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Research Article

Note on Colon-Multiplication Domains

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Let R be an integral domain with quotient field L. Call a nonzero (fractional) ideal A of R a colon-multiplication ideal any ideal A, such that $B(A : B) = A$ for every nonzero (fractional) ideal B of R. In this note, we characterize integral domains for which every maximal ideal (resp., every nonzero ideal) is a colon-multiplication ideal. It turns that this notion unifies Dedekind and *MTP* domains.

1. Introduction

Let R be an integral domain which is not a field with quotient field L. For any nonzero (fractional) ideals A and B, $B(A : B) \subseteq A$ and the inclusion may be strict. We say that A is B-colon-multiplication if equality holds, that is, $A = B(A : B)$. A nonzero (fractional) ideal A is said to be a colon-multiplication ideal if A is B-colon-multiplication for every nonzero (fractional) ideal B of R, and the domain R is called a colon-multiplication domain if all its nonzero (fractional) ideals are colon-multiplication ideals. The purpose of this note is to characterize integral domains R that are colon-multiplication domains. This notion unifies the notions of Dedekind domains and MTP domains (i.e., domains R such that for every nonzero (fractional) ideal I, either I is invertible or II^{-1} is a maximal ideal of R). Precisely we prove that for a domain R, every maximal ideal is a colon-multiplication ideal if and only if either R is a Dedekind domain or a local MTP domain (Theorem 2.2), and a domain R is a colon-multiplication domain if and only if R is a Dedekind domain (Theorem 2.4). We also provide an example showing that the notions of colon-multiplication ideals and multiplication ideals (i.e., ideals A such that for every ideal $B \subseteq A$, there exists an ideal C such that $B = AC$) do not imply each other; however, over Noetherian domains, multiplication domains and colon-multiplication domains collapse to Dedekind domains.

Throughout, R is an integral domain with quotient field L, $\mathrm{Spec}(R)$ denotes the set of all prime ideals of R, and $F(R)$ denotes the set of all nonzero fractional ideals of R, that is, R-submodules of L such that $dA \subseteq R$ for some nonzero $d \in R$. For $A, B \in F(R)$, $(A : B) = \{x \in L \mid xB \subseteq A\}$ and $A^{-1} = (R : A)$. Unreferenced material is standard, typically as in [1] or [2].

2. Main Results

Definition 2.1. (1) Let R be a domain, and A and B two nonzero (fractional) ideals of R. We say that A is B-colon-multiplication if $A = B(A : B)$.

(2) A nonzero (fractional) ideal A is said to be a colon-multiplication ideal if A is B-colon-multiplication for every nonzero (fractional) ideal B of R.

(3) A domain R is said to be a colon-multiplication domain if every nonzero (fractional) ideal A of R is colon-multiplication.

Our first main theorem characterizes integral domains for which every maximal ideal is colon-multiplication. Before stating the result, we recall that a domain R is said to be an MTP domain (MTP stands for maximal trace property) if for every nonzero (fractional) ideal I of R either $II^{-1} = R$ or $II^{-1} = M$ is a maximal ideal of R [3]. For more details on the trace properties see [4].

Theorem 2.2. *Let R be an integral domain. The following statements are equivalent.*

(1) *Every nonzero prime ideal of R is colon-multiplication;*

(2) *Every maximal ideal of R is colon-multiplication;*

(3) *Either R is a Dedekind domain or a local MTP domain.*

We need the following lemma.

Lemma 2.3. *Let R be an integral domain and I a nonzero invertible (fractional) ideal of R. Then every nonzero (fractional) ideal A of R is I-colon-multiplication.*

Proof. This follows immediately from the (easily verified) fact that if I is invertible, then $(A : I) = AI^{-1}$ for each nonzero ideal A. □

Proof of Theorem 2.2. (1) ⇒ (2) Trivial.

(2) ⇒ (3) First we claim that R is an MTP domain. Indeed, let I be a nonzero (fractional) ideal of R. Assume that $II^{-1} \subsetneq R$ and let M be a maximal ideal such that $II^{-1} \subseteq M$. Then $I^{-1} \subseteq (M : I) \subseteq I^{-1}$ and so $I^{-1} = (M : I)$. Since M is I-colon-multiplication, $M = I(M : I) = II^{-1}$, and therefore R is an MTP domain. Now, if R is a Dedekind domain, we are done. Assume that R is not Dedekind. Then R is an MTP domain with a unique noninvertible maximal ideal M [4, Corollary 2.11]. Then $MM^{-1} = M$. Now if N is a maximal ideal of R, by (2) N is M-colon-multiplication. So $N = M(N : M) \subseteq MM^{-1} = M$ and, by maximality, $N = M$. It follows that R is a local MTP domain, as desired.

