous positive function, and $\lim_{t\to\infty} k(t) = 0$.

(i) M has almost nonnegative Ricci curvature if $\mathrm{Ric}(x) \geq (n-1)k(d(p,x))$.

Furthermore,

(ii) M has asymptotically nonnegative Ricci curvature if $\mathrm{Ric}(x) \geq (n-1)k(d(p,x))$ and $C(k) := \int_0^\infty tk(t)dt < \infty$.

(iii) M has non-asymptotically almost nonnegative Ricci curvature if $\mathrm{Ric}(x) \geq (n-1)k(d(p,x))$ and $C(k) := \int_0^\infty tk(t)dt = \infty$.

The following is a volume comparison estimate for manifolds with general radially symmetric Ricci curvature lower bound, which is a generalization of that for manifolds with asymptotically nonnegative Ricci curvature and quadratic Ricci curvature decay by Zhu in [10] and Lott and Shen in [7], respectively.

Theorem 1.3. *Let M be a complete n-Riemannian manifold $(n \geq 2)$ with a radially symmetric Ricci curvature lower bound $k : [0,l) \to R$ at the point $p \in M$, and let $r \leq R$, $s \leq S$, $r \leq s$, $R \leq S$, Γ be a measurable subset of the unit sphere in the tangent space T_pM:*

$$A^{\Gamma}_{r,R}(p) := \{x \in M \mid r \leq d(p,x) \leq R, \dot{\gamma}(0) \in \Gamma \text{ for any minimal geodesic } \gamma \text{ from } p \text{ to } x\}. \tag{1.2}$$

Then

$$\frac{\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right)}{\operatorname{vol}\left(A^{\Gamma}_{r,R}(p)\right)} \leq \frac{\int_s^S y^{n-1}(t)dt}{\int_r^R y^{n-1}(t)dt}, \tag{1.3}$$

where $y(t)$ is the unique solution of one of the following two equations:

$$\begin{gathered} y'' - k(t)y = 0, \\ y(0) = 0, \qquad y'(0) = 1, \end{gathered} \tag{1.4}$$

$$\begin{gathered} y'' - k(t)y = 0, \\ y(0) = 0, \qquad y'(0) = 1, \\ y > 0 \quad \text{on } (0, l). \end{gathered} \tag{1.5}$$

In particular, (1) *if $y(t)$ is the unique solution of* (1.4)*, then*

$$\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right) \leq \operatorname{vol}(\Gamma) \int_s^S y^{n-1}(t)dt. \tag{1.6}$$

if $y(t)$ is the unique solution of (1.5)*, then*

$$\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right) \leq \frac{\operatorname{vol}(A^{\Gamma}_{s,s}(p))}{y^{n-1}(s)} \int_s^S y^{n-1}(t)dt. \tag{1.7}$$

(2) *If there exist constants $C_1, C_2, K, L > 0$ $(0 \leq L - K < 1/(n-1))$ such that the unique solution of* (1.4) *or* (1.5) *satisfies $C_1 t^K \leq y(t) \leq C_2 t^L$, then one has a constant $C(n,k) > 0$ depending only on n and k such that*

$$\operatorname{vol}(B_p(r+1) - B_p(r-1)) \leq C(n,k) \frac{\operatorname{vol}(B_p(r-1))}{(r-1)^{1-(n-1)(L-K)}}. \tag{1.8}$$

Remark 1.4. The condition $y > 0$ on $(0, l)$ in (1.5) constitutes an extra assumption imposed on the unique solution y of (1.4). In Theorem 1.3, we do not require that the radially symmetric Ricci curvature lower bound $k : [0, l) \to R$ corresponds to the generalized space forms with radially symmetric curvature lower bound. Our purpose is to establish a volume comparison estimate effectively.

Applying the generalized volume comparison estimate, we can now investigate the volume growth, total Betti number, and finite topological type of manifolds with non-asymptotically almost nonnegative Ricci curvature.

