

## Research Article

# Fuzzy Goal Programming Procedure to Bilevel Multiobjective Linear Fractional Programming Problems

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This paper presents a fuzzy goal programming (FGP) procedure for solving bilevel multiobjective linear fractional programming (BL-MOLFP) problems. It makes an extension work of Moitra and Pal (2002) and Pal et al. (2003). In the proposed procedure, the membership functions for the defined fuzzy goals of the decision makers (DMs) objective functions at both levels as well as the membership functions for vector of fuzzy goals of the decision variables controlled by first-level decision maker are developed first in the model formulation of the problem. Then a fuzzy goal programming model to minimize the group regret of degree of satisfactions of both the decision makers is developed to achieve the highest degree (unity) of each of the defined membership function goals to the extent possible by minimizing their deviational variables and thereby obtaining the most satisfactory solution for both decision makers. The method of variable change on the under- and over-deviational variables of the membership goals associated with the fuzzy goals of the model is introduced to solve the problem efficiently by using linear goal programming (LGP) methodology. Illustrative numerical example is given to demonstrate the procedure.

## 1. Introduction

Bi-level mathematical programming (BLMP) is identified as mathematical programming that solves decentralized planning problems with two decision makers (DMs) in a two-level or hierarchical organization [1]. The basic connect of the BLMP technique is that a first-level decision maker (FLDM) (the leader) sets his goals and/or decisions and then asks each subordinate level of the organization for their optima which are calculated in

isolation; the second-level DM (SLDM) (the follower) decisions are then submitted and modified by the FLDM with consideration of the overall benefit for the organization; the process continued until a satisfactory solution is reached. In other words, although the FLDM independently optimizes its own benefits, the decision may be affected by the reaction of the SLDM. As a consequence, decision deadlock arises frequently and the problem of distribution of proper decision power is encountered in most of the practical decision situations.

Most of the developments on BLMP problems focus on bi-level linear programming [2–5], and many others for bilevel nonlinear programming and bi-level multiobjective programming [2, 6–11]. A bibliography of references on bi-level programming in both linear and non-linear cases, which is updated biannually, can be found in [12].

The use of the fuzzy set theory [13] for decision problems with several conflicting objectives was first introduced by Zimmermann [14]. Thereafter, various versions of fuzzy programming (FP) have been investigated and widely circulated in literature. In a hierarchical decision making context, it has been realized that each DM should have a motivation to cooperate with other, and a minimum level of satisfaction of the DM at a lower-level must be considered for overall benefit of the organization. The use of the concept of membership function of fuzzy set theory to BLMP problems for satisfactory decisions was first introduced by Lai [15] in 1996. Thereafter, Lai's satisfactory solution concept was extended by Shih et al. [1] and a supervised search procedure with the use of max-min operator of Bellman and Zadeh [16] was proposed. The basic concept of these fuzzy programming (FP) approaches is the same as it implies that the SLDM optimizes his/her objective function, taking a goal or preference of the FLDM into consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the FLDM, the SLDM solves an FP problem with a constraint on an overall satisfactory degree of the FLDM. If the proposed solution is not satisfactory to the FLDM, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached [17, 18].

The main difficulty that arises with the FP approach of Shih et al. is that there is possibility of rejecting the solution again and again by the FLDM and reevaluation of the problem is repeatedly needed to reach the satisfactory decision, where the objectives of the DMs are overconflicting. Even inconsistency between the fuzzy goals of the objectives and the decision variables may arise. This makes the solution process a lengthy one [17, 18]. To overcome the above undesirable situation, fuzzy goal programming (FGP) technique introduced by Mohamed [19] is extended in this article to BL-MOLFP problems [17, 18, 20].

To formulate the FGP Model of the BL-MOLFP problem, the fuzzy goals of the objectives are determined by determining individual optimal solution. The fuzzy goals are then characterized by the associated membership functions which are transformed into fuzzy flexible membership goals by means of introducing over- and underdeviational variables and assigning highest membership value (unity) as aspiration level to each of them. To elicit the membership functions of the decision vectors controlled by the FLDM, the optimal solution of the first-level MOLFP problem is separately determined. A relaxation of the FLDM decisions is considered for avoiding decision deadlock.

The method of variable change on the under- and overdeviational variables of the membership goals associated with the fuzzy goals of the model is introduced to solve the problem efficiently by using linear goal programming (LGP) methodology.

