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Research Article

A Note on Four-Variable Reciprocity Theorem

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We give new proof of a four-variable reciprocity theorem using Heine's transformation, Watson's transformation, and Ramanujan's ${}_1\psi_1$-summation formula. We also obtain a generalization of Jacobi's triple product identity.

1. Introduction

Throughout the paper, we let $|q| < 1$ and we employ the standard notation:

$$\begin{aligned}
(a)_0 &:= (a;q)_0 = 1, \\
(a)_\infty &:= (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \\
(a)_n &:= (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad -\infty < n < \infty.
\end{aligned} \tag{1.1}$$

Ramanujan [1] stated several q-series identities in his "lost" notebook. One of the beautiful identities is the two-variable reciprocity theorem.

Theorem 1.1 (see [2]). *For $ab \neq 0$,*

$$\rho(a,b) - \rho(b,a) = \left(\frac{1}{b} - \frac{1}{a}\right)\frac{(aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-aq)_\infty (-bq)_\infty}, \tag{1.2}$$

where

$$\rho(a,b) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}. \tag{1.3}$$

In the recent past many new proofs of (1.2) have been found. The first proof of (1.2) was given by Andrews [3] using four-free-variable identity and Jacobi's triple product identity. Further, Andrews [4] applied (1.2) in proving Euler partition identity analogues stated in [1]. Somashekara and Fathima [5] established an equivalent version of (1.2) using Ramanujan's ${}_1\psi_1$ summation formula [6] and Heine's transformation [7, 8]. Berndt et al. [9] also derived (1.2) using the same above mentioned two transformations. In fact, Berndt et al. [9] in the same paper have given two more proofs of (1.2) one employing the Rogers-Fine identity [10] and the other is purely combinatorial. Using the q-binomial theorem:

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}t^n = \frac{(at)_\infty}{(t)_\infty}, \quad |t|<1,\ |q|<1, \tag{1.4}$$

Kim et al. [11] gave a much different proof of (1.2). Guruprasad and Pradeep [12] also devised a proof of (1.2) using the q-binomial theorem. Adiga and Anitha [13] established (1.2) along the lines of Ismail's proof [14] of Ramanujan's ${}_1\psi_1$ summation formula. Further, they showed that the reciprocity theorem (1.2) leads to a q-integral extension of the classical gamma function. Kang [2] constructed a proof of (1.2) along the lines of Venkatachaliengar's proof of the Ramanujan ${}_1\psi_1$ summation formula [6, 15].

In [2] Kang proved the following three- and four-variable generalizations of (1.2).

For $|c|<|a|<1$ and $|c|<|b|<1$,

$$\rho_3(a,b,c)-\rho_3(b,a,c) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c)_\infty(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-c/a)_\infty(-c/b)_\infty(-aq)_\infty(-bq)_\infty}, \tag{1.5}$$

where

$$\rho_3(a,b,c) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(c)_n(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n(-c/b)_{n+1}}, \quad a,\frac{c}{b}\neq -q^{-n}, \tag{1.6}$$

and for $|c|,|d|<|a|,|b|<1$,

$$\rho_4(a,b,c,d)-\rho_4(b,a,c,d) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(d)_\infty(c)_\infty(cd/ab)_\infty(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-d/a)_\infty(-d/b)_\infty(-c/a)_\infty(-c/b)_\infty(-aq)_\infty(-bq)_\infty}, \tag{1.7}$$

where

$$\rho_4(a,b,c,d) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(d)_n(c)_n(cd/ab)_n(1+cdq^{2n}/b)(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n(-c/b)_{n+1}(-d/b)_{n+1}}, \quad a,\frac{c}{b},\frac{d}{b}\neq -q^{-n}. \tag{1.8}$$

Kang [2] established (1.5) on employing Ramanujan's ${}_1\psi_1$ summation formula and Jackson's transformation of ${}_2\phi_1$ and ${}_2\phi_2$-series. Recently (1.5) was derived by Adiga and Guruprasad [16] using q-binomial theorem and Gauss summation formula. Somashekara and Mamta [17, 18] obtained (1.5) using the two-variable reciprocity theorem (1.2), Jackson's transformation, and again two-variable reciprocity theorem by parameter augmentation. Zhang [19] also established (1.5).