Theorem 1.5. *Let M be a complete n-Riemannian manifold $(n \geq 2)$ with non-asymptotically almost nonnegative Ricci curvature $k : [0, l) \to R$ at the point $p \in M$, and let M be noncollapsing, that* *is,* $\inf_{x\in M}\mathrm{vol}(B_x(1)) \geq v > 0$. *If there exist constants* $C_1, C_2, K, L > 0$ $(0 \leq L - K < 1/(n-1))$ *such that, the unique solution of* (1.4) *or* (1.5) *satisfies* $C_1 t^K \leq y(t) \leq C_2 t^L$*, then one has a constant* $C(n,k) > 0$ *depending only on* n *and* k *such that for* $r > 1$*,*

$$\mathrm{vol}(B_p(r)) \geq C(n,k,v) r^{1-(n-1)(L-K)}. \tag{1.9}$$

Theorem 1.6. *Let* M *be a complete* n*-Riemannian manifold* $(n \geq 2)$ *with non-asymptotically almost nonnegative Ricci curvature* $k : [0,l) \to R$ *at the point* $p \in M$*, and* M *has weakly bounded geometry, that is,* $\sec(M) \geq -1$*, and* $\inf_{x\in M}\mathrm{vol}(B_x(7^{-n-1}/2)) \geq v > 0$.

(1) *If there exist constants* $C, L > 0$ *such that the unique solution of* (1.4) *satisfies* $y(t) \leq C_2 t^L$*, then one has a constant* $C(n,k,v) > 0$ *depending only on* n*,* k*, and* v *such that, for* $r > 0$*,*

$$\sum_{i=0}^{n} b_i(p,r) \leq C(n,k,v)(1+r)^{(n^2-1)L+n+1}. \tag{1.10}$$

(2) *If there exist constants* $C_1, C_2, K, L > 0$ *such that the unique solution of* (1.5) *satisfies* $C_1 t^K \leq y(t) \leq C_2 t^L$*, then one has a constant* $C(n,k,v,\mathrm{vol}(S_p(1)),\mathrm{vol}(B_p(1))) > 0$ *depending only on* n*,* k*,* v*,* $\mathrm{vol}(S_p(1))$*, and* $\mathrm{vol}(B_p(1))$ *such that, for* $r > 1$*,*

$$\sum_{i=0}^{n} b_i(p,r) \leq C(n,C,v,\mathrm{vol}(S_p(1)),\mathrm{vol}(B_p(1)))(1+r)^{(n^2-1)L+n+1}. \tag{1.11}$$

Theorem 1.7. *Let* M *be a complete* n*-Riemannian manifold* $(n \geq 2)$ *with non-asymptotically almost nonnegative Ricci curvature* $k : [0,l) \to R$ *at the point* $p \in M$*,* $\sec(x) \geq -(C/d(p,x)^{\alpha})$ *where* $C > 0$*,* $0 \leq \alpha \leq 2$*, and let* M *be non-collapsing, that is,* $\inf_{x\in M}\mathrm{vol}(B_x(1)) \geq v > 0$. *If there exist constants* $C_1, C_2, K, L > 0$ $(0 \leq L - K < (1/(n-1))$ *such that the unique solution of* (1.4) *or* (1.5) *satisfies* $C_1 t^K \leq y(t) \leq C_2 t^L$*, then* M *is of finite topological type with the additional assumption that*

$$\limsup_{r\to\infty} \frac{\mathrm{vol}(B_p(r))}{r^{1+(\alpha/2)-(n-1)(L-K)}} < C(n,k,C,\alpha,v) \tag{1.12}$$

for some constant $\widetilde{C}(n,k,C,\alpha,v) > 0$ *depending only on* n*,* k*,* C*,* α*, and* v*.*