## 2. Problem Formulation

Assume that there are two levels in a hierarchy structure with first-level decision maker (FLDM) and second-level decision maker (SLDM). Let the vector of decision variables  $x = (x_1, x_2) \in R^n$  be partitioned between the two planners. The first-level decision maker has control over the vector  $x_1 \in R^{n_1}$  and the second-level decision maker has control over the vector  $x_2 \in R^{n_2}$ , where  $n = n_1 + n_2$ . Furthermore, assume that

$$F_i(x_1, x_2) : R^{n_1} \times R^{n_2} \longrightarrow R^{m_i}, \quad i = 1, 2 \quad (2.1)$$

are the first-level and second-level vector objective functions, respectively. So the BL-MOLFP problem of minimization type may be formulated as follows [21–24]:  
[1st Level]

$$\text{Min}_{x_1} F_1(x_1, x_2) = \text{Min}_{x_1} (f_{11}(x_1, x_2), f_{12}(x_1, x_2), \dots, f_{1m_1}(x_1, x_2)), \quad (2.2)$$

where  $x_2$  solves  
[2nd Level]

$$\text{Min}_{x_2} F_2(x_1, x_2) = \text{Min}_{x_2} (f_{21}(x_1, x_2), f_{22}(x_1, x_2), \dots, f_{2m_2}(x_1, x_2)) \quad (2.3)$$

subject to

$$x \in \mathbf{G} = \left\{ x = (x_1, x_2) \in R^n \mid A_1 x_1 + A_2 x_2 \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \mathbf{b}, x \geq 0, \mathbf{b} \in R^m \right\} \neq \phi, \quad (2.4)$$

where

$$f_{ij}(x_1, x_2) = \frac{c_{ij}x + \alpha_{ij}}{d_{ij}x + \beta_{ij}}, \quad (2.5)$$

$$j = 1, 2, \dots, m_1, \quad i = 1 \quad \text{for FLDM objective functions,}$$

$$j = 1, 2, \dots, m_2, \quad i = 2 \quad \text{for SLDM objective functions,}$$

and where

- (i)  $x_1 = (x_1^1, x_1^2, x_1^3, \dots, x_1^{n_1})$ ,  $x_2 = (x_2^1, x_2^2, x_2^3, \dots, x_2^{n_2})$ ,
- (ii)  $\mathbf{G}$  is the the bi-level convex constraints feasible choice set,
- (iii)  $m_1$  is the number of first-level objective functions,
- (iv)  $m_2$  is the number of second-level objective functions,
- (v)  $m$  is the number of the constraints,
- (vi)  $A_i$ :  $m \times n_i$  matrix,  $i = 1, 2$ ,

- (vii)  $c_{ij}, \mathbf{d}_{ij} \in R^n, \mathbf{d}_{ij}\mathbf{x} + \beta_{ij} > 0$  for all  $\mathbf{x} \in \mathbf{G}$ ,
- (viii)  $\beta_{ij}, \alpha_{ij}$  are constants.

### 3. Fuzzy Goal Programming Formulation of BL-MOLFP

In BL-MOLFP problems, if an imprecise aspiration level is assigned to each of the objectives in each level of the BL-MOLFP, then these fuzzy objectives are termed as fuzzy goals. They are characterized by their associated membership functions by defining the tolerance limits for achievement of their aspired levels.

#### 3.1. Construction of Membership Functions

Since the FLDM and the SLDM both are interested of minimizing their own objective functions over the same feasible region defined by the system of constraints (2.4), the optimal solutions of both of them calculated in isolation can be taken as the aspiration levels of their associated fuzzy goals.

Let  $(x_1^{1j}, x_2^{1j}; f_{1j}^{\min}, j = 1, 2, \dots, m_1)$  and  $(x_1^{2j}, x_2^{2j}; f_{2j}^{\min}, j = 1, 2, \dots, m_2)$  be the optimal solutions of FLDM and SLDM objective functions, respectively, when calculated in isolation. Let  $g_{ij} \geq f_{ij}^{\min}$  be the aspiration level assigned to the  $ij$ th objective  $f_{ij}(x_1, x_2)$  (the subscript  $ij$  means that  $j = 1, 2, \dots, m_1$  when  $i = 1$  for FLDM problem, and  $j = 1, 2, \dots, m_2$  when  $i = 2$  for SLDM problem). Also, let  $x^* = (x_1^*, x_2^*), x_1^* = (x_1^{1*}, x_1^{2*}, \dots, x_1^{m_1*})$  and  $x_2^* = (x_2^{1*}, x_2^{2*}, \dots, x_2^{m_2*})$ , be the optimal solution of the FLDM MOLFP problem. Then, the fuzzy goals of the decision makers objective functions at both levels and the vector of fuzzy goals of the decision variables controlled by first-level decision maker appear as

$$f_{ij}(x_1, x_2) \lesssim g_{ij}, \quad i = 1, 2, j = 1, 2, \dots, m_i, x_1 \underset{\sim}{=} x_1^*, \quad (3.1)$$

where " $\lesssim$ " and " $\underset{\sim}{=}$ " indicate the fuzziness of the aspiration levels and are to be understood as "essentially less than" and "essentially equal to" [14, 25].