Kang [2] established (1.7) on employing Andrews's generalization of ${}_1\psi_1$ summation formula, Sears's transformation of ${}_3\phi_2$-series, and a limiting case of Watson's transformation for a terminating very well-poised ${}_8\phi_7$-series [8]:

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\alpha)_n(\beta)_n(\gamma)_n(\delta)_n(\epsilon)_n(1-\alpha q^{2n})q^{n(n+3)/2}}{(\alpha q/\beta)_n(\alpha q/\gamma)_n(\alpha q/\delta)_n(\alpha q/\epsilon)_n(q)_n(1-\alpha)}\left(\frac{-\alpha^2}{\beta\gamma\delta\epsilon}\right)^n \\ &= \frac{(\alpha q)_\infty(\alpha q/\delta\epsilon)_\infty}{(\alpha q/\delta)_\infty(\alpha q/\epsilon)_\infty}\sum_{n=0}^{\infty}\frac{(\delta)_n(\epsilon)_n(\alpha q/\beta\gamma)_n}{(\alpha q/\beta)_n(\alpha q/\gamma)_n(q)_n}\left(\frac{\alpha q}{\delta\epsilon}\right)^n.\end{aligned} \tag{1.9}$$

Recently Ma [20, 21] proved a six-variable generalization and a five-variable generalization of (1.2). The main purpose of this paper is to provide a new proof of (1.7) using (1.9), Heine's transformation:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(q)_n(\gamma)_n}z^n = \frac{(\gamma/\beta)_\infty(\beta z)_\infty}{(\gamma)_\infty(z)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha\beta z/\gamma)_n(\beta)_n}{(\beta z)_n(q)_n}\left(\frac{\gamma}{\beta}\right)^n, \quad |q|<1,\ |z|<1,\ |\gamma|<|\beta|<1 \tag{1.10}$$

and Ramanujan's ${}_1\psi_1$ summation formula:

$${}_1\psi_1(a;b;z) := \sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n = \frac{(b/a)_\infty(az)_\infty(q/az)_\infty(q)_\infty}{(q/a)_\infty(b/az)_\infty(b)_\infty(z)_\infty}, \quad |q|<1,\ \left|\frac{b}{a}\right|<|z|<1. \tag{1.11}$$

Jacobi's triple product identity states that

$$\sum_{n=-\infty}^{\infty}q^{n(n+1)/2}z^n = (q)_\infty(-zq)_\infty\left(-\frac{1}{z}\right)_\infty, \quad z\neq 0,\ |q|<1. \tag{1.12}$$

Andrews [22] gave a proof of (1.12) using Euler identities. Combinatorial proofs of Jacobi's triple product identity were given by Wright [23], Cheema [24], and Sudler [25]. We can also find a proof of (1.12) in [26]. Using (1.12), Hirschhorn [27, 28] established Jacobi's two-square and four-square theorems.

Somashekara and Fathima [5] and Kim et al. [11] established

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^{-n} q^{n(n+1)/2}}{(-a)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n a^{-(n+1)} b^{n+1} q^{n(n+1)/2}}{(-b)_{n+1}}=\frac{(aq/b)_\infty (b/a)_\infty (q)_\infty}{(-a)_\infty (-b)_\infty}. \tag{1.13}$$

Note that (1.13) which is equivalent to (1.2) may be considered as a two-variable generalization of (1.12). Corteel and Lovejoy [29, equation (1.5)] have given a bijective proof of (1.13) using representations of over partitions. All the reciprocity theorems (1.2), (1.5), and (1.7) are generalizations of Jacobi's triple product identity (1.12).