2. A Volume Comparison Estimate with Radially Symmetric Ricci Curvature Lower Bound

Proof of Theorem 1.3. Choose polar coordinate (r,θ) at p. Define the function $J(r,\theta)$ by the formula

$$dv_M = J^{n-1} dr\, d\theta. \tag{2.1}$$

Then

$$\mathrm{vol}\left(A^{\Gamma}_{s,S}(p)\right) = \int_{\Gamma} d\theta \int_{\min\{s,\mathrm{cut}(\theta)\}}^{\min\{S,\mathrm{cut}(\theta)\}} J^{n-1} dr, \tag{2.2}$$

where $\mathrm{cut}(\theta)$ is the distance from p to the cut point in direction θ. It is well known (e.g., [11]) that J satisfies the following:

$$\begin{aligned} J'' - kJ \le 0, &\qquad 0 \le t \le \mathrm{cut}(\theta), \\ J(0) = 0, &\qquad J'(0) = 1. \end{aligned} \tag{2.3}$$

Let $y(t)$ be the unique solution of one of (1.4) and (1.5) (Note that, by the uniqueness of the solution of ordinary differential equation, the solution of (1.4) always exists.).

Then in the interval of $y > 0$, $J''y - y''J \le 0$, that is, $(J'y - y'J)' \le 0$. By the initial condition of J and y, $J'y - y'J \le 0$. Thus, when $y > 0$,

$$\left(\frac{J}{y}\right)' = \frac{1}{y^2}(J'y - y'J) \le 0. \tag{2.4}$$

This shows that J/y is nonincreasing in the interval of $y > 0$.

Note that in the interval of $J > 0$ we must have $y > 0$. Thus it suffices to consider that $y(t)$ is the unique solution of (1.4).

Otherwise, suppose that $t_0 > 0$ is the first point such that $y > 0$ in $(0, t_0)$, $y(t_0) = 0$, and $J > 0$ in $(0, t_0]$. By $J(0) = y(0) = 0$, $J'(0) = y'(0) = 1$, J/y is non-increasing in $(0, t_0)$:

$$\begin{gathered} \frac{J}{y}(0) := \lim_{t \to 0} \frac{J(t)}{y(t)} = \lim_{t \to 0} \frac{J'(t)}{y'(t)} = 1, \\ \frac{J}{y}(t) \le \frac{J}{y}(0) = 1, \quad t \in (0, t_0), \\ J(t) \le y(t), \quad t \in (0, t_0). \end{gathered} \tag{2.5}$$

Let $t \longrightarrow t_0$, then $J(t_0) \le y(t_0) = 0$. This is a contradiction. □

Thus consider the following lemma.

Lemma 2.1 (see [12]). *Let f, g be positive functions on $[0, +\infty)$; if f/g is nonincreasing, then for all $R > r > 0$, $S > s > 0$, $s > r$, $S > R$, one has*

$$\frac{\int_s^S f(t)dt}{\int_r^R f(t)dt} \le \frac{\int_s^S g(t)dt}{\int_r^R g(t)dt}. \tag{2.6}$$

We have

$$\begin{aligned}
\frac{\int_{\min\{r,\mathrm{cut}(\theta)\}}^{\min\{S,\mathrm{cut}(\theta)\}} J^{n-1}(t,\theta)dt}{\int_{\min\{r,\mathrm{cut}(\theta)\}}^{\min\{R,\mathrm{cut}(\theta)\}} J^{n-1}(t,\theta)dt} &\le \frac{\int_{\min\{r,\mathrm{cut}(\theta)\}}^{\min\{S,\mathrm{cut}(\theta)\}} y^{n-1}(t)dt}{\int_{\min\{r,\mathrm{cut}(\theta)\}}^{\min\{R,\mathrm{cut}(\theta)\}} y^{n-1}(t)dt} \\
&\le \frac{\int_{r}^{\min\{S,\mathrm{cut}(\theta)\}} y^{n-1}(t)dt}{\int_{r}^{\min\{R,\mathrm{cut}(\theta)\}} y^{n-1}(t)dt} \\
&\le \frac{\int_r^S y^{n-1}(t)dt}{\int_r^R y^{n-1}(t)dt}, \quad (\text{because } R < S),
\end{aligned} \tag{2.7}$$