It may be noted that the solutions  $(x_1^{1j}, x_2^{1j}), j = 1, 2, \dots, m_1, x^* = (x_1^*, x_2^*),$  and  $(x_1^{2j}, x_2^{2j}), j = 1, 2, \dots, m_2$  are usually different because the objectives of FLDM and the objectives of the SLDM are conflicting in nature. Therefore, it can reasonably be assumed that the values  $f_{\ell m}(x_1^{\ell m}, x_2^{\ell m}) \geq f_{ij}^{\min}$  for all  $\ell = 1, 2, m = 1, 2, \dots, m_i,$  and  $ij \neq \ell m$  and all values greater than  $f_{\ell m}^u = \max[f_{ij}(x_1^{\ell m}, x_2^{\ell m}), i = 1, 2, j = 1, 2, \dots, m_i,$  and  $ij \neq \ell m]$  are absolutely unacceptable to the objective function  $f_{\ell m}(x_1, x_2)$ . As such,  $f_{\ell m}^u$  can be considered as the upper tolerance limit  $u_{\ell m}$  of the fuzzy goal to the objective functions  $f_{\ell m}(x_1, x_2)$ . Then, membership functions  $\mu_{f_{ij}}(f_{ij}(x_1, x_2))$  for the  $ij$ th fuzzy goal can be formulated as in Figure 1:

$$\mu_{f_{ij}}(f_{ij}(x_1, x_2)) = \begin{cases} 1, & \text{if } f_{ij} \leq g_{ij}, \\ \frac{u_{ij} - f_{ij}(x_1, x_2)}{u_{ij} - g_{ij}}, & \text{if } g_{ij} \leq f_{ij} \leq u_{ij}, \quad i = 1, 2, j = 1, 2, \dots, m_i, \\ 0, & \text{if } f_{ij} \geq u_{ij}. \end{cases} \quad (3.2)$$

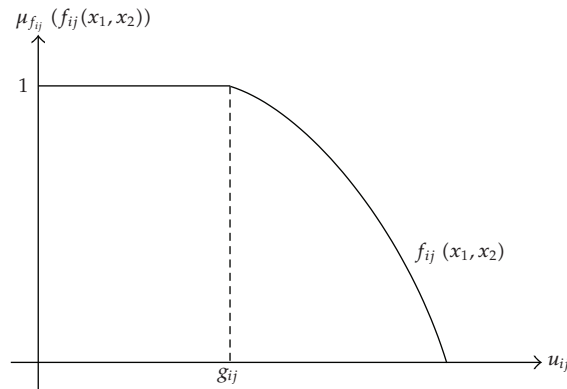


Figure 1: Membership function of minimization-type objective functions.

Following Lai [15] and Shih et al. [1], we include the membership functions for the fuzzy goals of the decision variables controlled by first-level decision maker,  $x_1 = (x_1^1, x_1^2, x_1^3, \dots, x_1^{n_1})$ , in the proposed model in this article. To build these membership functions, the optimal solution  $x^* = (x_1^*, x_2^*)$  of the first-level MOLFP problem should be determined first. Following Pal et al. approach [26], the optimal solution  $x^* = (x_1^*, x_2^*)$  could be obtained. It may be noted that any other approaches for solving MOLFP problems can be used in solving the first-level MOLFP problem [27–31]. In Section 4, the FGP model of Pal et al. [26], for solving the first-level MOLFP problem, is presented to facilitate the achievement of  $x^* = (x_1^*, x_2^*)$ .

Let  $t_k^L$  and  $t_k^R$ ,  $k = 1, 2, \dots, n_1$  be the maximum negative and positive tolerance values on the decision vector considered by the FLDM. The tolerance  $t_k^L$  and  $t_k^R$  are not necessarily same. The linear membership functions (Figure 2) for the decision vector  $x_1 = (x_1^1, x_1^2, x_1^3, \dots, x_1^{n_1})$  can be formulated as

$$\mu_{x_1^k}(x_1^k) = \begin{cases} \frac{x_1^k - (x_1^{k*} - t_k^L)}{t_k^L}, & \text{if } x_1^{k*} - t_k^L \leq x_1^k \leq x_1^{k*}, \\ \frac{(x_1^{k*} + t_k^R) - x_1^k}{t_k^R}, & \text{if } x_1^{k*} \leq x_1^k \leq x_1^{k*} + t_k^R, \\ 0, & \text{if otherwise.} \end{cases} \quad k = 1, 2, \dots, n_1, \quad (3.3)$$

It may be noted that the decision maker may desire to shift the range of  $x_1^k$ . Following Pramanik and Roy [20], this shift can be achieved.

Now, in a fuzzy decision environment, the achievement of the objective goals to their aspired levels to the extent possible is actually represented by the possible achievement of their respective membership values to the highest degree. Regarding this aspect of fuzzy programming problems, a goal programming approach seems to be most appropriate for the solution of the first-level multiobjective linear fractional programming problem and the bi-level multi-objective linear fractional programming problems [26].

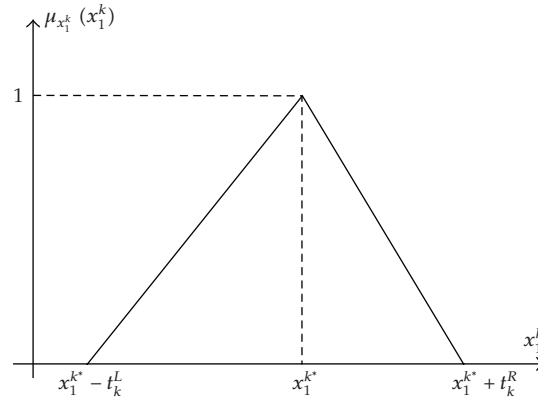


Figure 2: Membership function of decision vectors  $x_1^k$ .