We also obtain a generalization of Jacobi's triple product identity (1.12) which is due to Kang [2].

2. Proof of (1.7)—The Four-Variable Reciprocity Theorem

On employing q-binomial theorem, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-cq)_n(-dq)_n}{(-aq)_n(-bq)_n}q^n &= \frac{(-dq)_\infty}{(-bq)_\infty}\sum_{n=0}^{\infty}\frac{(-cq)_n(-bq^{n+1})_\infty}{(-aq)_n(-dq^{n+1})_\infty}q^n \\ &= \frac{(-dq)_\infty}{(-bq)_\infty}\sum_{m=0}^{\infty}\frac{(b/d)_m}{(q)_m}(-dq)^m\sum_{n=0}^{\infty}\frac{(-cq)_n}{(-aq)_n}\left(q^{m+1}\right)^n.\end{aligned} \tag{2.1}$$

On using Heine's transformation (1.10) with $\alpha=-cq$, $\beta=q$, $\gamma=-aq$, $z=q^{m+1}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-cq)_n}{(-aq)_n}\left(q^{m+1}\right)^n &= \frac{(q^{m+2})_\infty(-a)_\infty}{(q^{m+1})_\infty(-aq)_\infty}\sum_{n=0}^{\infty}\frac{(cq^{m+2}/a)_n}{(q^{m+2})_n}(-a)^n \\ &= \frac{(q)_m(1+a)(-a)^{-m-1}}{(cq/a)_{m+1}}\sum_{n=0}^{\infty}\frac{(cq/a)_{n+m+1}}{(q)_{n+m+1}}(-a)^{n+m+1} \\ &= \frac{(q)_m(1+a)(-a)^{-m-1}}{(cq/a)_{m+1}}\left[\sum_{n=0}^{\infty}\frac{(cq/a)_n}{(q)_n}(-a)^n-\sum_{n=0}^{m}\frac{(cq/a)_n}{(q)_n}(-a)^n\right] \\ &= \frac{(q)_m(-a)^{-m-1}(-cq)_\infty}{(cq/a)_{m+1}(-aq)_\infty}-\frac{(q)_m(1+a)(-a)^{-m-1}}{(cq/a)_{m+1}}\sum_{n=0}^{m}\frac{(cq/a)_n}{(q)_n}(-a)^n.\end{aligned} \tag{2.2}$$

Substituting this in (2.1), we obtain

$$\sum_{n=0}^{\infty}\frac{(-cq)_n(-dq)_n}{(-aq)_n(-bq)_n}q^n = \frac{(-dq)_\infty(-cq)_\infty}{(-a)(-bq)_\infty(-aq)_\infty}\sum_{m=0}^{\infty}\frac{(b/d)_m}{(cq/a)_{m+1}}\left(\frac{dq}{a}\right)^m + \frac{(1+a^{-1})(-dq)_\infty}{(-bq)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(b/d)_m(dq/a)^m(cq/a)_n(-a)^n}{(cq/a)_{m+1}(q)_n}. \tag{2.3}$$