where the last equality is due to

$$\frac{\int_{r}^{\min\{S,\mathrm{cut}(\theta)\}} y^{n-1}(t)dt}{\int_{r}^{\min\{R,\mathrm{cut}(\theta)\}} y^{n-1}(t)dt} = \begin{cases} \dfrac{\int_r^{\mathrm{cut}(\theta)} y^{n-1}(t)dt}{\int_r^{\mathrm{cut}(\theta)} y^{n-1}(t)dt} = 1, & \text{when } \mathrm{cut}(\theta) \le R \le S, \\[2ex] \dfrac{\int_r^{\mathrm{cut}(\theta)} y^{n-1}(t)dt}{\int_r^R y^{n-1}(t)dt}, & \text{when } R \le \mathrm{cut}(\theta) \le S, \\[2ex] \dfrac{\int_r^S y^{n-1}(t)dt}{\int_r^R y^{n-1}(t)dt}, & \text{when } R \le S \le \mathrm{cut}(\theta). \end{cases} \tag{2.8}$$

Then by integration on Γ, we have

$$\frac{\mathrm{vol}\left(A_{r,S}^{\Gamma}(p)\right)}{\mathrm{vol}\left(A_{r,R}^{\Gamma}(p)\right)} \le \frac{\int_r^S y^{n-1}(t)dt}{\int_r^R y^{n-1}(t)dt}. \tag{2.9}$$

Similarly,

$$\frac{\mathrm{vol}\left(A_{s,S}^{\Gamma}(p)\right)}{\mathrm{vol}\left(A_{r,R}^{\Gamma}(p)\right)} \le \frac{\int_s^S y^{n-1}(t)dt}{\int_r^R y^{n-1}(t)dt}. \tag{2.10}$$

In particular, (1)

$$\begin{aligned}
\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right) &\le \frac{\operatorname{vol}\left(A^{\Gamma}_{r,R}(p)\right)}{\int_r^R y^{n-1}(t)dt}\int_s^S y^{n-1}(t)dt \\
&= \frac{\int_\Gamma \int_{\min\{r,\mathrm{cut}(\theta)\}}^{\min\{R,\mathrm{cut}(\theta)\}} J^{n-1}(t,\theta)dt\,d\theta}{\int_r^R y^{n-1}(t)dt}\int_s^S y^{n-1}(t)dt \\
&= \int_\Gamma \frac{\int_{\min\{r,\mathrm{cut}(\theta)\}}^{\min\{R,\mathrm{cut}(\theta)\}} J^{n-1}(t,\theta)dt}{\int_r^R y^{n-1}(t)dt}d\theta \int_s^S y^{n-1}(t)dt.
\end{aligned} \tag{2.11}$$

Let $R \to r$, then

$$\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right) \le \int_\Gamma \frac{J^{n-1}(r,\theta)}{y^{n-1}(r)}d\theta \int_s^S y^{n-1}(t)dt. \tag{2.12}$$

When $y(t)$ is the unique solution of (1.4), let $r \to 0$; by $J(0) = y(0) = 0$, $J'(0) = y'(0) = 1$; then we have

$$\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right) \le \operatorname{vol}(\Gamma)\int_s^S y^{n-1}(t)dt. \tag{2.13}$$

When $y(t)$ is the unique solution of (1.5), let $r = s$, we have

$$\begin{aligned}
\operatorname{vol}\left(A^{\Gamma}_{s,S}(p)\right) &\le \int_\Gamma \frac{J^{n-1}(s,\theta)}{y^{n-1}(s)}d\theta \int_s^S y^{n-1}(t)dt \\
&= \frac{\operatorname{vol}\left(A^{\Gamma}_{s,s}(p)\right)}{y^{n-1}(s)}\int_s^S y^{n-1}(t)dt.
\end{aligned} \tag{2.14}$$