### 3.2. Fuzzy Goal Programming Approach

In fuzzy programming approaches, the highest degree of membership function is 1. So, as in [19], for the defined membership functions in (3.2) and (3.3), the flexible membership goals with the aspired level 1 can be presented as

$$\begin{aligned} \mu_{f_{ij}}(f_{ij}(x_1, x_2)) + d_{ij}^- - d_{ij}^+ &= 1, \quad i = 1, 2, j = 1, 2, \dots, m_i, \\ \mu_{x_1^k}(x_1^k) + d_k^- - d_k^+ &= 1, \quad k = 1, 2, \dots, n_1, \end{aligned} \quad (3.4)$$

or equivalently as

$$\begin{aligned} \frac{u_{ij} - f_{ij}(x_1, x_2)}{u_{ij} - g_{ij}} + d_{ij}^- - d_{ij}^+ &= 1, \quad i = 1, 2, j = 1, 2, \dots, m_i, \\ \frac{x_1^k - (x_1^{k*} - t_k^L)}{t_k^L} + d_k^{L-} - d_k^{L+} &= 1, \quad k = 1, 2, \dots, n_1, \\ \frac{(x_1^{k*} + t_k^R) - x_1^k}{t_k^R} + d_k^{R-} - d_k^{R+} &= 1, \quad k = 1, 2, \dots, n_1, \end{aligned} \quad (3.5)$$

where  $d_k^- = (d_k^{L-}, d_k^{R-})$ ,  $d_k^+ = (d_k^{L+}, d_k^{R+})$ , and  $d_{ij}^-, d_k^{L-}, d_k^{R-}, d_{ij}^+, d_k^{L+}, d_k^{R+} \geq 0$  with  $d_{ij}^- \times d_{ij}^+ = 0$ ,  $d_k^{L-} \times d_k^{L+} = 0$ , and  $d_k^{R-} \times d_k^{R+} = 0$  represent the under- and overdeviations, respectively, from the aspired levels.

In conventional GP, the under- and/or overdeviational variables are included in the achievement function for minimizing them and that depend upon the type of the objective functions to be optimized. In this approach, the over-deviational variables for the fuzzy goals of objective functions,  $d_{ij}^+$ , and the over-deviational and the underdeviational variables for the fuzzy goals of the decision variables,  $d_k^{L-}$ ,  $d_k^{L+}$ ,  $d_k^{R-}$ , and  $d_k^{R+}$ , are required to be minimized to

achieve the aspired levels of the fuzzy goals. It may be noted that any under-deviation from a fuzzy goal indicates the full achievement of the membership value [26].

It can be easily realized that the membership goals in (3.2) are inherently nonlinear in nature and this may create computational difficulties in the solution process. To avoid such problems, a linearization procedure is presented in the following section.

The FGP approach to multiobjective programming problems presented by Mohamed [19] is extended here to formulate the FGP approach to bi-level multi-objective linear fractional programming. Therefore, considering the goal achievement problem of the goals at the same priority level, the equivalent fuzzy bilevel multiobjective linear fractional goal programming model of the problem can be presented as

$$\begin{aligned}
 \min \quad & Z = \sum_{j=1}^{m_1} w_{1j}^+ d_{1j}^+ + \sum_{k=1}^{n_1} \left[ w_k^L (d_k^{L+} + d_k^{L-}) + w_k^R (d_k^{R+} + d_k^{R-}) \right] + \sum_{j=1}^{m_2} w_{2j}^+ d_{2j}^+ \\
 \text{subject to} \quad & \mu_{f_{1j}}(f_{1j}(x_1, x_2)) + d_{1j}^- - d_{1j}^+ = 1, \quad j = 1, 2, \dots, m_1, \\
 & \mu_{f_{2j}}(f_{2j}(x_1, x_2)) + d_{2j}^- - d_{2j}^+ = 1, \quad j = 1, 2, \dots, m_2, \\
 & \mu_{x_1^k}(x_1^k) + d_k^- - d_k^+ = I, \quad k = 1, 2, \dots, n_1, \\
 & A_1 x_1 + A_2 x_2 \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \mathbf{b}, \quad \mathbf{x} \geq 0, \\
 & d_{ij}^-, d_{ij}^+ \geq 0, \quad \text{with } d_{ij}^- \times d_{ij}^+ = 0, \\
 & d_k^-, d_k^+ \geq 0, \quad \text{with } d_k^- \times d_k^+ = 0,
 \end{aligned} \tag{3.6}$$