Now,

$$\begin{aligned}
&\frac{(1+a^{-1})(-dq)_\infty}{(-bq)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(b/d)_m(dq/a)^m(cq/a)_n(-a)^n}{(cq/a)_{m+1}(q)_n}\\
&\quad= \frac{(1+a^{-1})(-dq)_\infty}{(-bq)_\infty}\sum_{n=0}^{\infty}\frac{(b/d)_n(-dq)^n}{(q)_n(1-cq^{n+1}/a)}\sum_{m=0}^{\infty}\frac{(bq^n/d)_m(dq/a)^m}{(cq^{n+2}/a)_m}\\
&\quad= \frac{(1+a^{-1})(-dq)_\infty}{(-bq)_\infty}\sum_{n=0}^{\infty}\frac{(b/d)_n(-dq)^n}{(q)_n(1-cq^{n+1}/a)}\cdot\frac{(1-cq^{n+1}/a)}{(1-dq/a)}\sum_{m=0}^{\infty}\frac{(b/c)_m(cq^{n+1}/a)^m}{(dq^2/a)_m},\\
&\qquad\qquad\left(\text{on using (1.10) with } \alpha=\frac{bq^n}{d},\ \beta=q,\ \gamma=\frac{cq^{n+2}}{a},\ z=\frac{dq}{a}\right)\\
&\quad= \frac{(1+a^{-1})(-dq)_\infty}{(-bq)_\infty}\sum_{m=0}^{\infty}\frac{(b/c)_m(cq/a)^m}{(dq/a)_{m+1}}\sum_{n=0}^{\infty}\frac{(b/d)_n(-dq^{m+1})^n}{(q)_n}\\
&\quad= \frac{(1+a^{-1})(-dq)_\infty}{(-bq)_\infty}\sum_{m=0}^{\infty}\frac{(b/c)_m(cq/a)^m}{(dq/a)_{m+1}}\cdot\frac{(-bq^{m+1})_\infty}{(-dq^{m+1})_\infty}\\
&\quad= \left(1+a^{-1}\right)\sum_{m=0}^{\infty}\frac{(b/c)_m(-dq)_m}{(-bq)_m(dq/a)_{m+1}}\left(\frac{cq}{a}\right)^m.
\end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-cq)_n(-dq)_n}{(-aq)_n(-bq)_n}q^n &= \frac{(-dq)_\infty(-cq)_\infty}{(-a)(-bq)_\infty(-aq)_\infty}\sum_{m=0}^{\infty}\frac{(b/d)_m}{(cq/a)_{m+1}}\left(\frac{dq}{a}\right)^m\\
&\quad+\left(1+a^{-1}\right)\sum_{m=0}^{\infty}\frac{(b/c)_m(-dq)_m}{(-bq)_m(dq/a)_{m+1}}\left(\frac{cq}{a}\right)^m\\
&= \frac{(-dq)_\infty(-cq)_\infty}{(-a)(-bq)_\infty(-aq)_\infty}\sum_{m=0}^{\infty}\frac{(b/c)_m}{(dq/a)_{m+1}}\left(\frac{cq}{a}\right)^m\\
&\quad+\left(1+a^{-1}\right)\sum_{m=0}^{\infty}\frac{(b/c)_m(-dq)_m}{(-bq)_m(dq/a)_{m+1}}\left(\frac{cq}{a}\right)^m.
\end{aligned} \tag{2.5}$$

(Here, we used (1.10) with $\alpha = b/d$, $\beta = q$, $\gamma = cq^2/a$, $z = dq/a$.)

Changing c to $-c/q$, d to $-d/q$ in (2.5), we get

$$\sum_{n=0}^{\infty}\frac{(c)_n(d)_n}{(-aq)_n(-bq)_n}q^n=\frac{(d)_\infty(c)_\infty}{(-a)(-bq)_\infty(-aq)_\infty}\sum_{m=0}^{\infty}\frac{(-bq/c)_m}{(-d/a)_{m+1}}\left(-\frac{c}{a}\right)^m+\left(1+a^{-1}\right)\sum_{m=0}^{\infty}\frac{(-bq/c)_m(d)_m}{(-bq)_m(-d/a)_{m+1}}\left(-\frac{c}{a}\right)^m. \tag{2.6}$$

Interchanging a and b in (2.6), we have

$$\sum_{n=0}^{\infty}\frac{(c)_n(d)_n}{(-aq)_n(-bq)_n}q^n=\frac{(d)_\infty(c)_\infty}{(-b)(-bq)_\infty(-aq)_\infty}\sum_{m=0}^{\infty}\frac{(-aq/c)_m}{(-d/b)_{m+1}}\left(-\frac{c}{b}\right)^m+\left(1+b^{-1}\right)\sum_{m=0}^{\infty}\frac{(-aq/c)_m(d)_m}{(-aq)_m(-d/b)_{m+1}}\left(-\frac{c}{b}\right)^m. \tag{2.7}$$