(2) Choose $\Gamma = S_p^{n-1}$; for $r \geq 3$, an easy computation shows that

$$\begin{aligned}
&\frac{\operatorname{vol}\big(B_p(r+1)-B_p(r-1)\big)}{\operatorname{vol}\big(B_p(r-1)-B_p(1)\big)}\\
&\quad\leq \frac{\int_{r-1}^{r+1} y^{n-1}(t)\,dt}{\int_{1}^{r-1} y^{n-1}(t)\,dt}\\
&\quad\leq \frac{\int_{r-1}^{r+1} \left(C_2 t^{L}\right)^{n-1} dt}{\int_{1}^{r-1} \left(C_1 t^{K}\right)^{n-1} dt}\\
&\quad= \frac{C_2^{n-1}((n-1)K+1)}{C_1^{n-1}((n-1)L+1)}\cdot\frac{(r+1)^{(n-1)L+1}-(r-1)^{(n-1)L+1}}{(r-1)^{(n-1)K+1}-1}\\
&\quad\leq \frac{2C_2^{n-1}((n-1)K+1)}{C_1^{n-1}((n-1)L+1)}\cdot\frac{(r+1)^{(n-1)L+1}-(r-1)^{(n-1)L+1}}{(r-1)^{(n-1)K+1}}\\
&\quad= C(n,C_1,C_2,K,L)\left(\left(1+\frac{2}{r-1}\right)^{(n-1)K+1}\cdot(r+1)^{(n-1)(L-K)}-(r-1)^{(n-1)(L-K)}\right)\\
&\quad\leq C(n,C_1,C_2,K,L)\left(\left(1+\frac{C(n,K)}{r-1}\right)\cdot(r+1)^{(n-1)(L-K)}-(r-1)^{(n-1)(L-K)}\right)\\
&\quad= C(n,C_1,C_2,K,L)\cdot(r-1)^{(n-1)(L-K)}\cdot\left(\left(1+\frac{C(n,K)}{r-1}\right)\cdot\left(\frac{r+1}{r-1}\right)^{(n-1)(L-K)}-1\right)\\
&\quad\leq C(n,C_1,C_2,K,L)\cdot(r-1)^{(n-1)(L-K)}\cdot\left(\left(1+\frac{C(n,K)}{r-1}\right)\cdot\left(1+\frac{C(n,K,L)}{r-1}\right)-1\right)\\
&\quad= C(n,C_1,C_2,K,L)\cdot(r-1)^{(n-1)(L-K)}\cdot\left(\frac{C(n,K)+C(n,K,L)}{r-1}+\frac{C(n,K)\cdot C(n,K,L)}{(r-1)^2}\right)\\
&\quad\leq C(n,C_1,C_2,K,L)\frac{1}{(r-1)^{1-(n-1)(L-K)}}.
\end{aligned} \tag{2.15}$$

3. Proof of Theorem 1.5

Proof of Theorem 1.5. Note that for $r \geq 3$ there exists a point $q \in S_p(r)$ such that $(B_p(r+1) - B_p(r-1)) \supseteq B_q(1)$; thus

$$\operatorname{vol}\big(B_p(r+1)-B_p(r-1)\big) \geq \operatorname{vol} B_q(1). \tag{3.1}$$

And since M does not collapse at infinity, that is, $\inf_{x\in M}\operatorname{vol}(B_x(1)) \geq v > 0$, for $r \geq 3$, we have

$$\operatorname{vol}\big(B_p(r+1)-B_p(r-1)\big) \geq v. \tag{3.2}$$

Thus, for $r \geq 3$, by Theorem 1.3(2), there is some constant $C(n,k,v)$ such that $\mathrm{vol}(B_p(r)) \geq C(n,k,v)r^{1-(n-1)(K-L)}$. And note that for $1 < r \leq 3$,

$$B_p(r) \supseteq B_p(1). \tag{3.3}$$

Theorem 1.5 is obtained. □

4. Proof of Theorem 1.6

First let us recall Gromov's theorems [2]; one can refer to [13] for the details.