(3) ⇒ (1) If R is a Dedekind domain, then (1) it holds by Lemma 2.3. Assume that R is a local MTP domain. Then R is a one-dimensional domain [3, Proposition 2.10]. Hence $\mathrm{Spec}(R) = \{(0) \subsetneq M\}$ and so M is the unique nonzero prime ideal of R. Now, let A be a nonzero (fractional) ideal of R. If A is invertible, by Lemma 2.3, M is A-colon-multiplication. Assume that $AA^{-1} \subsetneq R$. Then necessarily $AA^{-1} = M$. Hence $A^{-1} = (M : A)$ and therefore $M = AA^{-1} = A(M : A)$, as desired. □

The next result shows that colon-multiplication domains collapse to Dedekind domains.

Theorem 2.4. *Let R be an integral domain. The following statements are equivalent.*

(1) *R is a colon-multiplication domain;*

(2) *Every nonzero principal (fractional) ideal of R is colon-multiplication;*

(3) *R has a nonzero principal (fractional) ideal that is colon-multiplication;*

(4) *R is a Dedekind domain.*

Proof. (1)⇒(2)⇒(3) are trivial.

(3)⇒(4) Suppose that R has a nonzero principal (fractional) ideal $I = aR$ that is colon-multiplication. Let J be any nonzero ideal of R. Then I is J-colon-multiplication. Hence $aR = I = J(I : J) = J(aR : J) = aJJ^{-1}$ and therefore $R = JJ^{-1}$, as desired.

(4)⇒(1). it Follows immediately from Lemma 2.3. □

We recall that an ideal A of a commutative ring R is a multiplication ideal if for every ideal $B \subseteq A$ there exists an ideal C such that $B = AC$, and the ring R is a multiplication ring if each ideal of R is a multiplication ideal. Note that from the equation $B = AC$, we have $C \subseteq (B : A)$. Thus $B = AC \subseteq A(B : A)$, and so we have $B = A(B : A)$. Hence if A is a multiplication ideal of an integral domain R, then every subideal B of A is A-colon-multiplication. According to [5], a multiplication ideal is locally principal, but not conversely. However, a finitely generated locally principal ideal is a multiplication ideal [6]. In particular, in Noetherian domain, multiplication domain and colon-multiplication domain collapse to Dedekind domain. However, the two notions (multiplication and colon-multiplication) do not imply each other as is shown by the following example.

Example 2.5. (1) It provides a maximal ideal M of a domain R which is colon-multiplication but not a multiplication ideal.

Let k be a field and X and Y indeterminates over k. Set $R = k + Yk(X)[[Y]] = k + M$. Clearly R is a one-dimensional PVD (pseudovaluation domain) and therefore a local MTP domain (here note that pseudovaluation domains have the trace property, [3, Example 2.12], and so the maximal trace property if $\dim R = 1$). By Theorem 2.2, M is colon-multiplication. However, M is not a multiplication ideal since M is not “locally” principal [5].

(2) Let R be a non-Dedekind domain. By Theorem 2.4, not every nonzero principal ideal is colon-multiplication. However, every principal ideal is a multiplication ideal [6].

Given a nonzero (fractional) ideal A of an integral domain, we define the map $\varphi_A : F(R) \to F(R), B \mapsto A(B : A)$. The next proposition characterizes maps φ_A that are surjective.

Proposition 2.6. *Let R be an integral domain and A a nonzero (fractional) ideal of R. The following conditions are equivalent.*

(1) *$\varphi_A = id$ (i.e., B is A-colon-multiplication for each $B \in F(R)$);*

(2) *φ_A is surjective;*

(3) *A is invertible.*

Proof. (1)⇒(2) Trivial.

(2)⇒(3) Assume that φ_A is surjective. Then there exists $B \in F(R)$ such that $A(B : A) = \varphi_A(B) = R$. Hence A is invertible.

(3)⇒(1) Assume that A is invertible. By Lemma 2.3, every $B \in F(R)$ is A-colon-multiplication. Hence $\varphi_A(B) = A(B : A) = B$ and so $\varphi_A = id$. □

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