Subtracting (2.6) from (2.7), we deduce that

$$\frac{(d)_\infty(c)_\infty}{(-bq)_\infty(-aq)_\infty}\left[\frac{1}{b}\sum_{m=0}^{\infty}\frac{(-aq/c)_m}{(-d/b)_{m+1}}\left(-\frac{c}{b}\right)^m-\frac{1}{a}\sum_{m=0}^{\infty}\frac{(-bq/c)_m}{(-d/a)_{m+1}}\left(-\frac{c}{a}\right)^m\right]=\left(1+b^{-1}\right)\sum_{m=0}^{\infty}\frac{(-aq/c)_m(d)_m}{(-aq)_m(-d/b)_{m+1}}\left(-\frac{c}{b}\right)^m-\left(1+a^{-1}\right)\sum_{m=0}^{\infty}\frac{(-bq/c)_m(d)_m}{(-bq)_m(-d/a)_{m+1}}\left(-\frac{c}{a}\right)^m. \tag{2.8}$$

Now change a to $-b/d$, b to $-c/a$, and z to $-d/a$ in (1.11) to obtain

$$\sum_{n=1}^{\infty}\frac{(-b/d)_n}{(-c/a)_n}\left(-\frac{d}{a}\right)^n+\sum_{n=0}^{\infty}\frac{(-aq/c)_n}{(-dq/b)_n}\left(-\frac{c}{b}\right)^n=\frac{(cd/ab)_\infty(b/a)_\infty(aq/b)_\infty(q)_\infty}{(-c/a)_\infty(-c/b)_\infty(-d/a)_\infty(-dq/b)_\infty}. \tag{2.9}$$

Changing n to $n+1$ in the first summation of the above identity and then multiplying both sides by $(1+d/b)^{-1}$, we find that

$$\frac{1}{(1+d/b)}\sum_{n=0}^{\infty}\frac{(-b/d)_{n+1}}{(-c/a)_{n+1}}\left(-\frac{d}{a}\right)^{n+1}+\sum_{n=0}^{\infty}\frac{(-aq/c)_n}{(-d/b)_{n+1}}\left(-\frac{c}{b}\right)^n=\left(1-\frac{b}{a}\right)\frac{(cd/ab)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty}{(-c/a)_\infty(-c/b)_\infty(-d/a)_\infty(-d/b)_\infty}. \tag{2.10}$$

Using (1.10) with $\alpha = -bq/c$, $\beta = q$, $\gamma = -dq/a$, and $z = -c/a$ in the first summation of the above identity and then multiplying both sides by $1/b$, we get

$$\begin{aligned}\frac{1}{b}\sum_{n=0}^{\infty}\frac{(-aq/c)_n}{(-d/b)_{n+1}}\left(-\frac{c}{b}\right)^n - \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-bq/c)_n}{(-d/a)_{n+1}}\left(-\frac{c}{a}\right)^n \\ = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(cd/ab)_\infty (bq/a)_\infty (aq/b)_\infty (q)_\infty}{(-c/a)_\infty(-c/b)_\infty(-d/a)_\infty(-d/b)_\infty}.\end{aligned} \tag{2.11}$$

Substituting (2.11) in (2.8), we see that

$$\begin{aligned}&\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(cd/ab)_\infty(c)_\infty(d)_\infty(bq/a)_\infty(aq/b)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty(-c/a)_\infty(-c/b)_\infty(-d/a)_\infty(-d/b)_\infty} \\ &= \left(1+b^{-1}\right)\sum_{m=0}^{\infty}\frac{(-aq/c)_m(d)_m}{(-aq)_m(-d/b)_{m+1}}\left(-\frac{c}{b}\right)^m - \left(1+a^{-1}\right)\sum_{m=0}^{\infty}\frac{(-bq/c)_m(d)_m}{(-bq)_m(-d/a)_{m+1}}\left(-\frac{c}{a}\right)^m.\end{aligned} \tag{2.12}$$