Theorem 4.1 (see [2]). *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geq -1$. Then there is a constant $C(n) > 1$ depending only on n such that, for any $0 < \epsilon < 1$ and any bounded subset $X \subset M$,*

$$\sum_{k=0}^{n} b_k(X, T_\epsilon X) \leq \left(1 + \mathrm{diam}(X)\epsilon^{-1}\right)^n C(n)^{1+dia(X)}, \tag{4.1}$$

where $T_\epsilon X$ denotes the ϵ-neighborhood of X in M.

Theorem 4.2 (see [2]). *Let M be an n-dimensional complete Riemannian manifold and let $p \in M$. For any fixed numbers $r > 0$ and $r_0 \leq 7^{-n-1}$, let $B_j^0 := B(p_j, r_0)$, $j = 1, \ldots, N$, be a ball covering of $B(p,r)$ with $p_j \in B(p,r)$. Let $B_j^k := 7^k B_j^0 := B(p_j, 7^k r_0)$, $k = 0, \ldots, n+1$. Then*

$$\sum_{i=0}^{n} b_i(B(p,r), B(p,r+1)) \leq (e-1)Nt^n \sup\left\{\sum_{i=0}^{n} b_i\left(B_j^k, 5B_j^k\right) : 0 \leq k \leq n,\ 1 \leq j \leq N\right\}, \tag{4.2}$$

where t is the smallest number such that, each ball B_j^n intersects at most t other balls $B_{j'}^n$.

Proof of Theorem 1.6. By Theorem 4.1, there is a constant $C(n)$ depending only on n such that for all balls $B(x,r)$ with radius $r \leq 1$ in M,

$$\sum_{i=0}^{n} b_i(B(x,r), B(x,5r)) \leq C_1(n). \tag{4.3}$$

Take $r_0 = 7^{-n-1}$, and let $B(p_j, (1/2)r_0)$, $j = 1, \ldots, N$, be a maximal set of disjoint balls with $p_j \in B(p,r)$, and let B_j^k, $j = 1, \ldots, N$, $k = 0, \ldots, n+1$, be the same as in Theorem 4.2. Then B_j^0, $j = 1, \ldots, N$, is a covering of $B(p,r)$. And let t, N be the same as in Theorem 4.2.

If there exist constants $C, L > 0$ such that the unique solution of (1.4) satisfies $y(t) \leq C_2 t^L$, choosing $\Gamma = S_p^{n-1}$, $s = 0$ in Theorem 1.3(1), then, for $S \geq 0$,

$$\begin{aligned}
\operatorname{vol}(B_p(S)) &\leq \operatorname{vol}\left(S_1^{n-1}\right) \int_0^S \left(C_2 t^L\right)^{n-1} dt \\
&= \operatorname{vol}\left(S_1^{n-1}\right) \frac{C_2}{(n-1)L+1} S^{(n-1)L+1} \\
&=: C(n,k) S^{(n-1)L+1}.
\end{aligned} \tag{4.4}$$

Then by the assumption that $\inf_{x \in M} \operatorname{vol}(B_x(7^{-n-1}/2)) \geq v > 0$,

$$\begin{aligned}
N &\leq \frac{\operatorname{vol}(B_x(r + (r_0/2)))}{\min_j \operatorname{vol}\left(B_{p_j}(r_0/2)\right)} \\
&\leq \frac{C(n,k)(r + (r_0/2))^{(n-1)L+1}}{v} \\
&\leq \frac{C(n,k)(r+1)^{(n-1)L+1}}{v} \\
t &\leq \frac{\operatorname{vol}\left(B_{p_j}((2/7) + (r_0/2))\right)}{\min_{j'} \operatorname{vol}\left(B_{p_{j'}}(r_0/2)\right)} \\
&\leq \frac{\operatorname{vol}(B_p(r + (2/7) + (r_0/2)))}{v} \\
&\leq \frac{C(n,k)(r + (2/7) + (r_0/2))^{(n-1)L+1}}{v} \\
&\leq \frac{C(n,k)(r+1)^{(n-1)L+1}}{v}.
\end{aligned} \tag{4.5}$$