and the above problem can be rewritten as

$$\begin{aligned}
 \min \quad & Z = \sum_{j=1}^{m_1} w_{1j}^+ d_{1j}^+ + \sum_{k=1}^{n_1} \left[ w_k^L (d_k^{L+} + d_k^{L-}) + w_k^R (d_k^{R+} + d_k^{R-}) \right] + \sum_{j=1}^{m_2} w_{2j}^+ d_{2j}^+ \\
 \text{subject to} \quad & \frac{u_{1j} - f_{1j}(x_1, x_2)}{u_{1j} - g_{1j}} + d_{1j}^- - d_{1j}^+ = 1, \quad j = 1, 2, \dots, m_1, \\
 & \frac{u_{2j} - f_{2j}(x_1, x_2)}{u_{2j} - g_{2j}} + d_{2j}^- - d_{2j}^+ = 1, \quad j = 1, 2, \dots, m_2, \\
 & \frac{x_1^k - (x_1^{k*} - t_k^L)}{t_k^L} + d_k^{L-} - d_k^{L+} = 1, \quad k = 1, 2, \dots, n_1, \\
 & \frac{(x_1^{k*} + t_k^R) - x_1^k}{t_k^R} + d_k^{R-} - d_k^{R+} = 1, \quad k = 1, 2, \dots, n_1,
 \end{aligned}$$

$$A_1x_1 + A_2x_2 \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \mathbf{b}, \quad \mathbf{x} \geq 0,$$

$$d_{ij}^-, d_{ij}^+ \geq 0, \quad \text{with } d_{ij}^- \times d_{ij}^+ = 0, \quad i = 1, 2, \quad j = 1, 2, \dots, m_i,$$

$$d_k^{L-}, d_k^{L+} \geq 0, \quad \text{with } d_k^{L-} \times d_k^{L+} = 0, \quad k = 1, 2, \dots, n_1,$$

$$d_k^{R-}, d_k^{R+} \geq 0, \quad \text{with } d_k^{R-} \times d_k^{R+} = 0, \quad k = 1, 2, \dots, n_1.$$
(3.7)

### 3.3. Linearization of Membership Goals

Following Pal et al. [26], the  $ij$ th membership goal in (3.5) can be presented as

$$L_{ij}u_{ij} - L_{ij}f_{ij}(x_1, x_2) + d_{ij}^- - d_{ij}^+ = 1, \quad \text{where } L_{ij} = \frac{1}{u_{ij} - g_{ij}}. \quad (3.8)$$

Introducing the expression of  $f_{ij}(x_1, x_2)$  from (2.5), the above goal can be presented as

$$L_{ij} u_{ij} - L_{ij} \frac{\mathbf{c}_{ij}\mathbf{x} + \alpha_{ij}}{\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}} + d_{ij}^- - d_{ij}^+ = 1$$

$$\implies L_{ij}u_{ij}(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) - L_{ij}(\mathbf{c}_{ij}\mathbf{x} + \alpha_{ij}) + d_{ij}^-(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) - d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) = (\mathbf{d}_{ij}\mathbf{x} + \beta_{ij})$$

$$\implies -L_{ij}(\mathbf{c}_{ij}\mathbf{x} + \alpha_{ij}) + d_{ij}^-(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) - d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) = [1 - L_{ij}u_{ij}](\mathbf{d}_{ij}\mathbf{x} + \beta_{ij})$$

$$\implies -L_{ij}(\mathbf{c}_{ij}\mathbf{x} + \alpha_{ij}) + d_{ij}^-(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) - d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) = L'_{ij}(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}), \quad \text{where } L'_{ij} = 1 - L_{ij}u_{ij}$$

$$\implies (-L_{ij}\mathbf{c}_{ij} - L'_{ij}\mathbf{d}_{ij})\mathbf{x} + d_{ij}^-(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) - d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) = L_{ij}\alpha_{ij} + L'_{ij}\beta_{ij}$$

$$\implies \mathbf{C}_{ij}\mathbf{x} + d_{ij}^-(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) - d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}) = \mathbf{G}_{ij},$$
(3.9)

where

$$\mathbf{C}_{ij} = -L_{ij}\mathbf{c}_{ij} - L'_{ij}\mathbf{d}_{ij}, \quad \mathbf{G}_{ij} = L_{ij}\alpha_{ij} + L'_{ij}\beta_{ij}. \quad (3.10)$$

Now, using the method of variable change as presented by Kornbluth and Steuer [29], Pal et al. [26], and Steuer [32], the goal expression in (3.9) can be linearized as follows.

Let  $D_{ij}^- = d_{ij}^-(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij})$ , and  $D_{ij}^+ = d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij})$ ; the linear form of the expression in (3.9) is obtained as

$$\mathbf{C}_{ij}\mathbf{x} + D_{ij}^- - D_{ij}^+ = \mathbf{G}_{ij} \quad (3.11)$$

with  $D_{ij}^-, D_{ij}^+ \geq 0$  and  $D_{ij}^- \times D_{ij}^+ = 0$  since  $d_{ij}^-, d_{ij}^+ \geq 0$  and  $\mathbf{d}_{ij}\mathbf{x} + \beta_{ij} > 0$ .



Now, in making decision, minimization of  $d_{ij}^+$  means minimization of  $D_{ij}^+ = d_{ij}^+(\mathbf{d}_{ij}\mathbf{x} + \beta_{ij})$ , which is also a non-linear one.