Now setting $\alpha = -cd/b$, $\beta = cd/ab$, $\gamma = c$, $\delta = q$, and $\epsilon = d$ in (1.9) and then multiplying both sides by $1/(1+d/b)(1+c/b)$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(cd/ab)_n(c)_n(d)_n\left(1+cdq^{2n}/b\right)q^{n(n+1)/2}(-1)^n a^n b^{-n}}{(-aq)_n(-c/b)_{n+1}(-d/b)_{n+1}} \\ = \sum_{n=0}^{\infty}\frac{(-aq/c)_n(d)_n}{(-aq)_n(-d/b)_{n+1}}\left(-\frac{c}{b}\right)^n.\end{aligned} \tag{2.13}$$

Interchanging a and b in (2.13), we have

$$\sum_{n=0}^{\infty}\frac{(cd/ab)_n(c)_n(d)_n\left(1+cdq^{2n}/a\right)q^{n(n+1)/2}(-1)^n b^n a^{-n}}{(-bq)_n(-c/a)_{n+1}(-d/a)_{n+1}} = \sum_{n=0}^{\infty}\frac{(-bq/c)_n(d)_n}{(-bq)_n(-d/a)_{n+1}}\left(-\frac{c}{a}\right)^n. \tag{2.14}$$

Substituting (2.13) and (2.14) in (2.12), we deduce (1.7).

Theorem 2.1 (A four-variable generalization of Jacobi's triple product identity). *For* $|c|, |d| < |a|, |b| < 1$,

$$\begin{aligned}&\frac{(cd/ab)_\infty(c)_\infty(d)_\infty(b/a)_\infty(aq/b)_\infty(q)_\infty}{(-a)_\infty(-b)_\infty(-c/a)_\infty(-c/b)_\infty(-d/a)_\infty(-d/b)_\infty} \\ &= \sum_{m=0}^{\infty}\frac{(d)_m\left(-cq^{-m}/a\right)_m(-1)^m a^m b^{-m} q^{m(m+1)/2}}{(-a)_{m+1}(-d/b)_{m+1}} \\ &\quad - \sum_{m=0}^{\infty}\frac{(d)_m\left(-cq^{-m}/b\right)_m(-1)^m a^{-(m+1)} b^{m+1} q^{m(m+1)/2}}{(-b)_{m+1}(-d/a)_{m+1}}.\end{aligned} \tag{2.15}$$

Proof. Employing

$$\left(-\frac{aq}{c}\right)_m = \left(\frac{a}{c}\right)^m q^{m(m+1)/2}\left(-\frac{cq^{-m}}{a}\right)_m,$$
$$\left(-\frac{bq}{c}\right)_m = \left(\frac{b}{c}\right)^m q^{m(m+1)/2}\left(-\frac{cq^{-m}}{b}\right)_m \tag{2.16}$$

in the right side of (2.12) and then multiplying both sides by $b/(1+a)(1+b)$, we obtain (2.15). □