Since each ball B_j^k has radius ≤ 1, it follows from (4.3) and Theorem 4.2 that

$$\begin{aligned}
\sum_{i=0}^{n} b_i(B_p(r), M) &\leq \sum_{i=0}^{n} b_i(B_p(r), B_p(r+1)) \\
&\leq (e-1)\left(\frac{C(n,k)(r+1)^{(n-1)L+1}}{v}\right)^{n+1} C(n) \\
&=: C(n,k,v)(1+r)^{((n-1)L+1)(n+1)}.
\end{aligned} \tag{4.6}$$

If there exist constants $C_1, C_2, K, L > 0$ such that the unique solution of (1.5) satisfies $C_1 t^K \leq y(t) \leq C_2 t^L$, choosing $\Gamma = S_p^{n-1}$, $s = 1$ in Theorem 1.3(1), then, for $S \geq 1$,

$$\begin{aligned}
\operatorname{vol}\left(B_p(S) - B_p(1)\right) &\leq \frac{\operatorname{vol}\left(S_p(1)\right)}{y^{n-1}(1)} \int_1^S y^{n-1}(t)dt \\
&\leq \frac{\operatorname{vol}\left(S_p(1)\right)}{C_1} \int_1^S \left(C_2 t^L\right)^{n-1} dt \\
&\leq \frac{\operatorname{vol}\left(S_p(1)\right)}{C_1} \frac{C_2}{(n-1)L+1} S^{(n-1)L+1} \\
&=: C\left(n, k, \operatorname{vol}\left(S_p(1)\right)\right) S^{(n-1)L+1}, \\
\operatorname{vol}\left(B_p(S)\right) &\leq C\left(n, k, \operatorname{vol}\left(S_p(1)\right)\right) S^{(n-1)L+1} + \operatorname{vol}\left(B_p(1)\right) \\
&\leq C\left(n, k, \operatorname{vol}\left(S_p(1)\right), \operatorname{vol}\left(B_p(1)\right)\right) S^{(n-1)L+1}.
\end{aligned} \tag{4.7}$$

Similar to the above, there exists a constant $C(n, k, \operatorname{vol}(S_p(1)), \operatorname{vol}(B_p(1))) > 0$ such that

$$\sum_{i=0}^{n} b_i\left(B_p(r), M\right) \leq C\left(n, k, \operatorname{vol}\left(S_p(1)\right), \operatorname{vol}\left(B_p(1)\right)\right)(1+r)^{((n-1)L+1)(n+1)}. \tag{4.8}$$

□

5. Proof of Theorem 1.7

We use critical point theory of the distance function to prove Theorem 1.7.

First of all, we recall some concepts (cf., e.g., [3, 7, 14]). Notice that the distance function $d_p(x) := d(p, x)$ is not a smooth function (on the cutlocus of p). Hence the critical points of d_p are not defined in a usual sense. The notion of critical points of d_p is introduced by Grove and Shiohama [15].

A point $x \in M$ is called a *critical point* of d_p if for any unit vector $v \in T_x M$ there is a minimizing geodesic σ from x to p such that $\angle(\sigma'(0), v) \leq \pi/2$.

For every r, the open set $M \setminus \overline{B(p, r)}$ contains only finitely many unbounded components, U_r. Each U_r has finitely many boundary components, $\Sigma_r \subset \partial B(p, r)$. In particular, Σ_r is a closed subset. Let us say that a connected component Σ_r of $S(p, r)$ is *good* if it is part of the boundary of an unbounded component of $M \setminus \overline{B(p, r)}$ and there is a ray from p passing through Σ_r.

Now we can introduce the following lemma.

Lemma 5.1 (see (Lemma 3.2, [7]); cf., also [14]). *Suppose that there is a $r_0 > 0$ such that if $r > r_0$ then there is no critical point of d_p on any good component Σ_r of $S(p, r)$. Then M has finite topological type.*

Another concept is the *diameter growth function* $\mathfrak{D}(p, r)$.