It may be noted that when a membership goal is fully achieved,  $d_{ij}^+ = 0$  and when its achievement is zero,  $d_{ij}^+ = 1$  are found in the solution [26]. So, involvement of  $d_{ij}^+ \leq 1$  in the solution leads to impose the following constraint to the model of the problem:

$$\frac{D_{ij}^+}{\mathbf{d}_{ij}\mathbf{x} + \beta_{ij}} \leq 1, \quad \text{that is, } -\mathbf{d}_{ij}\mathbf{x} + D_{ij}^+ \leq \beta_{ij}. \tag{3.12}$$

Here, on the basis of the previous discussion, it may be pointed out that any such constraint corresponding to  $d_{ij}^-$  does not arise in the model formulation [26].

Therefore, under the framework of *minsum* GP, the equivalent proposed FGP model of problem (3.7) becomes

$$\begin{aligned} \min \quad Z &= \sum_{j=1}^{m_1} w_{1j}^+ D_{1j}^+ + \sum_{k=1}^{n_1} \left[ w_k^L (d_k^{L+} + d_k^{L-}) + w_k^R (d_k^{R+} + d_k^{R-}) \right] + \sum_{j=1}^{m_2} w_{2j}^+ D_{2j}^+ \\ \text{subject to} \quad C_{1j}\mathbf{x} + D_{1j}^- - D_{1j}^+ &= \mathbf{G}_{1j}, \quad j = 1, 2, \dots, m_1, \\ C_{2j}\mathbf{x} + D_{2j}^- - D_{2j}^+ &= \mathbf{G}_{2j}, \quad j = 1, 2, \dots, m_2, \\ \frac{x_1^k - (x_1^{k*} - t_k^L)}{t_k^L} + d_k^{L-} - d_k^{L+} &= 1, \quad k = 1, 2, \dots, n_1, \\ \frac{(x_1^{k*} + t_k^R) - x_1^k}{t_k^R} + d_k^{R-} - d_k^{R+} &= 1, \quad k = 1, 2, \dots, n_1, \\ -d_{ij}\mathbf{x} + D_{ij}^+ &\leq \beta_{ij}, \quad i = 1, 2, \quad j = 1, 2, \dots, m_i, \\ A_1x_1 + A_2x_2 &\begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \mathbf{b}, \quad \mathbf{x} \geq 0, \\ D_{ij}^-, D_{ij}^+ &\geq 0, \quad i = 1, 2, \quad j = 1, 2, \dots, m_i, \\ d_k^{L-}, d_k^{L+} &\geq 0 \quad \text{with } d_k^{L-} \times d_k^{L+} = 0, \quad k = 1, 2, \dots, n_1, \\ d_k^{R-}, d_k^{R+} &\geq 0, \quad \text{with } d_k^{R-} \times d_k^{R+} = 0, \quad k = 1, 2, \dots, n_1, \end{aligned} \tag{3.13}$$

where  $Z$  represents the fuzzy achievement function consisting of the weighted over-deviational variables  $D_{ij}^+$  of the fuzzy goals  $g_{ij}$  and the underdeviational and the over-deviational variables  $d_k^{L-}, d_k^{R-}, d_k^{L+}$ , and  $d_k^{R+}$ ,  $k = 1, 2, \dots, n_1$  for the fuzzy goals of the decision variables  $x_1^1, x_1^2, x_1^3, \dots, x_1^m$ , where the numerical weights  $w_{ij}^+$ ,  $w_k^L$ , and  $w_k^R$  represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraints set in the decision situation.

To assess the relative importance of the fuzzy goals properly, the weighting scheme suggested by Mohamed [19] can be used to assign the values to  $w_{ij}^+$  and  $w_{ij}^-$ . In the present formulation, the values of  $w_{ij}^-$  and  $w_{ij}^+$  are determined as

$$\begin{aligned} w_{ij}^+ &= \frac{1}{u_{ij} - g_{ij}}, \quad i = 1, 2, \quad j = 1, 2, \dots, m_i, \\ w_k^L &= \frac{1}{t_k^L}, \quad w_k^R = \frac{1}{t_k^R}, \quad k = 1, 2, \dots, n_1, \end{aligned} \quad (3.14)$$

The FGP model (3.13) provides the most satisfactory decision for both the FLDM and the SLDM by achieving the aspired levels of the membership goals to the extent possible in the decision environment. The solution procedure is straightforward and illustrated via the following example.

#### 4. The FGP Model for MOLFP Problems

In this section, the FGP model of Pal et al. [26], for solving the first-level MOLFP problem, is presented here to facilitate the achievement of  $x^* = (x_1^*, x_2^*)$ . This solution is used to elicit the membership functions of the decision vectors  $x_1 = (x_1^1, x_1^2, x_1^3, \dots, x_1^{n_1})$ , that included in the FGP approach for solving BL-MOLFP problems proposed in this article.