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References

[1] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Springer, Berlin, Germany, 1988.
[2] S.-Y. Kang, "Generalizations of Ramanujan's reciprocity theorem and their applications," *Journal of the London Mathematical Society*, vol. 75, no. 1, pp. 18–34, 2007.
[3] G. E. Andrews, "Ramanujan's "lost" note book I: partial θ—functions," *Advances in Mathematics*, vol. 41, pp. 137–172, 1981.
[4] G. E. Andrews, "Ramanujan's "lost" notebook. V. Euler's partition identity," *Advances in Mathematics*, vol. 61, no. 2, pp. 156–164, 1986.
[5] D. D. Somashekara and S. N. Fathima, "An interesting generalization of Jacobi's triple product identity," *Far East Journal of Mathematical Sciences*, vol. 9, no. 3, pp. 255–259, 2003.
[6] B. C. Berndt, *Ramanujan's Notebooks. Part III*, Springer, New York, NY, USA, 1991.
[7] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, vol. 71 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1999.
[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, vol. 96 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 2nd edition, 2004.
[9] B. C. Berndt, S. H. Chan, B. P. Yeap, and A. J. Yee, "A reciprocity theorem for certain q-series found in Ramanujan's lost notebook," *Ramanujan Journal*, vol. 13, no. 1–3, pp. 27–37, 2007.
[10] N. J. Fine, *Basic Hypergeometric Series and Applications*, vol. 27 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1988.
[11] T. Kim, D. D. Somashekara, and S. N. Fathima, "On a generalization of Jacobi's triple product identity and its applications," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 165–174, 2004.
[12] P. S. Guruprasad and N. Pradeep, "A simple proof of Ramanujan's reciprocity theorem," *Proceedings of the Jangjeon Mathematical Society*, vol. 9, no. 2, pp. 121–124, 2006.
[13] C. Adiga and N. Anitha, "On a reciprocity theorem of Ramanujan," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 22, no. 1, pp. 9–15, 2006.
[14] M. E. H. Ismail, "A simple proof of Ramanujan's ${}_1\psi_1$ sum," *Proceedings of the American Mathematical Society*, vol. 63, no. 1, pp. 185–186, 1977.
[15] C. Adiga, B. C. Berndt, S. Bhargava, and G. N. Watson, "Chapter 16 of Ramanujan's second notebook: theta-functions and q-series," *Memoirs of the American Mathematical Society*, vol. 53, no. 315, 1985.
[16] C. Adiga and P. S. Guruprasad, "On a three variable reciprocity theorem," *South East Asian Journal of Mathematics and Mathematical Sciences*, vol. 6, no. 2, pp. 57–61, 2008.
[17] D. D. Somashekara and D. Mamta, "A note on three variable reciprocity theorem," to appear in *International Journal of Pure and Applied Mathematics*, 2009 .
[18] D. D. Somashekara and D. Mamta, "On the three variable reciprocity theorem and its applications," to appear in *The Australian Journal of Mathematical Analysis and Applications*.
[19] Z. Zhang, "An identity related to Ramanujan's and its applications," to appear in *Indian Journal of Pure and Applied Mathematics*.

[20] X. R. Ma, "Six-variable generalization of Ramanujan's reciprocity theorem and its variants," *Journal of Mathematical Analysis and Applications*, vol. 353, no. 1, pp. 320–328, 2009.
[21] X. R. Ma, "A five-variable generalization of Ramanujan's reciprocity theorem and its applications," to appear.
[22] G. E. Andrews, "A simple proof of Jacobi's triple product identity," *Proceedings of the American Mathematical Society*, vol. 16, pp. 333–334, 1965.
[23] E. M. Wright, "An enumerative proof of an identity of Jacobi," *Journal of the London Mathematical Society*, vol. 40, pp. 55–57, 1965.
[24] M. S. Cheema, "Vector partitions and combinatorial identities," *Mathematics of Computation*, vol. 18, pp. 414–420, 1964.
[25] C. Sudler Jr., "Two enumerative proofs of an identity of Jacobi," *Proceedings of the Edinburgh Mathematical Society*, vol. 15, pp. 67–71, 1966.
[26] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, New York, NY, USA, 5th edition, 1979.
[27] M. D. Hirschhorn, "A simple proof of Jacobi's two-square theorem," *The American Mathematical Monthly*, vol. 92, no. 8, pp. 579–580, 1985.
[28] M. D. Hirschhorn, "A simple proof of Jacobi's four-square theorem," *Proceedings of the American Mathematical Society*, vol. 101, no. 3, pp. 436–438, 1987.
[29] S. Corteel and J. Lovejoy, "Overpartitions," *Transactions of the American Mathematical Society*, vol. 356, no. 4, pp. 1623–1635, 2004.