Definition 5.2. The diameter growth function $\mathfrak{D}(p,r)$ is defined by

$$\mathfrak{D}(p,r) = \sup_{\Sigma_r} \operatorname{diam}\left(\sum_r\right), \tag{5.1}$$

where the supremum is taken over all good components Σ_t of S_t and the diameter is measured using the metric on M.

Proof of Theorem 1.7. (i) We first show that if a complete noncompact Riemannian manifold satisfies $K(x) \geq -C/d(p,x)^{\alpha}$, where $C > 0$, $0 \leq \alpha \leq 2$, and the following diameter growth condition

$$\limsup_{r\to\infty} \frac{\mathfrak{D}(p,r)}{r^{\alpha/2}} < \frac{\delta_1}{2}, \tag{5.2}$$

where

$$\delta_1 = \max_{0<\epsilon\leq 1/20} \left\{ 2\epsilon - \frac{\operatorname{arc\,cosh}\left(\cosh^2 2^{\alpha} C^{1/2}\epsilon\right)}{2^{\alpha}C^{1/2}} > 0 \right\} \tag{5.3}$$

then M is of finite topological type.

As (i) in the proof of Theorem 1.1 in [8], choose a good connected component Σ_r of $S(p,r)$, for any $x \in \Sigma_r$, and a ray γ from p passing through Σ_r, choose $q = \gamma(t)$ such that $t \geq 2d(p,x)$, and suppose that x is a critical point of d_p, then

$$e_{pq}(x) \geq \delta_1 d(p,x)^{\alpha/2}. \tag{5.4}$$

On the other hand, by the triangle inequality,

$$e_{pq}(x) \leq 2\mathfrak{D}(p,r), \tag{5.5}$$

thus,

$$\mathfrak{D}(p,r) \geq \frac{\delta_1}{2} d(p,x)^{\alpha/2}, \tag{5.6}$$

For r large enough, by the assumption on the diameter growth, this is a contradiction.

Thus, there does not exist a critical point of $d(p,\cdot)$ on any good connected component. By Lemma 5.1, M is of finite topological type.

(ii) Given that $r > 0$, choose a good connected component Σ_r, of the boundary of an unbounded component of $M \setminus \overline{B(p,r)}$. For any $x, y \in \Sigma_r$, there is a continuous curve $c : [0,s] \to \Sigma_r$ from x to y. Suppose that $d(x,y) > 2$. Then there is a partition

$0 = t_0 < t_1 < \cdots < t_k = r$ such that $\{B(c(t_i),1)\}_{i=0}^{k}$ are disjoint and $B(c(t_i),2) \bigcap B(c(t_{i+1}),2) \neq \emptyset$. Note that $B(c(t_i),1) \subset B(p,r+1) - \overline{B(p,r-1)}$. Thus

$$(k+1)v \leq \sum_{i=0}^{k} \mathrm{vol}(B(c(t_i),1)) \leq \mathrm{vol}\Big(B(p,r+1) - \overline{B(p,r-1)}\Big),$$
$$\mathrm{diam}\left(\sum_{r}\right) \leq \sum_{i=0}^{k-1} d(c(t_i),c(t_{i+1})) \leq C(n,k,v)\mathrm{vol}\Big(B_p(r+1) - \overline{B_p(r-1)}\Big). \tag{5.7}$$

Then, by Theorem 1.3(2), there is a constant $\widetilde{C}(n,k,C,\alpha,v)$ such that if the volume growth satisfies

$$\limsup_{r\to\infty} \frac{\mathrm{vol}(B_p(r))}{r^{1+(\alpha/2)-(n-1)(L-K)}} < \widetilde{C}(n,k,C,\alpha,v) \tag{5.8}$$

the diameter growth satisfies

$$\limsup_{r\to\infty} \frac{\mathfrak{D}(p,r)}{r^{\alpha/2}} < \frac{\delta_1}{2}. \tag{5.9}$$

Then by (i), Theorem 1.7 is obtained. □

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