The first-level MOLFP problem is

$$\begin{aligned} \text{Min} \quad & F_1(x_1, x_2) = \text{Min} (f_{11}(x_1, x_2), f_{12}(x_1, x_2), \dots, f_{1m_1}(x_1, x_2)) \\ \text{subject to} \quad & \mathbf{x} \in \mathbf{G} = \left\{ \mathbf{x} = (x_1, x_2) \in R^n \mid A_1x_1 + A_2x_2 \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \mathbf{b}, \mathbf{x} \geq 0, \mathbf{b} \in R^m \right\} \neq \phi. \end{aligned} \quad (4.1)$$

And the FGP model of Pal et al. [26] is

$$\begin{aligned} \text{min} \quad & Z = \sum_{j=1}^{m_1} w_{1j}^+ D_{1j}^+ \\ \text{subject to} \quad & C_{1j}\mathbf{x} + D_{1j}^- - D_{1j}^+ = G_{1j}, \quad j = 1, 2, \dots, m_1, \\ & -d_{1j}\mathbf{x} + D_{1j}^+ \leq \beta_{1j}, \quad j = 1, 2, \dots, m_1, \\ & A_1x_1 + A_2x_2 \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \mathbf{b}, \quad \mathbf{x} \geq 0, \\ & D_{1j}^-, D_{1j}^+ \geq 0, \quad j = 1, 2, \dots, m_i. \end{aligned} \quad (4.2)$$

## 5. The FGP Algorithm for BL-MOLFP Problems

Following the above discussion, we can now construct the proposed FGP algorithm for solving the BL-MOLFP problems.

*Step 1.* Calculate the individual minimum and maximum of each objective function in the two levels under the given constraints.

*Step 2.* Set the goals and the upper tolerance limits for all the objective functions in the two levels.

*Step 3.* Elicit the membership functions  $\mu_{f_{1j}}(f_{1j}(x_1, x_2))$ ,  $j = 1, 2, \dots, m_1$  for each of the objective functions in the first level.

*Step 4.* Formulate the Model (4.2) for the first level MOLFP problem.

*Step 5.* Solve the Model (4.2) to get  $x^* = (x_1^*, x_2^*)$ .

*Step 6.* Set the maximum negative and positive tolerance values on the decision vector  $x_1 = (x_1^1, x_1^2, x_1^3, \dots, x_1^{n_1})$ ,  $t_k^L$  and  $t_k^R$ ,  $k = 1, 2, \dots, n_1$ .

*Step 7.* Elicit the membership functions  $\mu_{x_1^k}(x_1^k)$  for decision vector  $x_1 = (x_1^1, x_1^2, x_1^3, \dots, x_1^{n_1})$ .

*Step 8.* Elicit the membership functions  $\mu_{f_{2j}}(f_{2j}(x_1, x_2))$ ,  $j = 1, 2, \dots, m_2$  for each of the objective functions in the second level.

*Step 9.* Formulate the Model (3.13) for the BL-MOLFP problem.

*Step 10.* Solve the Model (3.13) to get the satisfactory solution of the BL-MOLFP problem.

## 6. Numerical Example

To demonstrate proposed FGP procedure, consider the following bi-level multi-objective linear fractional programming problem:

[1st Level]

$$\text{Min}_{x_1} \left( f_{11} = \frac{x_1 + x_2 - 1}{x_1 - 2x_2 + 1}, f_{12} = \frac{2 - 2x_1 - x_2}{x_2 + 4} \right), \quad (6.1)$$

where  $x_2$  solves

[2nd Level]

$$\text{Min}_{x_2} \left( f_{21} = \frac{-x_1 + 4}{-x_2 + 3}, f_{22} = \frac{x_1 - 4}{x_2 + 1}, f_{23} = x_1 - x_2 \right) \quad (6.2)$$

**Table 1:** Coefficients  $\alpha_{ij}$ ,  $\beta_{ij}$ ,  $c_{ij}$ , and  $d_{ij}$  for the first- and second-level objectives of the BL-MOLFP problem.

	$f_{11}$	$f_{12}$	$f_{21}$	$f_{22}$	$f_{23}$
$\alpha_{ij}$	-1	2	4	-4	0
$\beta_{ij}$	7	4	3	1	1
$c_{ij}$	(-1,0)	(-1,0)	(-1,0)	(1,0)	(-1,0)
$d_{ij}$	(0,-1)	(0,-1)	(0,-1)	(0,1)	(0,-1)
$\min_G f_{ij}$	-0.143	-1.61	-0.308	-4	0
$\max_G f_{ij}$	0.51	0.5	1.3333	0.2667	4
$g_{ij}$	0	-0.5	-0.3	-4	0
$u_{ij}$	0.5	0.5	1.3	0	4
$L_{ij}$	2	1	0.625	0.25	0.25
$L'_{ij}$	0	0.5	0.1875	1	0
$C_{ij}$	(2,2)	(0.5,0.5)	(0.625, 0.1875)	(-0.25,-1)	(-0.25,0.25)
$G_{ij}$	-2	4	3.0625	0	0
$w_{ij}$	2	1	1	0.25	0.25

subject to

$$\begin{aligned}
 -x_1 + 4x_2 &\leq 0, \\
 x_1 - \frac{1}{2}x_2 &\leq 4, \\
 x_1, x_2 &\geq 0.
 \end{aligned} \tag{6.3}$$

Table 1 summarizes the coefficients  $\alpha_{ij}$ ,  $\beta_{ij}$ ,  $c_{ij}$ , and  $d_{ij}$  for the first- and second-level objectives of the BL-MOLFP problem. Also, the optimal minimum and maximum separate solutions of these objectives subjected to given constraints. The decided aspiration levels and upper tolerance limits to the objective functions are also mentioned. The values  $L_{ij}$ ,  $L'_{ij}$ ,  $C_{ij}$ ,  $G_{ij}$  and the weights  $w_{ij}$  are calculated and also contained in the table.

Following the procedure, the FGP model for the first-level multi-objective linear fractional programming problem is obtained as

$$\begin{aligned}
 \min \quad Z &= 2D_{11}^+ + D_{12}^+ \\
 \text{subject to} \quad &-2x_1 - 2x_2 + D_{11}^- - D_{11}^+ = -2, \\
 &0.5x_1 + 0.5x_2 + D_{11}^- - D_{11}^+ = 4, \\
 &-x_1 + 2x_2 + D_{11}^+ \leq 7, \\
 &-x_2 + D_{12}^+ \leq 4, \\
 &-x_1 + 4x_2 \leq 0, \\
 &x_1 - \frac{1}{2}x_2 \leq 4, \\
 &x_1, x_2 \geq 0, \quad D_{11}^-, D_{11}^+, D_{12}^-, D_{12}^+ \geq 0.
 \end{aligned} \tag{6.4}$$

Using the LP-ILP linear and integer programming software program, version 1 for windows, the optimal solution of the problem is  $(x_1, x_2) = (0.8, 0.2)$ . Let the first level DM decide  $x_1^* = 0.8$  with the negative and positive tolerance  $t_1^R = t_1^L = 0.4$ .

Then, following the procedure, the proposed FGP model for the bi-level multi-objective linear fractional programming problem is obtained as

$$\begin{aligned}
 \min \quad & Z = 2D_{11}^+ + D_{12}^+ + D_{21}^+ + 0.25D_{22}^+ + 0.25D_{23}^+ + 2.5(d_1^{L-} + d_1^{L+}) + 2.5(d_1^{R-} + d_1^{R+}) \\
 \text{subject to} \quad & -2x_1 - 2x_2 + D_{11}^- - D_{11}^+ = -2, \\
 & 0.5x_1 + 0.5x_2 + D_{11}^- - D_{11}^+ = 4, \\
 & x_1 + D_{21}^- - D_{21}^+ = 4, \\
 & -0.25x_1 - x_2 + D_{22}^- - D_{22}^+ = 0, \\
 & -0.25x_1 + 0.25x_2 + D_{23}^- - D_{23}^+ = 0, \\
 & -x_1 + 2x_2 + D_{11}^+ \leq 7, \\
 & -x_2 + D_{12}^+ \leq 4, \\
 & x_2 + D_{21}^+ \leq 3, \\
 & -x_2 + D_{22}^+ \leq 1, \\
 & D_{23}^+ \leq 1, \\
 & 2.5x_1 + d_1^{L-} - d_1^{L+} = 2, \\
 & 2.5x_1 - d_1^{R-} + d_1^{R+} = 2, \\
 & -x_1 + 4x_2 \leq 0, \\
 & x_1 - \frac{1}{2}x_2 \leq 4, \\
 & x_1, x_2 \geq 0, \\
 & d_1^{L-}, d_1^{L+}, d_1^{R-}, d_1^{R+} \geq 0, \\
 & D_{11}^-, D_{11}^+, D_{12}^-, D_{12}^+, D_{21}^-, D_{21}^+, D_{22}^-, D_{22}^+, D_{23}^-, D_{23}^+ \geq 0.
 \end{aligned} \tag{6.5}$$

The optimal satisfactory of the problem is  $(x_1, x_2) = (0.8, 0.2)$  with objective functions values  $f_{11} = 0$ ,  $f_{12} = 0.048$ ,  $f_{21} = 1.143$ ,  $f_{22} = -2.667$ , and  $f_{23} = 0.6$ , with membership functions values  $\mu_{11} = 1$ ,  $\mu_{12} = 0.45$ ,  $\mu_{21} = 0.1$ ,  $\mu_{22} = 0.667$ , and  $\mu_{23} = 0.1$ .

## 7. Conclusion

This paper presents a fuzzy goal programming procedure for solving bi-level multi-objective linear fractional programming (BL-MOLFP) problems. A fuzzy goal programming model

to minimize the group regret of degree of satisfactions of both the decision makers is developed to achieve the highest degree (unity) of each of the defined membership function goals to the extent possible by minimizing their deviational variables and thereby obtaining the most satisfactory solution for both decision makers. The main advantage of the proposed fuzzy goal programming procedure is that the possibility of rejecting the solution again and again by the FLDM and reevaluation of the problem repeatedly, by redefining the elicited membership functions, needed to reach the satisfactory decision does not arise. A linearization process of solving BL-MOLFP problems via *minsum* FGP is investigated. An illustrative numerical example is given to demonstrate the procedure.

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